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Finite Element Orthogonal Collocation Approach for Time Fractional Telegraph Equation with Mamadu-Njoseh Polynomials

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Abstract

Finite element method (FEM) is an efficient numerical tool for the solution of partial differential equations (PDEs). It is one of the most general methods when compared to other numerical techniques. PDEs posed in a variational form over a given space, say a Hilbert space, are better numerically handled with the FEM. The FEM algorithm is used in various applications which includes fluid flow, heat transfer, acoustics, structural mechanics and dynamics, electric and magnetic field, etc. Thus, in this paper, the Finite Element Orthogonal Collocation Approach (FEOCA) is established for the approximate solution of Time Fractional Telegraph Equation (TFTE) with Mamadu-Njoseh polynomials as grid points corresponding to new basis functions constructed in the finite element space. The FEOCA is an elegant mixture of the Finite Element Method (FEM) and the Orthogonal Collocation Method (OCM). Two numerical examples are experimented on to verify the accuracy and rate of convergence of the method as compared with the theoretical results, and other methods in literature.

Keywords

Sobolev Space, Finite Element Method, Mamadu-Njoseh Polynomials, Orthogonal Collocation Method, Telegraph Equation

1. Introduction

A well defined fractional derivative operator denotes the generalization of derivatives of integer order that allows the introduction of any value of α ($\alpha \geq 0$). Since the dawn of fractional calculus, two fundamental definitions and concepts are applied in practice: Caputo derivative and Riemann-Liouville derivative [1].

These two main definitions are presented in (1.1) and (1.2) respectively

$$ {}_{0}^{C}D_{t}^{\alpha}u(t)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}\frac{u^{(m)}(s)}{(t-s)^{\alpha+1-m}}\mathrm{d}s \tag{1.1}$$

$$ {}_{0}^{R}D_{t}^{\alpha}u(t)=\frac{\mathrm{d}}{\mathrm{d}t^{m}}\left[\frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}\frac{u(s)}{(t-s)^{\alpha+1-m}}\mathrm{d}s\right] \tag{1.2}$$

where $\Gamma(t)$ is a factorial function, α denotes the fractional order with $\alpha < m$, and m is the smallest integer. Two special cases of (1.1) and (1.2) are obtained when the fractional order α assumes different integer values, that is,

$$\begin{cases} {}_{0}^{C}D_{t}^{\alpha}u(t)=\int_{0}^{t}\cdots\int_{0}^{t_m}u(t_n)\mathrm{d}t_n\cdots\mathrm{d}t_1 & \text{if } \alpha\in\mathbb{Z}>0 \\ {}_{0}^{R}D_{t}^{\alpha}u(t)=\dfrac{\partial^{\alpha}u(t)}{\partial t^{\alpha}} & \text{if } \alpha\in\mathbb{Z}>0 \end{cases} \tag{1.3}$$

The Time Fractional Telegraph Equation (TFTE) with fractional order α has the form [1] [2] [3]:

$$\begin{cases} {}_{0}^{C}D_{t}^{\alpha}u(x,t)+u_t(x,t)-u_{xx}(x,t)=f(x,t), & (x,t)\in\Omega\times(0,T], \\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T], \\ u(x,0)=u_0(x), & x\in\partial\Omega, \end{cases} \tag{1.4}$$

where $f(x,t)$ and $u_0(x)$ are functions defined in Sobolev space, $\partial\Omega$ defines the boundary with convex domain $\Omega\subset\mathbb{R}^2$, and ${}_{0}^{C}D_{t}^{\alpha}u(x,t)$ is fractional derivative of the Caputo type.

Many relevant frequencies related problems in real life can be modeled using the TFTE. Analytic procedures to seek the solution of the TFTE seem complicated, and almost impossible due to complex mathematical perturbations and transformations. Thus, different numerical schemes have been developed and implemented over the years by various researchers for solving the TFTE. For instance, Orsinger and Beghin [4] [5] considered the time-fractional telegraph equation perturbed by a Brownian time. The study centered on seeking the fundamental analytic solution to time fractional telegraph equation of fractional order 2α. It was observed that for any α = 0.5, the fundamental solution represents a uniform distribution of a telegraph process perturbed by time. Similarly, Deresse [6] applied the reduced differential transform method to seek the closed form solution of the one dimensional space-time nonlinear comformable fractional telegraph with relevant prescribed initial conditions. The procedure requires no form of transformation, linearization, discretizing, and weak assumptions. The resulting numerical evidence showed absolute convergent.

Gary and Sharma [7] obtained the closed form solutions of a fully discretized space-time fractional telegraph equation via the Adomain decomposition method (ADM). Here, the space-time fractional derivatives were defined in the Caputo fractional sense, with solutions expressed in terms of Mittage-Leffler functions. Prakash [8] presented an Homotopy perturbation transform method (HPTM) to solve space fractional telegraph equation. The author presented the numerical solutions in terms of a convergent series. The method proved to be eloquent and computationally attractive. Similarly, Kamran *et al.* [9] considered a non-mesh method called the hybrid transform based method, to construct the solution of time fractional telegraph equation. The authors applied the Laplace transform method to reduce the finite fractional telegraph equation to a set of finite elliptic equations. The local radial basis functions were then applied to solve the finite set of elliptic equations in parallel, and the solution is represented in terms of an integral in a smooth curve pathway along the complex plain. A major advantage of the method lies in the absence of instability which may have resulted in a time stepping procedure.

Ahmad *et al.* [10] expressed the space-time telegraph equation as a system of linear differential equations. Then, the Adomian decomposition method was used to seek the solution of the resultant system of equations. It was observed that the method converges favourably with more terms in the series. In like manner, Wei *et al.* [11] presented a finite difference scheme for the solution of time-fractional telegraph equation. The authors proved the convergence and stability of the method using the energy algorithm approach. Numerical evidences presented showed that the method is accurate and reliable.

With the advancement of science and technology, there is growing demand for better and efficient numerical techniques for solving the time telegraph equation, hence the need for this paper. Thus, this paper will focus on the application of certain orthogonal polynomials called Mamadu-Njoseh polynomials (see, [12]-[17]) as basis functions in a Finite Element Orthogonal Collocation Approach (FEOCA) for the solution of the TFTE. For the understanding of the method's foundations and other structural elements, readers are advised to consult the authors Mamadu *et al.* [1] [2] [3].

2. Basis Functions and Subspace

Let

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = T \tag{2.1}$$

be the partition of [0, *T*].

Define

$$I_k = \{\varphi_{j-1}(x), \varphi_j(x)\}, \quad h_j = \varphi_j(x) - \varphi_{j-1}(x), \quad 1 \le j \le n,$$

and $h = \max_{1 \le j \le n} h_j$.

Let a finite-dimensional subspace of u be defined as

$$U_h = \{u \in U : u \text{ in each } I_k \text{ is a polynomial of degree} \le r\}. \tag{2.2}$$

Now, basis functions are formulated depending on the degree of the polynomial involved, and also on the nodal points. It should be of note that each basis functions corresponds to a nodal point. Here, the Mamadu-Njoseh polynomials (see, [1] [3] [12] [16] [17]) are treated as grid points $\{\varphi_j(x)\}$ in a finite element space.

Basically, when $r = 1$ (Linear finite element method), the grid points $\{\varphi_j(x)\}$ coincide with the nodal points. Thus, the basis function $\phi_j(x)$ associated with $\varphi_j(x)$ is defined as

$$\phi_j(x) = \begin{cases} 0, & \varphi_1(x) - \varphi_{j-1}(x) < 0 \\ \frac{1}{h_j}\left(\varphi_1(x) - \varphi_{j-1}(x)\right), & x \in \left[\varphi_1(x), \varphi_{j-1}(x)\right] \\ \frac{1}{h_{j+1}}\left(\varphi_{j+1}(x) - \varphi_1(x)\right), & x \in \left[\varphi_1(x), \varphi_{j+1}(x)\right] \end{cases} \tag{2.3}$$

Similarly, for $r = 2$ (Quadratic FEM), it is required to estimate three coefficients from a quadratic basis functions. We must define the three nodal points on each subinterval of $\left[\varphi_{j-1}, \varphi_j\right]$. The centre becomes the extra nodal point since the two endpoints are obvious nodal points.

Thus, the basis functions $\phi_j(x)$ associated with $\varphi_j(x)$ are defined as:

$$\phi_{2j-1}(x) = \begin{cases} 0, & \varphi_1(x) - \varphi_{j-1}(x) < 0 \\ \frac{-4}{h_j^2}\left(\varphi_1(x) - \varphi_{j-1}(x)\right)\left(\varphi_1(x) - \varphi_j(x)\right), & x \in \left[\varphi_{j-1}(x), \varphi_j(x)\right] \\ 0\left(\varphi_{j+1}(x) - \varphi_1(x)\right), & x \in \left[\varphi_1(x), \varphi_{j+1}(x)\right] \end{cases} \tag{2.4}$$

$$\phi_{2j}(x) = \begin{cases} 0, & \varphi_1(x) - \varphi_{j-1}(x) < 0 \\ \frac{2}{h_j^2}\left(\varphi_1 - \varphi_{j-1}\right)\left(\varphi_1 - \varphi_{j-1/2}\right), & x \in \left[\varphi_{j-1}(x), \varphi_j(x)\right] \\ \frac{2}{h_{j+1}^2}\left(\varphi_1 - \varphi_{j+1/2}\right)\left(\varphi_1 - \varphi_{j+1}\right), & x \in \left[\varphi_j(x), \varphi_{j+1}(x)\right] \\ 0 & \varphi_{j+1}(x) < \varphi_1(x) \end{cases} \tag{2.5}$$

3. The Finite Element Orthogonal Collocation Approach (FEOCA)

We need to first show how the fractional derivative in (1.4) can be discretized. By the analogy of Diethlem [19], we transform the Caputo type fractional order to Riemann-Liouville type to enhance the validity of the operator so as to savage the requirements for higher smoothness. Thus, for $m = 1, u(t) = y_0$ (y_0, a constant) in (1.4), we have

$${}_0^R D_t^\alpha y_0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{y_0}{(t-s)^\alpha}\mathrm{d}s\right] = \frac{y_0}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{(1-\alpha)t^\alpha}\right) = \frac{y_0}{\Gamma(1-\alpha)t^\alpha}.$$

Thus,

$${}_0^R D_t^\alpha u(t) = \frac{1}{\Gamma(-\alpha)}\int_0^t \frac{1}{(t-s)^{\alpha-1}} u(s)\mathrm{d}s. \tag{3.1}$$

Let $t_j = \frac{j}{n}, j = 1(2)n$, such that $t \in [0, T]$ is partitioned as $0 = t_0 < \cdots < t_n = T$. Then (3.1) can be approximated in time as

$$ {}_0^R D_t^\alpha u(x,t_j) = \frac{1}{\Gamma(-\alpha)} \int_0^{t_j} \frac{1}{(t_j - s)^{\alpha-1}} u(s)\, \mathrm{d}s . \tag{3.2} $$

Suppose $s = (1-\vartheta)t_j$, then

$$ {}_0^R D_t^\alpha u(x,t_j) = \frac{1}{t_j^\alpha \Gamma(-\alpha)} \int_0^1 \frac{u((1-\vartheta)t_j) - u(0)}{\vartheta^{\alpha-1}} \mathrm{d}w . \tag{3.3} $$

Thus, (3.3) can be rewritten via the quadrature formula as

$$ {}_0^R D_t^\alpha u(x,t_j) = \frac{1}{\Gamma(-\alpha)} \left[\sum\nolimits_{i=0}^{j} \alpha_{ij} u(t_j - t_i) + Y_j(f) \right], \tag{3.4} $$

where

$$ \left\| Y_j(f) \right\| \le a_j^{\alpha-2} \sup\nolimits_{0 \le t \le T} \left\| u''(t_j - t_j t) \right\| . $$

Now, the FEOCA is an elegant mixture of the finite element method and the orthogonal collocation method [2] [3]. The mathematical formulation of the method as applied to TFTE is as follows:

Let $U_h = \{ V_h(x) : V_h(x) \in [0,T] \}$ be linear and continuous on the convex domain $\Omega \subset \mathbb{R}^2$. The weak formulation for the TFTE is to approximate $v(t) \in H^2(\Omega)$ such that

$$ \left({}_0^R D_t^\alpha u(x,t_j), v \right) + (u_t, v) - (u_{xx}, v) = (f, v), v \in H_0^1 . \tag{3.5} $$

By the finite element method (FEM), we compute $V_h(t) \in U_h$, such that,

$$ \left({}_0^R D_t^\alpha u(x,t_j), \beta \right) + (u_t, \beta) = (\beta, \beta_x) + (u_x, \beta_x), \alpha \in U_h , \tag{3.6} $$

which in its abstract sense becomes,

$$ \left({}_0^R D_t^\alpha u(x,t_j), \beta \right) + A_h V_h = H_h f ,\quad t > 0 , \tag{3.7} $$

with $(A_h V_h, \beta) = (u_t, \beta) - (u_x, \beta_x)$, $\alpha \in U_h$, $H_h : H \to U_h$ defined by $(H_h v, \beta) = (v, \beta)$, $\forall v \in H_0^1$, $v \in L_2$, such that $\left\| H_j \right\| \le a^{\alpha-2} \sup_{0 \le t \le T} \left\| u''(t_j - t_j t) \right\|$ for $t_j = \frac{j}{n}, j = 1, 2, \cdots, n$.

Now, let

$$ u = U_j \approx V_h(t_j) = \sum\nolimits_{j=1}^{M-1} \gamma_j \phi_j(t) , \tag{3.8} $$

be an approximation of $V_h(t_j)$, where $\phi_j(t)$, $j = 0, 1, 2, \cdots, M$, are either linear or quadratic finite element basis functions depending on M and γ_j's are unknown parameters. Substituting (3.8) into (3.7), we have,

$$ \left(\frac{\partial^\alpha}{\partial t^\alpha} \left(\sum\nolimits_{j=1}^{M-1} \gamma_j \phi_j(t) \right), \beta \right) + A_h \left(\sum\nolimits_{j=1}^{M-1} \gamma_j \phi_j(t) \right) = H_h f . \tag{3.9} $$

Interpolating (3.9) for $M > 1$, and collocating orthogonally at $\phi_j(t)$ for $j = 0, 1, \cdots, M-1$, yield system of nonlinear equations which on solving via MAPLE 18 yields the approximate solution.

4. Numerical Illustrations

After Here, the FEOCA is experimented on TFTE with the examples below for accuracy and convergence.

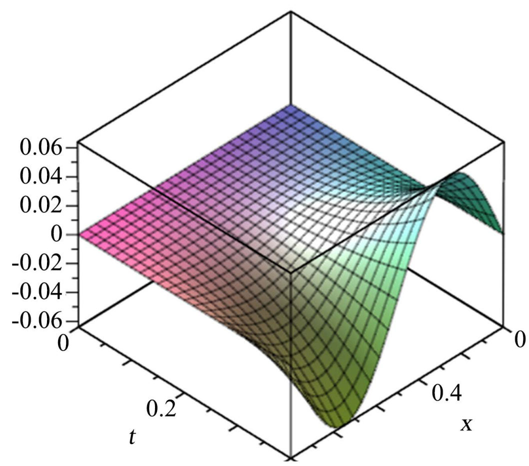
Example 4.1: Consider the time fractional telegraph equation:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^{2} u(x,t)}{\partial x^{2}} = \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} \sin 2\pi x + \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)} \sin 2\pi x + 4\pi^{2} t^{3} \sin 2\pi x, (x,t) \in (0,1) \times (0,T]. \tag{4.1}$$

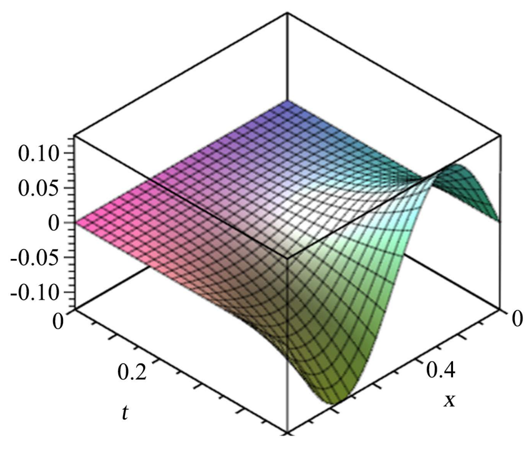
The initial and boundary values conditions can be computed directly from the exact solution given as

$$u(x,t) = t^{3} \sin 2\pi x. \tag{4.2}$$

Applying the scheme (3.1) - (3.9) on (4.1) at $j = 3$ and $N = 3$ with parameters $\alpha = 1.1$, at $t = 0.5$ and 1, we obtained the following results presented in Tables 1-5 and Figures 1-3 via MAPLE 18.

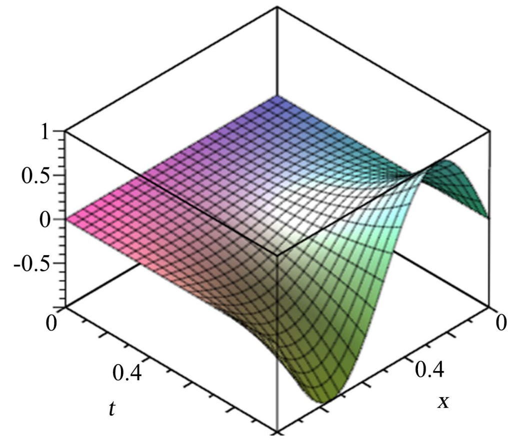


(a)

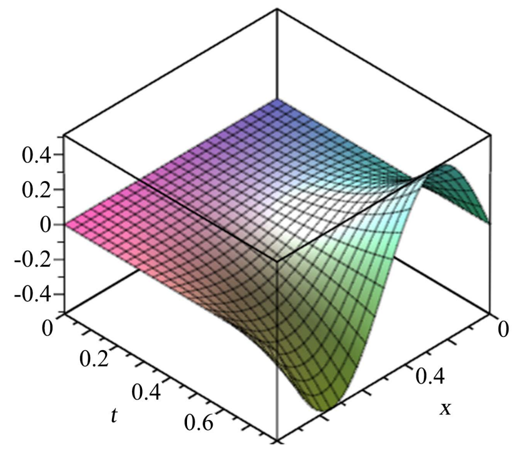


(b)

Figure 1. Solutions on mesh 10 × 10 at $t = 0.5$. (a) Exact solution $u(x,t)$. (b) Computed solution V_h.



(a)



(b)

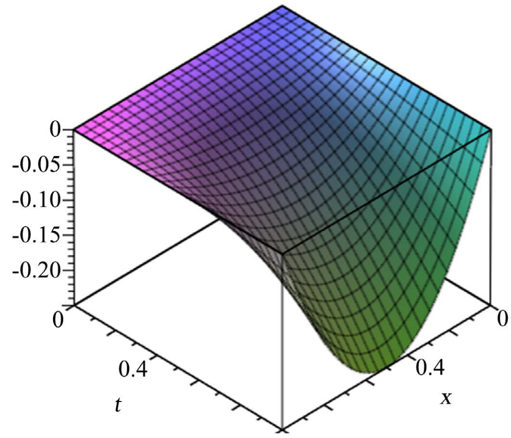
Figure 2. Solutions on mesh 10 × 10 at $t = 1$. (a). Exact solution $u(x,t)$. (b) Computed solution V_h.

Table 1. Maximum error at $\alpha = 1.1$ at $t = 0.5$ for L_2.

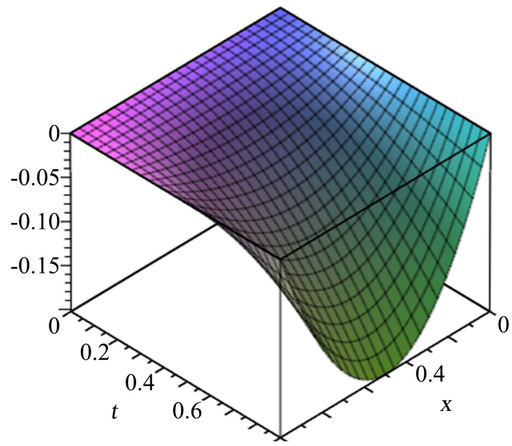
j	L_2 (FEOCA)	L_2 (Wei *et al.* [11])
5	3.712902E−003	0.264327E+00
10	2.543082E−003	0.129217E+00
20	1.058264E−004	6.424721E−002
40	1.069594E−005	3.207865E−002

Table 2. Maximum error at $\alpha = 1.1$ at $t = 0.5$ for L_∞.

j	L_∞ (FEOCA)	L_∞ (Wei *et al.*, [11])
5	0.137545E−004	0.147325E+00
10	3.382301E−005	0.301007E+00
20	1.179475E−005	2.437212E−002
40	0.510544E−006	1.198709E−003



(a)



(b)

Figure 3. Solutions on mesh 10 × 10 at $\alpha = 1.5$, $t = 1$. (a) Exact solution $u(x,t)$; (b) Computed solution V_h.

Table 3. Maximum error at $\alpha = 1.1$ at $t = 1$ for L_2.

j	L_2 (FEOCA)	L_2 (Wei *et al.*, [11])
5	0.3660861E−002	0.363762E+00
10	1.635798E−002	1.080140E+00
20	5.240419E−004	4.536763E−002
40	1.429174E−007	2.553056E−005

Table 4. Maximum error at $\alpha = 1.1$ at $t = 1$ for L_∞.

j	L_∞ (New method)	L_∞ (Wei *et al.*, [11])
5	7.121584E−002	0.302761E+00
10	0.087854E−003	0.414531E+00
20	1.279014E−005	1.384370E−003
40	4.216778E−007	4.112162E−004

Table 5. Comparison of exact and approximate solutions.

x	t = 1, α = 1.1		
	Exact	Computed	Errors
5	0.9510565165	0.965514558	4.04224835E−03
10	0.5877852524	1.993754227	1.4951094E+00
20	0.3090169944	1.0458932356	7.43947536E−01
40	0.1564344651	0.16529663197	3.25478331E−03

Example 4.2: Consider the time fractional telegraph equation:

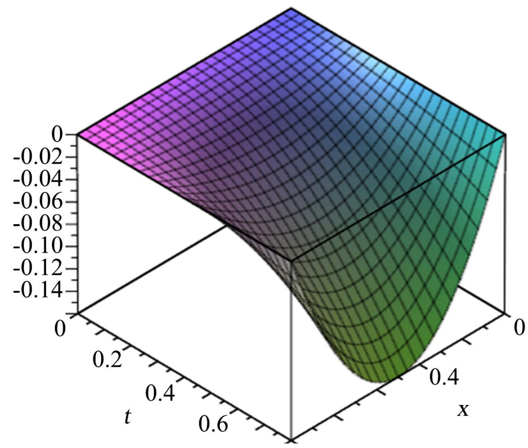
$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 2(x^2 - x)t\left(\frac{\Gamma(3-\alpha)+t^{1-\alpha}}{\Gamma(3-\alpha)}\right) - 2t^2, \; x \in [0,1], \; t \in (0,1], \\ u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, \quad t > 0, \end{cases} \tag{4.3}$$

The exact solution is given as $u(x,t) = (x^2 - x)t^2$.

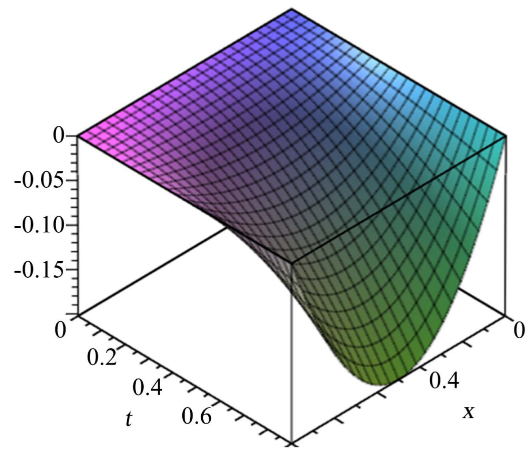
Using (3.1) - (3.9) on (4.3) at $j = N = 3$ with $t = 1$, results are presented below.

5. Discussion of Results

The resulting numerical evidence for **Example 4.1** is expressed in L_2 and L_∞ error norms and compared with Wei *et al.* [17] as shown in **Tables 1-5**. Consequently, maximum errors of order 10^{-5} and 10^{-6} were obtained for $t = 0.5$ with fractional order, $\alpha = 1.1$, as shown in **Table 1** and **Table 2**, respectively. Similarly, maximum errors of order 10^{-7} and 10^{-7} were obtained for $t = 1$ with fractional order, $\alpha = 1.1$, as shown in **Table 3** and **Table 4**, respectively. Also, comparing the results with the standard finite difference method by Wei *et al.* [17] showed that the method FEOCA converges faster and more accurately. Similarly, evaluating **Example 4.2** at $t = 1$, $\alpha = 1.5$ and 1.8, the FEOCA attained maximum error norms of order 10^{-7} and 10^{-7} for L_∞ error norms as shown in **Table 6** and **Table 7**, respectively. Comparison of results between the exact and approximate solutions gave maximum errors of order 10^{-3} (when $\alpha = 1.5$) and 10^{-5} (when $\alpha = 1.8$) at $t = 1$, as shown in **Table 8**. Also, graphical comparison of solutions showed that the computed solutions and the exact solution are in agreement as shown the **Figures 1-4**. In conclusion, we observed that there is a better convergence of the new method as t decreases when error when terms are defined in L_2 and L_∞. However, when it is expressed in absolute error, the new method converges as t increases. Computationally, FEOCA is more efficient than other numerical techniques as seen in Wei *et al.* [11] and Liu *et al.* [18] where they used the standard finite element method and the finite difference method, respectively.



(a)



(b)

Figure 4. Solutions on mesh 10 × 10 at α = 1.8, t = 1. (a) Exact solution $u(x,t)$; (b) Computed solution V_h.

Table 6. Maximum error at $\alpha = 1.5$, $t = 1$ for L_∞.

j	L_∞ (New Method)	L_∞ (Liu *et al.* [18])
20	5.6345E−007	7.011822E−003
40	4.5448E−005	3.255871E−003
80	2.5002E−004	4.323357E−003
160	5.7727E−005	4.547658E−003

Table 7. Maximum error at $\alpha = 1.8$, $t = 1$ for L_∞.

j	L_∞ (New Method)	L_∞ (Liu *et al.* [18])
20	1.25879E−003	1.25781E−002
40	4.78156E−003	1.58784E−002
80	1.12568E−006	1.08504E−002
160	5.5687E−007	9.96507E−003

Table 8. Comparison of exact and approximate solutions at $t = 1$.

x	$t = 1, \alpha = 1.5$			$t = 1, \alpha = 1.8$		
	Exact	Computed	Errors	Exact	Computed	Errors
20	0.047500	0.00237565	4.56250E−02	0.047500	0.04999946	2.49355E−03
40	0.024375	0.000608375	2.37886E−02	0.024375	0.02499954	6.2361E−04
80	0.012344	0.0000015429697	1.22895E−02	0.01234	0.012499955	1.5346E−04
160	0.003211	0.000038818455	6.12212E−03	0.006211	0.000649975	3.1061E−05

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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