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Hopf Bifurcation of Nonresident Computer Virus Model with Age Structure and Two Delays Effects

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Abstract

This paper constructed and studied a nonresident computer virus model with age structure and two delays effects. The non-negativity and boundedness of the solution of the model have been discussed, and then gave the basic regeneration number, and obtained the conditions for the existence and the stability of the virus-free equilibrium and the computer virus equilibrium. Theoretical analysis shows the conditions under which the model undergoes Hopf bifurcation in three different cases. The numerical examples are provided to demonstrate the theoretical results.

Keywords

The Computer Virus Model, Age-Structure, Two Delays, Stability, Hopf Bifurcation

1. Introduction

With the rapid development of computer network technology, computer risks such as hackers, viruses, phishing emails, and other threats to information security are becoming increasingly serious. Computer viruses may damage computer data, occupy computer space and memory resources, damage computer hardware and software, and are known as the "biggest hidden danger of the 21st century". Understand the essence and characteristics of computer virus transmission, strengthen research on network risk prevention strategies, and ensure information security. Therefore, understanding the nature and characteristics of the spread of computer viruses and strengthening research on network risk prevention strategies can help ensure network information security.

There is a high similarity between the transmission process of viruses in the network and the spread of diseases within the population [1]. It leads to more and more scholars constructing computer virus transmission dynamics models based on epidemic compartment models such as SIR models [2], SIRS models [3] [4] [5] [6], SLBS models [7] [8], and then meanwhile a variety of computer virus models have also been built [9]-[15]. In recent years, more experts and scholars have begun to focus their attention on nonresident computer viruses that do not execute themselves from or in computer memory [16] [17] [18]. In [16], the authors constructed the following nonresident computer virus SLAS epidemiological model in the network:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = b - \mu_1 S(t) - \beta_1 S(t)L(t) - \beta_2 S(t)A(t) + \gamma_1 L(t) + \gamma_2 A(t), \\ \dfrac{\mathrm{d}L(t)}{\mathrm{d}t} = \beta_1 S(t)L(t) + \beta_2 S(t)A(t) + \alpha_2 A(t) - (\mu_2 + \alpha_1 + \gamma_1)L(t), \\ \dfrac{\mathrm{d}A(t)}{\mathrm{d}t} = \alpha_1 L(t) - (\mu_3 + \alpha_2 + \gamma_2)A(t). \end{cases} \tag{1}$$

where $S(t)$ represents the numbers of the uninfected computer having no immunity in S class at time t, $L(t)$ represents the numbers of infected computers in which viruses are not yet loaded in their memory (latent computers, for short) in L class at time t, $A(t)$ represents the numbers of infected computers in which viruses are located in memory (infectious computers, for short) in A class at time t, and μ_1 is the rate of the uninfected computers disconnects from the network. And then, give the following assumptions. First, all newly accessed computers are virus-free. Second, all viruses staying in computers are nonresident. Third, users of latent computers cannot perceive the existence of virus, so latent computers cannot get cured.

The model (1) divides the overall process of virus transmission in a computer into three stages. However, combined with the actual situation, we found that the virus in the latent class into the memory at different times, will lead to its transformation into the infection rate is also different. At the same time, the time of virus infection is different, and the time of computer cure is different. Therefore, based on the above analysis, we make improvements to the model and carry out a specific analysis of the model below.

In fact, nonresident viruses in the latent class of computers are loaded into memory after a period of time before they become infected. Therefore, we use the piecewise function $\alpha_1(a)$ to characterize the rate at which a nonresident virus in a latent computer is loaded into memory to become an infection class computer, that is,

$$\alpha_1(a) = \begin{cases} \alpha_1, & a > \tau_1, \\ 0, & a \le \tau_1, \end{cases} \tag{2}$$

where $\alpha_1(a) \in L_+^\infty((0,+\infty),\mathbb{R})$, $\alpha_1 > 0$, a is the length of time the computer stays in the latent compartment, and τ_1 is the minimum time when the latent computers in L class enters in A class. Correspondingly, the outflow of the latent computers in Equation (1) can be rewritten by the partial differential equation as

$$\frac{\partial L(t,a)}{\partial t}+\frac{\partial L(t,a)}{\partial a}=-\left(\mu_2+\alpha_1(a)+\gamma_1\right)L(t,a),\quad t\ge 0,a\ge 0, \tag{3}$$

with $L(t,a)$ is the density of the latent computers with the latent age a at time *t*, μ_2 is the rate of the latent computers disconnects from the network, and γ_1 is the recovery rate of latent computers returns to *S* class.

In addition, it takes time for the infected computers in *A* class to be cured and become the uninfected computers in *S* class. Therefore, we use the piecewise function $\gamma_2(b)$ to characterize the recovery rate of the infected computers, that is,

$$\gamma_2(b)=\begin{cases}\gamma_2, & b>\tau_2,\\ 0, & b\le\tau_2,\end{cases} \tag{4}$$

where $\gamma_2(b)\in L_+^\infty((0,+\infty),\mathbb{R})$, $\gamma_2>0$, *b* refers to the length of time the nonresident virus is in the infection compartment, and τ_2 the maximum time for an infected computer in *A* class to be cured of being the uninfected computers in *S* class. Correspondingly, the outflow of the infected computers in the Equation (1) also can be rewritten by the partial differential equation as

$$\frac{\partial A(t,b)}{\partial t}+\frac{\partial A(t,b)}{\partial b}=-\left(\mu_3+\alpha_2+\gamma_2(b)\right)A(t,b),\quad t\ge 0,b\ge 0, \tag{5}$$

where $A(t,b)$ is the density of the infected computers with the immunity age *b* at time *t*, μ_3 is the rate of the infected computers disconnects from the network, α_2 is the rate of return of infected computers in *A* class to latent computers in *L* class.

Therefore, we have constructed the computer virus epidemic model with age structure and two delays effects as follows. Where infection age and immunity age refers to the time the computer spent in the infected class compartment and the recovery class compartment, respectively, and is a class age rather than the actual age.

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{\mathrm{d}t}=\Lambda-\mu_1S(t)-\beta_1S(t)\displaystyle\int_0^{+\infty}L(t,a)\mathrm{d}a-\beta_2S(t)\int_0^{+\infty}A(t,b)\mathrm{d}b\\ \qquad\qquad+\gamma_1\displaystyle\int_0^{+\infty}L(t,a)\mathrm{d}a+\int_0^{+\infty}\gamma_2(b)A(t,b)\mathrm{d}b,\\ \dfrac{\partial L(t,a)}{\partial t}+\dfrac{\partial L(t,a)}{\partial a}=-\left(\mu_2+\alpha_1(a)+\gamma_1\right)L(t,a),\\ \dfrac{\partial A(t,b)}{\partial t}+\dfrac{\partial A(t,b)}{\partial b}=-\left(\mu_3+\alpha_2+\gamma_2(b)\right)A(t,b),\\ L(t,0)=\beta_1S(t)\displaystyle\int_0^{+\infty}L(t,a)\mathrm{d}a+\beta_2S(t)\int_0^{+\infty}A(t,b)\mathrm{d}b+\alpha_2\int_0^{+\infty}A(t,b)\mathrm{d}b,\quad t\ge 0,\\ A(t,0)=\displaystyle\int_0^{+\infty}\alpha_1(a)L(t,a)\mathrm{d}a,\quad t\ge 0,\end{cases} \tag{6}$$

where the initial condition

$$S(0)=S_0>0, L(0,\cdot)=L_0(\cdot)\in L_+^1(0,+\infty), A(0,\cdot)=A_0(\cdot)\in L_+^1(0,+\infty).$$

Here, Λ is the external computers are accessed to the Internet at positive constant number, β_1, β_2 are the nonresident virus transmission rate, It's worth noting that $\mu_1 \le \min\{\mu_2, \mu_3\}$.

The overall construction of this paper is as follows. In section 2, we study the non-negativity and the boundedness of the solution of the system. In section 3, we investigate the existence of all the feasible equilibria, including the virus-free equilibrium P_0 and the computer virus equilibrium P_*. In section 4, we explore the local stability of the virus-free equilibrium P_0, and the local stability of the computer virus equilibrium P_* when $\tau_1 = \tau_2 = 0$. In section 5, we study the existence of Hopf bifurcation of the system under three different cases, which are 1) $\tau_1 > 0$ and $\tau_2 = 0$, 2) $\tau_1 = 0$ and $\tau_2 > 0$, and 3) $\tau_1 = \tau_2 > 0$, respectively. In section 6, we present some numerical examples and conclusions.

2. Preliminaries

In this section, we focus on the non-negativity and consistent boundedness of the system(6) solution under any non-negative initial value condition.

Theorem 1 If $\phi = \left(S_0, L_0(\cdot), A_0(\cdot)\right) \in \mathbb{R}_+ \times L_+^1(0,+\infty) \times L_+^1(0,+\infty)$, then the solution of the system (6) is non-negative for all $t \ge 0$, and it is ultimately bounded for t large enough.

Proof. For any $S_0, A_0(\cdot) \in \mathbb{R}_+ \times L_+^1$, assume that $S(t)$, $A(t,b)$ remains non-negative for $t \ge 0$. Suppose the assumption does not hold, then the continuity of the solution of the system (6) concerning the initial value shows that the following two cases may occur:

1) There exists a $t_1 > 0$ such that $S(t) > 0$ and $A(t,b) > 0$ for $t \in [0, t_1)$, $A(t_1, b) > 0$, $S(t_1) = 0$ and $S'(t_1) < 0$;

2) There exists a $t_2 > 0$ such that $S(t) > 0$ and $A(t,b) > 0$ for $t \in [0, t_2)$, $S(t_2) > 0$, $A(t_2, b) = 0$ and $\left.\frac{\partial A(t,b)}{\partial t}\right|_{t=t_2} < 0$;

For case 1), by using of the second and fourth equation of system (6), we can get

$$L(t_1, a) = \begin{cases} L(t_1 - a, 0)\mathrm{e}^{-\int_0^a (\mu_2 + \alpha_1(\theta) + \gamma_1)\mathrm{d}\theta}, & a \le t_1, \\ L_0(a - t_1)\mathrm{e}^{-\int_{a-t_1}^a (\mu_2 + \alpha_1(\theta) + \gamma_1)\mathrm{d}\theta}, & a > t_1. \end{cases}$$

Since $S(t_1 - a) > 0$, and $A(t_1 - a, b) > 0$, we can obtain $L(t_1, a) \ge 0$. And then, the first equation of system (6) implies that

$$\left.\frac{\mathrm{d}S(t)}{\mathrm{d}t}\right|_{t=t_1} = \Lambda + \gamma_1 \int_0^{+\infty} L(t_1, a)\mathrm{d}a + \int_0^{+\infty} \gamma_2(b) A(t_1, b)\mathrm{d}b > 0,$$

which contradicts with $S'(t_1) < 0$.

Similarly, for case 2), from the third equation of the system (6) it follows that

$$\left.\frac{\partial A(t,b)}{\partial t}\right|_{t=t_2} = -\frac{\partial A(t_2, b)}{\partial b} - \left(\mu_3 + \alpha_2 + \gamma_2(b)\right) A(t_2, b) = 0,$$

which also contracts with $\left.\frac{\partial A(t,b)}{\partial t}\right|_{t=t_2} < 0$. So the assumption is valid.

Based on the above analysis, for any nonnegative initial values $S_0 \ge 0$, $A_0(\cdot) \ge 0$, the solution of the system (6) is guaranteed to be $S(t) \ge 0$, $A(t,b) \ge 0$, always for all $t \ge 0$. We directly integrate the second equation in system (6) along the characteristic line yields that

$$L(t,a) = \begin{cases} L(t-a,0)\mathrm{e}^{-\int_0^a(\mu_2+\alpha_1(\theta)+\gamma_1)\mathrm{d}\theta}, & a \le t, \\ L_0(a-t)\mathrm{e}^{-\int_{a-t}^a(\mu_2+\alpha_1(\theta)+\gamma_1)\mathrm{d}\theta}, & a > t. \end{cases}$$

It is clear that $L(t,a)$ remains nonnegative for any $t \ge 0$ when $S(t) \ge 0$, $A(t,b) \ge 0$ and $L_0(\cdot) \in L_+^1(0,+\infty)$.

Next, we prove the boundedness of solution of system (6). To this end, we denote $\bar{L}(t) = \int_0^{+\infty} L(t,a)\mathrm{d}a$, and $\bar{A}(t) = \int_0^{+\infty} A(t,b)\mathrm{d}b$, which represents the total number of the latent computers and the infectious computers at time *t*, respectively. Suppose that $\lim_{a\to+\infty} L(t,a) = 0$, and $\lim_{b\to+\infty} A(t,b) = 0$. It is well known these assumptions are in line with practical biological significance, as the lifespan of computers is limited. Then,

$$\begin{aligned}
&\frac{\mathrm{d}\left(S(t)+\bar{L}(t)+\bar{A}(t)\right)}{\mathrm{d}t} \\
&= \Lambda - \mu_1 S(t) - \mu_2 \bar{L}(t) - \mu_3 \bar{A}(t) - \beta_1 S(t)\bar{L}(t) - \beta_2 S(t)\bar{A}(t) - \alpha_2 \bar{A}(t) \\
&\quad - \lim_{a\to+\infty} L(t,a) + L(t,0) - \int_0^{+\infty} \alpha_1(a) L(t,a)\mathrm{d}a - \lim_{b\to+\infty} A(t,b) + A(t,0) \\
&\le \Lambda - \mu_1 S(t) - \mu_2 \bar{L}(t) - \mu_3 \bar{A}(t) \\
&\le \Lambda - \mu_1 \left(S(t) + \bar{L}(t) + \bar{A}(t)\right).
\end{aligned}$$

It implies that $\limsup_{t\to+\infty}\left(S(t)+\bar{L}(t)+\bar{A}(t)\right) \le \frac{\Lambda}{\mu_1}$. That is, the set

$$\Gamma = \left\{ (S,L,A) \in \mathbb{R}_+ \times L_+^1\left((0,+\infty),\mathbb{R}\right) \times L_+^1\left((0,+\infty),\mathbb{R}\right) : \right. \\ \left. S(t) + \int_0^{+\infty} L(t,a)\mathrm{d}a + \int_0^{+\infty} A(t,b)\mathrm{d}b \le \frac{\Lambda}{\mu_1} \right\}$$

is positively invariant with respect to system (6). □

3. The Existence of the Equilibria

In this section can be divided into two main parts, in the first part, we prove the existence of the virus-free equilibrium of the system (6); in the second part, we prove the existence of the computer virus equilibrium and also give an explicit expression for the basic reproduction number of the system (6).

First, a direct calculation shows that there always exists virus-free equilibrium $P_0(S_0,0,0)$ of the system (6), where $S_0 = \frac{\Lambda}{\mu_1}$.

Next, in the second part, in order to obtain the existence of the system (6) computer virus equilibrium $P_*\left(S_*, L_*(a), A_*(b)\right)$, we need to determine the existence of nonnegative solutions to the following system of equations.

$$\begin{cases} \Lambda-\mu_1 S-\beta_1 S\int_0^{+\infty} L(a)\mathrm{d}a-\beta_2 S\int_0^{+\infty} A(b)\mathrm{d}b+\gamma_1\int_0^{+\infty} L(a)\mathrm{d}a \\ +\int_0^{+\infty}\gamma_2(b)A(b)\mathrm{d}b=0, \\ \dfrac{\mathrm{d}L(a)}{\mathrm{d}a}=-\left(\mu_2+\alpha_1(a)+\gamma_1\right)L(a), \\ \dfrac{\mathrm{d}A(b)}{\mathrm{d}b}=-\left(\mu_3+\alpha_2+\gamma_2(b)\right)A(b), \\ L(0)=\beta_1 S\int_0^{+\infty} L(a)\mathrm{d}a+\beta_2 S\int_0^{+\infty} A(b)\mathrm{d}b+\alpha_2\int_0^{+\infty} A(b)\mathrm{d}b, \\ A(0)=\int_0^{+\infty}\alpha_1(a)L(a)\mathrm{d}a. \end{cases} \tag{7}$$

The second of Equation (7) implies that

$$L(a)=L(0)\mathrm{e}^{-\int_0^a\left(\mu_2+\alpha_1(\theta)+\gamma_1\right)\mathrm{d}\theta}, \tag{8}$$

Further from the combination of the third and fifth equations of the system of Equation (7) we can obtain

$$\begin{aligned} A(b)&=\mathrm{e}^{-\int_0^b\left(\mu_3+\alpha_2+\gamma_2(\vartheta)\right)\mathrm{d}\vartheta}\int_0^{+\infty}\alpha_1(a)L(a)\mathrm{d}a \\ &=L(0)\mathrm{e}^{-\int_0^b\left(\mu_3+\alpha_2+\gamma_2(\vartheta)\right)\mathrm{d}\vartheta}\int_0^{+\infty}\alpha_1(a)\mathrm{e}^{-\int_0^a\left(\mu_2+\alpha_1(\theta)+\gamma_1\right)\mathrm{d}\theta}\mathrm{d}a. \end{aligned} \tag{9}$$

Substituting Equation (8) and Equation (9) into the forth equation in Equation (7), we get

$$S=\frac{M_1(\tau_1,\tau_2)}{M_2(\tau_1,\tau_2)}=\frac{\Lambda}{\mu_1\mathcal{R}_0(\tau_1,\tau_2)} \tag{10}$$

with

$$\begin{aligned} &M_1(\tau_1,\tau_2) \\ &=1-\alpha_2\int_0^{+\infty}\mathrm{e}^{-\int_0^b\left(\mu_3+\alpha_2+\gamma_2(\vartheta)\right)\mathrm{d}\vartheta}\mathrm{d}b\int_0^{+\infty}\alpha_1(a)\mathrm{e}^{-\int_0^a\left(\mu_2+\alpha_1(\theta)+\gamma_1\right)\mathrm{d}\theta}\mathrm{d}a \\ &=1-\frac{\alpha_2\mathrm{e}^{\gamma_2\tau_2}}{\mu_3+\alpha_2+\gamma_2}\times\frac{\alpha_2\mathrm{e}^{-(\mu_2+\gamma_1)\tau_1}}{\mu_2+\alpha_1+\gamma_1}, \\ &M_2(\tau_1,\tau_2) \\ &=\beta_2\int_0^{+\infty}\mathrm{e}^{-\int_0^b\left(\mu_3+\alpha_2+\gamma_2(\vartheta)\right)\mathrm{d}\vartheta}\mathrm{d}b\int_0^{+\infty}\alpha_1(a)\mathrm{e}^{-\int_0^a\left(\mu_2+\alpha_1(\theta)+\gamma_1\right)\mathrm{d}\theta}\mathrm{d}a \\ &\quad+\beta_1\int_0^{+\infty}\mathrm{e}^{-\int_0^a\left(\mu_2+\alpha_1(\theta)+\gamma_1\right)\mathrm{d}\theta}\mathrm{d}a \\ &=\beta_1\times\frac{\mathrm{e}^{\alpha_1\tau_1}}{\mu_2+\alpha_1+\gamma_1}+\beta_2\times\frac{\mathrm{e}^{\gamma_2\tau_2}}{\mu_3+\alpha_2+\gamma_2}\times\frac{\alpha_2\mathrm{e}^{-(\mu_2+\gamma_1)\tau_1}}{\mu_2+\alpha_1+\gamma_1}, \\ &\mathcal{R}_0(\tau_1,\tau_2)=\frac{\Lambda}{\mu_1}\frac{M_2(\tau_1,\tau_2)}{M_1(\tau_1,\tau_2)}. \end{aligned}$$

Substituting Equation (8), Equation (9) and Equation (10), into the first equation in Equation (7), we get

$$L(0)=\frac{\Lambda-\mu_1 S}{M_3(\tau_1,\tau_2)}=\frac{\mu_1 S}{M_3(\tau_1,\tau_2)}\left(\mathcal{R}_0(\tau_1,\tau_2)-1\right)$$

with

$$M_3(\tau_1,\tau_2)=1-\frac{\gamma_1 e^{\alpha_1\tau_1}}{\mu_2+\alpha_1+\gamma_1}-\frac{\alpha_2 e^{-(\mu_2+\gamma_1)\tau_1}}{\mu_2+\alpha_1+\gamma_1}\left[\frac{\alpha_2 e^{\gamma_2\tau_2}}{\mu_3+\alpha_2+\gamma_2}+\frac{\gamma_2 e^{-(\mu_3+\gamma_2)\tau_2}}{\mu_3+\alpha_2+\gamma_2}\right].$$

It is clear that $\mathcal{R}_0(\tau_1,\tau_2)>1$ and $M_3(\tau_1,\tau_2)>0$ can ensure $L(0)>0$. That is, system (6) exists the computer virus equilibrium $P_*(S_*,L_*(a),A_*(b))$, where

$$S_*=\frac{\Lambda}{\mu_1\mathcal{R}_0(\tau_1,\tau_2)},\quad L_*(0)=\frac{\mu_1 S_*}{M_3(\tau_1,\tau_2)}(\mathcal{R}_0(\tau_1,\tau_2)-1),$$

$$L_*(a)=L_*(0)e^{-\int_0^a(\mu_2+\alpha_1(\theta)+\gamma_1)d\theta},$$

$$A_*(b)=L_*(0)e^{-\int_0^b(\mu_3+\alpha_2+\gamma_2(\vartheta))d\vartheta}\int_0^{+\infty}\alpha_1(a)e^{-\int_0^a(\mu_2+\alpha_1(\theta)+\gamma_1)d\theta}da.$$

Theorem 2. For the system (6), there always exists the virus-free equilibrium P_0, and when $\mathcal{R}_0(\tau_1,\tau_2)>1$ and $M_3(\tau_1,\tau_2)>0$, it exists the computer virus equilibrium P_*.

4. The Stability of the Equilibria

In this section, we focus on the stability of the virus-free equilibrium P_0 and the computer virus equilibrium P_* of the system (6).

4.1. The Stability of the Virus-Free Equilibrium

Theorem 3. When $\mathcal{R}_0(\tau_1,\tau_2)<1$, the virus-free equilibrium P_0 of the system (6) is locally asymptotically stable, and P_0 is unstable when $\mathcal{R}_0(\tau_1,\tau_2)>1$.

Proof. By linearizing system (6) at virus-free equilibrium P_0, we can obtain the characteristic equation as follows:

$$\begin{aligned}&(\lambda+\mu_1)\Big[1-\beta_1 S_0\int_0^{+\infty}e^{-\int_0^a(\lambda+\mu_2+\alpha_1(\theta)+\gamma_1)d\theta}da\\&-(\beta_2 S_0+\alpha_2)\int_0^{+\infty}e^{-\int_0^b(\lambda+\mu_3+\alpha_2+\gamma_2(\vartheta))d\vartheta}db\int_0^{+\infty}\alpha_1(a)e^{-\int_0^a(\lambda+\mu_2+\alpha_1(\theta)+\gamma_1)d\theta}da\Big]=0.\end{aligned}\tag{11}$$

Obviously, $\lambda_1=-\mu_1$ is a root of Equation (11). And, the remaining characteristic roots of the characteristic Equation (11) must satisfy the following equation:

$$\begin{aligned}\Delta_1(\lambda)=&1-\beta_1 S_0\int_0^{+\infty}e^{-\int_0^a(\lambda+\mu_2+\alpha_1(\theta)+\gamma_1)d\theta}da\\&-(\beta_2 S_0+\alpha_2)\int_0^{+\infty}e^{-\int_0^b(\lambda+\mu_3+\alpha_2+\gamma_2(\vartheta))d\vartheta}db\int_0^{+\infty}\alpha_1(a)e^{-\int_0^a(\lambda+\mu_2+\alpha_1(\theta)+\gamma_1)d\theta}da\\=&0.\end{aligned}\tag{12}$$

When $\lambda\in\mathbb{R}$, it is clear that $\Delta_1(\lambda)=0$ is a continuous strictly monotonically increasing real-valued function on λ, and satisfies

$$\Delta_1(0)=M_1(\tau_1,\tau_2)(1-\mathcal{R}_0(\tau_1,\tau_2)),\quad \lim_{\lambda\to+\infty}\Delta_1(\lambda)=1.$$

It implies that $\Delta_1(\lambda)=0$ has at least one positive real root when $\mathcal{R}_0(\tau_1,\tau_2)>1$, that is, P_0 is unstable. Obviously, $\Delta_1(\lambda)=0$ does not have a positive real root when $\mathcal{R}_0(\tau_1,\tau_2)<1$. We assert that $\Delta_1(\lambda)=0$ also does not have the complex roots with the real part greater than 0. Otherwise, suppose $\bar{\lambda}=\alpha+i\beta(\alpha\geq 0)$ is an arbitrary complex root of $\Delta_1(\lambda)=0$. Therefore,

$$0=\left|\Delta_1\left(\bar{\lambda}\right)\right|=\Delta_1(\alpha)\geq\Delta_1(0)$$
$$=M_1(\tau_1,\tau_2)\left(1-\mathcal{R}_0(\tau_1,\tau_2)\right)>0,$$

which is clearly contradictory. That is, the real parts of the roots of $\Delta_1(\lambda)=0$ all are less than 0.

In summary, when $\mathcal{R}_0(\tau_1,\tau_2)<1$, the virus-free equilibrium P_0 is locally asymptotically stable.

4.2. The Stability of the Computer Virus Equilibrium

In this subsection, we discuss stability of the computer virus equilibrium P_* of system (6) in the case where $\mathcal{R}_0(\tau_1,\tau_2)>1$ and $M_3(\tau_1,\tau_2)>0$.

By direct calculation, we can obtain the characteristic equation of system (6) at P_* as follows:

$$\begin{aligned} f(\lambda,\tau_1,\tau_2)=&\lambda^3+A_2(\tau_1,\tau_2)\lambda^2+A_1(\tau_1,\tau_2)\lambda+A_0(\tau_1,\tau_2)\\ &+B_0(\tau_1,\tau_2)\mathrm{e}^{-\lambda(\tau_1+\tau_2)}+\left(C_1(\tau_1,\tau_2)\lambda+C_0(\tau_1,\tau_2)\right)\mathrm{e}^{-\lambda\tau_1}\\ =&0, \end{aligned}\tag{13}$$

where

$$\begin{aligned} A_2(\tau_1,\tau_2)=&\mu_1+\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b+\mu_2+\alpha_1+\gamma_1\\ &-\beta_1S_*\mathrm{e}^{\alpha_1\tau_1}+\mu_3+\alpha_2+\gamma_2,\\ A_1(\tau_1,\tau_2)=&\left(\mu_1+\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right)\left[\mu_2+\alpha_1+\gamma_1-\beta_1S_*\mathrm{e}^{\alpha_1\tau_1}\right.\\ &\left.+\mu_3+\alpha_2+\gamma_2\right]+\left(\mu_2+\alpha_1+\gamma_1-\beta_1S_*\mathrm{e}^{\alpha_1\tau_1}\right)\left(\mu_3+\alpha_2+\gamma_2\right)\\ &+\left(\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right)\left(\beta_1S_*-\gamma_1\right)\mathrm{e}^{\alpha_1\tau_1}\\ A_0(\tau_1,\tau_2)=&\left(\mu_1+\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right)\left(\mu_2+\alpha_1+\gamma_1-\beta_1S_*\mathrm{e}^{\alpha_1\tau_1}\right)\\ &\times\left(\mu_3+\alpha_2+\gamma_2\right)+\left(\mu_3+\alpha_2+\gamma_2\right)\left(\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right)\\ &\times\left(\beta_1S_*-\gamma_1\right)\mathrm{e}^{\alpha_1\tau_1},\\ B_0(\tau_1,\tau_2)=&-\alpha_1\gamma_2\mathrm{e}^{-(\mu_2+\gamma_1)\tau_1-(\mu_3+\alpha_2)\tau_2}\left(\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right),\\ C_1(\tau_1,\tau_2)=&-\alpha_1\left(\beta_2S_*+\alpha_2\right)\mathrm{e}^{\gamma_2\tau_2-(\mu_2+\gamma_1)\tau_1},\\ C_0(\tau_1,\tau_2)=&-\mu_1\alpha_1\beta_2S_*\mathrm{e}^{\gamma_2\tau_2-(\mu_2+\gamma_1)\tau_1}-\alpha_1\alpha_2\mathrm{e}^{\gamma_2\tau_2-(\mu_2+\gamma_1)\tau_1}\\ &\times\left(\mu_1+\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right). \end{aligned}$$

Theorem 4. Suppose $\mathcal{R}_0(0,0)>1$, and $M_3(0,0)>0$. Then the computer virus equilibrium P_* of the system (6) is locally asymptotically stable.

Proof. When $\tau_1=\tau_2=0$, Equation (13) can be rewritten as

$$\begin{aligned} f(\lambda,0,0)=&\lambda^3+A_2(0,0)\lambda^2+\left(A_1(0,0)+C_1(0,0)\right)\lambda\\ &+A_0(0,0)+B_0(0,0)+C_0(0,0)\\ =&0. \end{aligned}$$

At this time,

$$\mu_2+\alpha_1+\gamma_1-\beta_1 S_*=\mu_2+\alpha_1+\gamma_1-\beta_1\frac{\mu_2+\alpha_1+\gamma_1-\dfrac{\alpha_1\alpha_2}{\mu_3+\alpha_2+\gamma_2}}{\beta_1+\dfrac{\beta_2\alpha_1}{\mu_3+\alpha_2+\gamma_2}}$$

$$=\frac{1}{\beta_1+\dfrac{\beta_2\alpha_1}{\mu_3+\alpha_2+\gamma_2}}\times\frac{1}{\mu_3+\alpha_2+\gamma_2}\times\left[\alpha_1\alpha_2\beta_1+\beta_2\alpha_1\left(\mu_2+\alpha_1+\gamma_1\right)\right]>0.$$

It is clear that $A_2(0,0)>0$.

Next,

$$\begin{aligned}&A_1(0,0)+C_1(0,0)\\&=\left(\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right)\left(\mu_2+\alpha_1+\mu_3+\alpha_2+\gamma_2\right)\\&\quad+\mu_2\left(\mu_2+\alpha_1+\gamma_1-\beta_1S_*+\mu_3+\alpha_2+\gamma_2\right)>0,\end{aligned}$$

$$\begin{aligned}&A_0(0,0)+B_0(0,0)+C_0(0,0)\\&=\left(\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right)\left[\mu_2\left(\mu_3+\alpha_2+\gamma_2\right)+\alpha_1\mu_2\right]\\&>0,\end{aligned}$$

where

$$\int_0^{+\infty}L_*(a)\mathrm{d}a=\frac{\mu_1}{\mu_2+\dfrac{\alpha_1\mu_3}{\mu_3+\alpha_2+\gamma_2}}\times\frac{\mu_2+\alpha_1+\gamma_1-\dfrac{\alpha_1\alpha_2}{\mu_3+\alpha_2+\gamma_2}}{\beta_1+\dfrac{\beta_2\alpha_1}{\mu_3+\alpha_2+\gamma_2}}\times\left(\mathcal{R}_0(0,0)-1\right),$$

$$\begin{aligned}\int_0^{+\infty}A_*(b)\mathrm{d}b=&\frac{\alpha_1\mu_1}{\mu_2\left(\mu_3+\alpha_2+\gamma_2\right)+\alpha_1\mu_3}\times\frac{\mu_2+\alpha_1+\gamma_1-\dfrac{\alpha_1\alpha_2}{\mu_3+\alpha_2+\gamma_2}}{\beta_1+\dfrac{\beta_2\alpha_1}{\mu_3+\alpha_2+\gamma_2}}\\&\times\left(\mathcal{R}_0(0,0)-1\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}&\left(A_1(0,0)+C_1(0,0)\right)A_2(0,0)-\left(A_0(0,0)+B_0(0,0)+C_0(0,0)\right)\\&=\left(\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right)^2\left(\mu_2+\alpha_1+\mu_3+\alpha_2+\gamma_2\right)\\&\quad+\left(\beta_1\int_0^{+\infty}L_*(a)\mathrm{d}a+\beta_2\int_0^{+\infty}A_*(b)\mathrm{d}b\right)\left[\left(\mu_2+\alpha_1+\gamma_1-\beta_1S_*+\mu_3\right.\right.\\&\quad\left.+\alpha_2+\gamma_2\right)\left(2\mu_2+\alpha_1+\mu_3+\alpha_2+\gamma_2\right)+\mu_1\left(\mu_2+\alpha_1+\mu_3+\alpha_2+\gamma_2\right)\\&\quad\left.-\alpha_1\mu_3-\mu_2\left(\mu_3+\alpha_2+\gamma_2\right)\right]+\mu_2\left(\mu_2+\alpha_1+\gamma_1-\beta_1S_*+\mu_3+\alpha_2+\gamma_2\right)\\&\quad\times\left(\mu_2+\alpha_1+\gamma_1-\beta_1S_*+\mu_3+\alpha_2+\gamma_2+\mu_1\right)\\&>0.\end{aligned}$$

The Rausch-Helwitz criterion implies that the real part of the roots of $f(\lambda,0,0)=0$ are all negative. It implies that the computer virus equilibrium state P_* of the system (6) is locally asymptotically stable.

5. Hopf Bifurcation

In this section, we will explore the dynamic behaviors of system (6) in three different cases, including 1) $\tau_1 > 0$, $\tau_2 = 0$; 2) $\tau_1 = 0$, $\tau_2 > 0$; and 3) $\tau_1 = \tau_2 = \tau > 0$, respectively. When an unstable computer virus equilibrium occurs in the system (6), at this point P_* bifurcates and thus changes from unstable to stable. In other words, the system transitions from one stable state to another with periodic oscillations.

Case.1 In the case where $\tau_1 > 0$, and $\tau_2 = 0$.

We will use the method in Section 2 of [19] to discuss the existence of Hopf bifurcation. To this end, Equation (13) can be rewritten as

$$P(\lambda,\tau_1) + Q(\lambda,\tau_1)\mathrm{e}^{-\lambda\tau_1} = 0. \tag{14}$$

where

$$P(\lambda,\tau_1) = \lambda^3 + A_2(\tau_1,0)\lambda^2 + A_1(\tau_1,0)\lambda + A_0(\tau_1,0),$$
$$Q(\lambda,\tau_1) = C_1(\tau_1,0)\lambda + C_0(\tau_1,0) + B_0(\tau_1,0),$$

and the following hypotheses need to be justified:

1) $P(0,\tau_1) + Q(0,\tau_1) \neq 0$;

2) $P(i\omega,\tau_1) + Q(i\omega,\tau_1) \neq 0$;

3) $\lim\sup\left\{\left|\frac{Q}{P}\right| : |\lambda| \to \infty, \operatorname{Re}\lambda \geq 0\right\} < 1$, $\forall \tau_1 \in R_+$;

4) $F(\omega,\tau_1) = |P(i\omega,\tau_1)|^2 - |Q(i\omega,\tau_1)|^2$;

5) Each positive roots $\omega(\tau_1)$ of $F(\omega,\tau_1) = 0$ is continuous and differentiable in τ_1 whenever it exists.

The rigorous calculations can help us obtain

$$\begin{aligned} &P(0,\tau_1) + Q(0,\tau_1) \\ &= \left[\beta_1 \mathrm{e}^{\alpha_1\tau_1}(\mu_3 + \alpha_2 + \gamma_2) + \beta_2\alpha_1\mathrm{e}^{-(\mu_2+\gamma_1)\tau_1}\right]\mu_1 S_*\left(\mathcal{R}_0(\tau_1,0) - 1\right) > 0, \end{aligned}$$

$$\begin{aligned} &P(i\omega,\tau_1) + Q(i\omega,\tau_1) \\ &= -A_2(\tau_1,0)\omega^2 + i\omega\left[A_1(\tau_1,0) + C_1(\tau_1,0) - \omega^2\right] \\ &\quad + A_0(\tau_1,0) + B_0(\tau_1,0) + C_0(\tau_1,0) \neq 0, \end{aligned}$$

$$\lim_{|\lambda|\to\infty}\left|\frac{Q(\lambda,\tau_1)}{P(\lambda,\tau_1)}\right| = \lim_{|\lambda|\to\infty}\left|\frac{C_1(\tau_1,0)\lambda + C_0(\tau_1,0) + B_0(\tau_1,0)}{\lambda^3 + A_2(\tau_1,0)\lambda^2 + A_1(\tau_1,0)\lambda + A_0(\tau_1,0)}\right| = 0.$$

It is clear that the above conditions 1), 2), and 3) are satisfied. Since

$$\begin{aligned} |P(i\omega,\tau_1)|^2 &= \omega^6 + \left[A_2^2(\tau_1,0) - 2A_1(\tau_1,0)\right]\omega^4 \\ &\quad + \left[A_1^2(\tau_1,0) - 2A_2(\tau_1,0)A_0(\tau_1,0)\right]\omega^2 + A_0^2(\tau_1,0), \end{aligned}$$

$$|Q(i\omega,\tau_1)|^2 = C_1^2(\tau_1,0)\omega^2 + \left(B_0(\tau_1,0) + C_0(\tau_1,0)\right)^2,$$

we can get

$$\begin{aligned} F(\omega,\tau_1) = \omega^6 &+ \left(A_2^2(\tau_1,0) - 2A_1(\tau_1,0)\right)\omega^4 + \left(A_1^2(\tau_1,0) - 2A_2(\tau_1,0)A_0(\tau_1,0)\right. \\ &\left. - C_1^2(\tau_1,0)\right)\omega^2 + A_0^2(\tau_1,0) - \left(B_0(\tau_1,0) + C_0(\tau_1,0)\right)^2. \end{aligned} \tag{15}$$

It means that condition 4) holds. Furthermore, since both $A_i(\tau_1,0)(i=0,1,2)$, $B_0(\tau_1,0)$ and $C_j(\tau_1,0)(j=0,1)$ are continuous differentiable functions with respect to τ_1 the implicit function theorem ensures condition 5) also satisfied.

Let $\lambda = i\omega(\omega>0)$ be a pure imaginary root of Equation (14), then

$$\begin{aligned}&-i\omega^3 - A_2(\tau_1,0)\omega^2 + A_1(\tau_1,0)i\omega + A_0(\tau_1,0)\\&+\left[C_1(\tau_1,0)i\omega + C_0(\tau_1,0) + B_0(\tau_1,0)\right]\left[\cos\omega\tau_1 - i\sin\omega\tau_1\right]=0.\end{aligned} \tag{16}$$

And then, let $\Theta=\omega^2$, then Equation (15) can be rewritten as

$$Q(\Theta) := \Theta^3 + q_2(\tau_1)\Theta^2 + q_1(\tau_1)\Theta + q_0(\tau_1), \tag{17}$$

where

$$\begin{aligned}q_2(\tau_1) &= A_2^2(\tau_1,0) - 2A_1(\tau_1,0),\\ q_1(\tau_1) &= A_1^2(\tau_1,0) - 2A_2(\tau_1,0)A_0(\tau_1,0) - C_1^2(\tau_1,0),\\ q_0(\tau_1) &= A_0^2(\tau_1,0) - \left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)^2,\end{aligned}$$

and $M_3(\tau_1,0)>0$ ensures $q_0(\tau_1)>0$ holds. Since the existence of pure imaginary root of Equation (15) is equivalent to the existence of the positive root of Equation (17), we first discuss the existence of the positive root of $Q(\Theta)=0$.

Let $Q'(\Theta)=3\Theta^2+2q_2(\tau_1)\Theta+q_1(\tau_1)$. On the one hand, if $q_2^2(\tau_1)-3q_1(\tau_1)<0$, then $Q'(\Theta)=0$ has no real root; on the other hand, if $q_2^2(\tau_1)-3q_1(\tau_1)\ge 0$, then $Q'(\Theta)=0$ at least has one real root, in which Θ_2 is a bigger root. Therefore, the following lemma gives the existence of the positive root of $Q(\Theta)=0$.

Lemma 5. 1) If $q_2^2(\tau_1)-3q_1(\tau_1)<0$, then $Q(\Theta)=0$ has no positive root; 2) if $q_2^2(\tau_1)-3q_1(\tau_1)\ge 0$, and $\Theta_2\le 0$, then $Q(\Theta)=0$ has no positive root; 3) if $q_2^2(\tau_1)-3q_1(\tau_1)\ge 0$, $\Theta_2>0$, and $Q(\Theta_2)>0$, then $Q(\Theta)=0$ has no positive root; 4) if $q_2^2(\tau_1)-3q_1(\tau_1)\ge 0$, $\Theta_2>0$, and $Q(\Theta_2)\le 0$, then $Q(\Theta)=0$ has the positive roots.

If Equation (17) has no positive roots, then the stability of the computer virus equilibrium P_* does not change as τ_1 increases. Conversely, if there exists the positive root in Equation (17), the stability of the computer virus equilibrium P_* may change when τ_1 reaches some critical value τ_{1*}. At this point, Hopf bifurcation may appear in the system. In summary, we have the following conclusions:

Theorem 6. Suppose than $\mathcal{R}_0(\tau_1,0)>1$, $M_3(\tau_1,0)>0$, and $\tau_1>0$ hold. 1) If $q_2^2(\tau_1)-3q_1(\tau_1)<0$, then the computer virus equilibrium P_* of system (6) is locally asymptotically stable; 2) if $q_2^2(\tau_1)-3q_1(\tau_1)\ge 0$, and $\Theta_2\le 0$, then the computer virus equilibrium P_* of system (6) is locally asymptotically stable; 3) if $q_2^2(\tau_1)-3q_1(\tau_1)\ge 0$, $\Theta_2>0$, and $Q(\Theta_2)>0$, then the computer virus equilibrium P_* of system (6) is locally asymptotically stable.

Next, we judge the Hopf bifurcation of the system. If $Q(\Theta)=0$ has a positive root, then the stability of the computer virus equilibrium P_* may change when τ_1 passes through some specific values. Let us consider whether the stability of the positive equilibrium changes when $Q(\Theta)=0$ has one positive root. In the case where $Q(\Theta)=0$ has two positive roots, the analysis is similar.

Let Θ_* be the positive root of $Q(\Theta)=0$, that is, $\omega(\tau_{1*})=\sqrt{\Theta_*}$ is the positive real root of $F(\omega,\tau_1)=0$. Then, we define a set by

$$\Sigma=\left\{\tau_1>0:q_2^2(\tau_1)-3q_1(\tau_1)\geq 0,\Theta_2>0,Q(\Theta_2)\leq 0\right\}.$$

That is, for $\tau_1\in\Sigma$, there exists $\omega=\omega(\tau_1)>0$ such that $F(\omega,\tau_1)=0$.

Separating the real part and the imaginary part from Equation (16)

$$-\omega^3+A_1(\tau_1,0)\omega=\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)\sin\omega\tau_1-C_1(\tau_1,0)\omega\cos\omega\tau_1,$$
$$A_2(\tau_1,0)\omega^2-A_0(\tau_1,0)=\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)\cos\omega\tau_1+C_1(\tau_1,0)\omega\cos\omega\tau_1.$$

Let $\theta(\tau_1)\in(0,2\pi]$ ($\tau_1\in\Sigma$) be a solution of the following equations:

$$\cos\theta(\tau_1)=\frac{C_1(\tau_1,0)\omega^4+\left[A_2(\tau_1,0)\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)-A_1(\tau_1,0)C_1(\tau_1,0)\right]\omega^2}{C_1^2(\tau_1,0)\omega^2+\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)^2}-\frac{A_0(\tau_1,0)\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)}{C_1^2(\tau_1,0)\omega^2+\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)^2},$$

$$\sin\theta(\tau_1)=\frac{\left[A_1(\tau_1,0)\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)-A_0(\tau_1,0)C_1(\tau_1,0)\right]\omega}{C_1^2(\tau_1,0)\omega^2+\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)^2}+\frac{\left[A_2(\tau_1,0)C_1(\tau_1,0)-B_0(\tau_1,0)-C_0(\tau_1,0)\right]\omega^3}{C_1^2(\tau_1,0)\omega^2+\left(B_0(\tau_1,0)+C_0(\tau_1,0)\right)^2}$$

We conclude that $\omega(\tau_1)\tau_1=\theta(\tau_1)+2n\pi$. Hence, $\omega_*=\omega(\tau_{1*})$ ($\omega_*>0$) is a purely imaginary root of Equation (14) if and only if τ_{1*} is a zero of $C_n(\tau_1)$ for some $n\in\mathbb{N}$, which is defined by

$$C_n(\tau_1)=\tau_1-\frac{\theta(\tau_1)+2n\pi}{\omega(\tau_1)},\quad \tau_1\in\Sigma,n\in\mathbb{N}.$$

Theorem 2.2 in [19] implies that the following lemma is true.

Lemma 7. ([19]): Assume that $\omega(\tau_1)$ is a positive real root of $F(\omega,\tau_1)=0$ for $\tau_1\in\Sigma$, and at some $\tau_{1*}\in\Sigma$,

$$C_n(\tau_{1*})=0,\ \text{for some } n\in\mathbb{N}.$$

Then a pair of simple conjugate pure imaginary roots $\lambda_+(\tau_{1*})=+i\omega(\tau_{1*})$, and $\lambda_-(\tau_{1*})=-i\omega(\tau_{1*})$ of the characteristic Equation (14) exists at $\tau=\tau_{1*}$ which crosses the imaginary axis from left to right if $H(\tau_{1*})>0$ and crosses the imaginary axis from right to left if $H(\tau_{1*})<0$, where

$$H(\tau_{1*})=sign\left\{\left.\frac{\mathrm{d}Re(\lambda)}{\mathrm{d}\tau_1}\right|_{\lambda=i\omega(\tau_{1*})}\right\}=sign\left\{F'_\omega\left(\omega(\tau_{1*}),\tau_{1*}\right)\right\}sign\left\{\left.\frac{\mathrm{d}C_n(\tau_1)}{\mathrm{d}\tau_1}\right|_{\tau_1=\tau_{1*}}\right\}.$$

Therefore

$$H(\tau_{1*}) = sign\left\{\frac{\mathrm{d}Re(\lambda)}{\mathrm{d}\tau_1}\bigg|_{\lambda=i\omega(\tau_{1*})}\right\}$$
$$= sign\{Q'(\Theta_*)\}\, sign\left\{\frac{\mathrm{d}C_n(\tau_1)}{\mathrm{d}\tau_1}\bigg|_{\tau_1=\tau_{1*}}\right\},$$

that means the transversality condition holds. So a Hopf bifurcation occurs when $Q'(\Theta_*) \neq 0$ and $\tau_1 = \tau_{1*}$, and the following conclusion is obtained according to the Hopf bifurcation theorem.

Theorem 8. Suppose $\mathcal{R}_0(\tau_1,0) > 1$, $M_3(\tau_1,0) > 0$, and $Q'(\Theta) \neq 0$. If $q_2^2(\tau_1) - 3q_1(\tau_1) \geq 0$, $\Theta_2 > 0$, and $Q(\Theta_2) \leq 0$, then the network virus equilibrium P_* of system (6) is locally asymptotically stable for $\tau_2 \in [0, \tau_{2*})$, and the system (6) undergoes a Hopf bifurcation at the network virus equilibrium P_* when $\tau_2 = \tau_{2*}$.

Case.2 In the case where $\tau_1 = 0$, $\tau_2 > 0$.

When $\tau_1 = 0$ and $\tau_2 > 0$, using the same approach to prove the stability analysis of the computer virus equilibrium P_* and the occurrence of Hopf bifurcation as τ_2 increases. To this end, Equation (13) can be rewritten as

$$P(\lambda,\tau_2) + Q(\lambda,\tau_2)\mathrm{e}^{-\lambda\tau_2} = 0. \tag{18}$$

where

$$P(\lambda,\tau_1) = \lambda^3 + A_2(0,\tau_2)\lambda^2 + \big(A_1(0,\tau_2) + C_1(0,\tau_2)\big)\lambda + A_0(0,\tau_2) + B_0(0,\tau_2),$$
$$Q(\lambda,\tau_1) = B_0(0,\tau_2),$$

and then

$$q_2(\tau_2) = A_2^2(0,\tau_2) - 2\big(A_1(0,\tau_2) + C_1(0,\tau_2)\big),$$
$$q_1(\tau_2) = \big(A_1(0,\tau_2) + C_1(0,\tau_2)\big)^2 - 2A_2(0,\tau_2)\big(A_0(0,\tau_2) + C_0(0,\tau_2)\big),$$
$$q_0(\tau_2) = \big(A_0(0,\tau_2) + C_0(0,\tau_2)\big)^2 - B_0^2(0,\tau_2).$$

Unlike in case 1, there exists $q_0(\tau_2) < 0$ which means that $Q(\Theta) = 0$ has a positive real root, and the system (6) also generates a Hopf bifurcation. In summary, we have the following conclusions:

Theorem 9. Suppose than $\mathcal{R}_0(0,\tau_2) > 1$, $M_3(0,\tau_2) > 0$, and $\tau_2 > 0$ hold. 1) If $q_2^2(\tau_2) - 3q_1(\tau_2) < 0$, then the computer virus equilibrium P_* of system (6) is locally asymptotically stable; 2) if $q_2^2(\tau_2) - 3q_1(\tau_2) \geq 0$, and $\Theta_2 \leq 0$, then the computer virus equilibrium P_* of system (6) is locally asymptotically stable; 3) if $q_2^2(\tau_2) - 3q_1(\tau_2) \geq 0$, $\Theta_2 > 0$, and $Q(\Theta_2) > 0$, then the computer virus equilibrium P_* of system (6) is locally asymptotically stable.

And the following conclusion is about the Hopf bifurcation.

Theorem 10. Suppose $\mathcal{R}_0(0,\tau_2) > 1$, $M_3(0,\tau_2) > 0$ and $\tau_2 > 0$ holds. If $q_0(\tau_2) < 0$, or $q_0(\tau_2) > 0$, $q_2^2(\tau_2) - 3q_1(\tau_2) \geq 0$, $\Theta_2 > 0$, and $Q(\Theta_2) \leq 0$, then the computer virus equilibrium P_* of system (6) is locally asymptotically stable for $\tau_2 \in [0, \tau_{2*})$, and the system (6) undergoes a Hopf bifurcation at the computer virus equilibrium P_* when $\tau_2 = \tau_{2*}$.

Case.3 In the case where $\tau_1 = \tau_2 = \tau > 0$.

The characteristic Equation (13) is rewritten as the transcendental equation

$$D(\lambda,\tau) := P_0(\lambda,\tau) + P_1(\lambda,\tau)e^{-\lambda\tau} + P_2(\lambda,\tau)e^{-2\lambda\tau} = 0, \tag{19}$$

where $\tau \in \mathbb{R}_+$, and

$$P_0(\lambda,\tau) = \lambda^3 + A_2(\tau)\lambda^2 + A_1(\tau)\lambda + A_0(\tau),$$
$$P_1(\lambda,\tau) = C_1(\tau)\lambda + C_0(\tau),$$
$$P_2(\lambda,\tau) = B_0(\tau).$$

When $P_l(\lambda,\tau)(l = 0,1,2)$ are independent of the delay τ. Following Section 2 in [20], we need to justify the following hypotheses:

1) $\deg(P_0(\lambda,\tau)) \ge \max\{\deg(P_1(\lambda,\tau)), \deg(P_2(\lambda,\tau))\}$;

2) $P_2(0,\tau) + P_1(0,\tau) + P_0(0,\tau) \ne 0$;

3) The polynomials $P_l(\lambda,\tau), l = 0,1,2$ have no common factor;

4) $\lim\limits_{\substack{Re\lambda\ge 0\\ |\lambda|\to+\infty}} \sup\left(\left|\frac{P_1(\lambda,\tau)}{P_0(\lambda,\tau)}\right| + \left|\frac{P_2(\lambda,\tau)}{P_0(\lambda,\tau)}\right|\right) < 1$;

5) $P_l(i\omega,\tau) \ne 0$, $l = 0,1,2$;

6) For any $\omega \in \mathbb{R}_+$, at least one of $|P_l(i\omega,\tau)|$, $l = 0,1,2$ tends to $+\infty$ as $\tau \to -\infty$.

The conditions 1) and 3) apparently established. Through a tedious manipulation, we can derive that

$$\begin{aligned}
&P_2(0,\tau) + P_1(0,\tau) + P_0(0,\tau) = A_0(\tau) + B_0(\tau) + C_0(\tau) \\
&= \Big[(\mu_3 + \alpha_2 + \gamma_2)(\mu_2 + \alpha_1 + \gamma_1 - \gamma_1 e^{\alpha_1\tau}) \\
&\quad - \alpha_1 e^{-(\mu_2+\gamma_1)\tau}\left(\alpha_2 e^{\gamma_2\tau} + \gamma_2 e^{-(\mu_3+\gamma_2)\tau}\right)\Big]\left(\beta_1\int_0^{+\infty} L_*(a)\,da + \beta_2\int_0^{+\infty} A_*(b)\,db\right) \\
&\ne 0,
\end{aligned}$$

$$\begin{aligned}
&\lim\limits_{\substack{Re\lambda\ge 0\\ |\lambda|\to+\infty}}\left(\left|\frac{P_1(\lambda,\tau)}{P_0(\lambda,\tau)}\right| + \left|\frac{P_2(\lambda,\tau)}{P_0(\lambda,\tau)}\right|\right) \\
&= \lim\limits_{\substack{Re\lambda\ge 0\\ |\lambda|\to+\infty}}\left(\left|\frac{C_1(\tau)\lambda + C_0(\tau)}{\lambda^3 + A_2(\tau)\lambda^2 + A_1(\tau)\lambda + A_0(\tau)}\right|\right) \\
&\quad + \lim\limits_{\substack{Re\lambda\ge 0\\ |\lambda|\to+\infty}}\left(\left|\frac{B_0(\tau)}{\lambda^3 + A_2(\tau)\lambda^2 + A_1(\tau)\lambda + A_0(\tau)}\right|\right) \\
&= 0 < 1,
\end{aligned}$$

$$P_0(i\omega,\tau) = -i\omega^3 - A_2(\tau)\omega^2 + A_1(\tau)i\omega + A_0(\tau) \ne 0,$$

$$P_1(i\omega,\tau) = C_1(\tau)i\omega + C_0(\tau) \ne 0,$$

$$P_2(i\omega,\tau) = B_0(\tau) \ne 0,$$

which implies that the above conditions 2), 4) and 5) are satisfied. Noting that

$$|P_0(i\omega,\tau)|^2 = \omega^6 + \left(A_2^2(\tau) - 2A_1(\tau)\right)\omega^4 + \left(A_1^2(\tau) - 2A_0(\tau)A_2(\tau)\right)\omega^2 + A_0^2(\tau),$$

$$\left|P_1(i\omega,\tau)\right|^2 = C_1^2(\tau)\omega^2 + C_0^2(\tau),$$

$$\begin{aligned}\left|P_2(i\omega,\tau)\right|^2 &= B_0^2(\tau) \\ &= \left[-\alpha_1\gamma_2 \mathrm{e}^{-(\mu_2+\gamma_1)\tau-(\mu_3+\alpha_2)\tau}\left(\beta_1\int_0^{+\infty} L_*(a)\mathrm{d}a + \beta_2\int_0^{+\infty} A_*(b)\mathrm{d}b\right)\right]^2.\end{aligned}$$

When $\tau \to -\infty$, we can get $\left|P_2(i\omega,\tau)\right| \to +\infty$. Obviously, 6) also satisfies.

In the following, we search the points $\tau \in \mathbb{R}_+$ such that $\lambda = i\omega(\omega > 0)$ is a zero of Equation (19). Let

$$a_1(\omega,\tau) = \frac{P_1(i\omega,\tau)}{P_0(i\omega,\tau)}, \quad a_2(\omega,\tau) = \frac{P_2(i\omega,\tau)}{P_0(i\omega,\tau)}.$$

Then, $(i\omega,\tau)$ is zero of Equation (19), if and only if

$$\mathcal{D}(\omega,\tau) \equiv 1 + a_1(\omega,\tau)\mathrm{e}^{-i\omega\tau} + a_2(\omega,\tau)\mathrm{e}^{-i\omega 2\tau} = 0. \tag{20}$$

Suppose that $(i\omega,\tau)$ is the zero of Equation (19), Then the 3 parts 1, $a_1(\omega,\tau)\mathrm{e}^{-i\omega\tau}$, and $a_2(\omega,\tau)\mathrm{e}^{-i\omega 2\tau}$ on the right side of the above equation must be connected and form a triangle in the complex plane, as shown in **Figure 1**. Therefore, we can obtain the feasible region of (ω,τ) in the following lemma.

Lemma 11. For $\mathcal{R}_0(\tau,\tau) > 1$, and $M_3(\tau,\tau) > 0$ holds, the feasible region Ω for (ω,τ), such that 1, $|a_1(\omega,\tau)|$ and $|a_2(\omega,\tau)|$ create a triangle, is

$$\Omega = \left\{(\omega,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+ : H_1 \ge 0, H_2 \le 0\right\}$$

where

$$\begin{aligned}H_1 &= \omega^6 + F_1(\tau)\omega^4 + F_2(\tau)\omega^2 + F_{31}(\tau)\omega + F_{41}(\tau), \\ H_2 &= \omega^6 + F_1(\tau)\omega^4 + F_2(\tau)\omega^2 - F_{32}(\tau)\omega + F_{42}(\tau),\end{aligned}$$

at the same time

$$F_1(\tau) = A_2^2(\tau) - 2A_1(\tau),$$

$$F_2(\tau) = A_1^2(\tau) - 2A_0(\tau)A_2(\tau) - B_0^2(\tau)N_1^2(\tau),$$

$$F_{31}(\tau) = 2N_1(\tau)B_0^2(\tau)\left(1 - N_2(\tau)\right),$$

$$F_{32}(\tau) = 2N_1(\tau)B_0^2(\tau)\left(1 + N_2(\tau)\right),$$

$$F_{41}(\tau) = A_0^2(\tau) - B_0^2(\tau)\left(1 - N_2(\tau)\right)^2,$$

$$F_{42}(\tau) = A_0^2(\tau) - B_0^2(\tau)\left(1 + N_2(\tau)\right)^2,$$

$$N_1(\tau) = \frac{\alpha_1}{B_0(\tau)}\left(\beta_2 S_* + \alpha_2\right)\mathrm{e}^{(\gamma_2-\mu_2-\gamma_1)\tau},$$

$$N_2(\tau) = \mu_1 N_1(\tau) + \frac{\alpha_2}{\gamma_2}\mathrm{e}^{(\mu_3+2\gamma_2)\tau}.$$

Proof. We let $\left|P_1(i\omega,\tau)\right| = \left(N_1(\tau)\omega + N_2(\tau)\right)\left|P_2(i\omega,\tau)\right|$.

First, $\left|P_0(\lambda,\tau)\right| + \left|P_1(\lambda,\tau)\right| \ge \left|P_2(\lambda,\tau)\right|$ is equivalent to

$$\omega^6 + F_1(\tau)\omega^4 + F_2(\tau)\omega^2 + F_{31}(\tau)\omega + F_{41}(\tau) \ge 0.$$

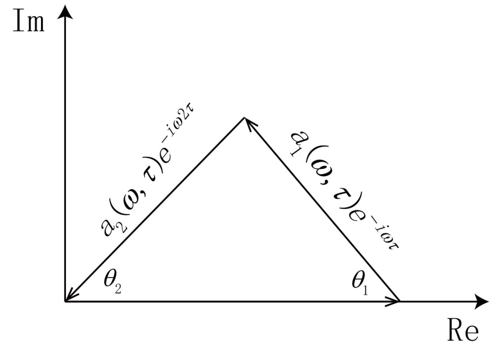


Figure 1. Triangle formed by 1, $a_1(\omega,\tau)\mathrm{e}^{-i\omega\tau}$, $a_2(\omega,\tau)\mathrm{e}^{-i\omega 2\tau}$.

Second, for $|P_0(\lambda,\tau)|+|P_2(\lambda,\tau)|\ge|P_1(\lambda,\tau)|$. The result of inequality is consistent with the above equation. Third, $|P_1(\lambda,\tau)|+|P_2(\lambda,\tau)|\ge|P_0(\lambda,\tau)|$ is equivalent to

$$\omega^6+F_1(\tau)\omega^4+F_2(\tau)\omega^2-F_{32}(\tau)\omega+F_{42}(\tau)\le 0.$$

Therefore, we can get the feasible region Ω.

Once again, we consider two possible scenarios:

1) If $Im\left(a_1(\omega,\tau)\mathrm{e}^{-i\omega\tau}\right)>0$, we can get

$$\arg\left(a_1(\omega,\tau)\mathrm{e}^{-i\omega\tau}\right)=\pi-\theta_1(\omega,\tau),\quad \arg\left(a_2(\omega,\tau)\mathrm{e}^{-i\omega 2\tau}\right)=\theta_2(\omega,\tau)-\pi,$$

where

$$\theta_1(\omega,\tau)=\arccos\left(\frac{1+|a_1(\omega,\tau)|^2-|a_2(\omega,\tau)|^2}{2|a_1(\omega,\tau)|}\right),$$

$$\theta_2(\omega,\tau)=\arccos\left(\frac{1+|a_2(\omega,\tau)|^2-|a_1(\omega,\tau)|^2}{2|a_2(\omega,\tau)|}\right).$$

Then, we can get

$$\arg\left(a_1(\omega,\tau)\right)-\omega\tau+2n\pi=\pi-\theta_1(\omega,\tau),\quad n\in\mathbb{Z},$$

and

$$\tau=\frac{1}{\omega}\left[\arg\left(a_1(\omega,\tau)\right)+\theta_1(\omega,\tau)+(2n-1)\pi\right],\quad n\in\mathbb{Z}.$$

2) If $Im\left(a_1(\omega,\tau)\mathrm{e}^{-i\omega\tau}\right)<0$, then we can get the triangular formed by 1, $a_1(\omega,\tau)\mathrm{e}^{-i\omega\tau}$, and $a_2(\omega,\tau)\mathrm{e}^{-i\omega 2\tau}$ is the mirror image of the one in **Figure 1** about the real axis. Therefore, we obtain

$$\arg\left(a_1(\omega,\tau)\right)-\omega\tau+2n\pi=\pi+\theta_1(\omega,\tau),\quad n\in\mathbb{Z},$$

and

$$\tau=\frac{1}{\omega}\left[\arg\left(a_1(\omega,\tau)\right)-\theta_1(\omega,\tau)+(2n-1)\pi\right],\quad n\in\mathbb{Z}.$$

We denote by I_ω the interval of ω for the feasible region Ω. Now, for fixed $\omega \in I_\omega$ and $n \in \mathbb{Z}$, we can introduce the functions of τ, say $S_n^{\pm}: I_\omega \to \mathbb{R}$, as

$$S_n^{\pm}(\omega,\tau) = \tau - \frac{1}{\omega}\left[\arg\left(a_1(\omega,\tau)\right) + (2n-1)\pi \pm \theta_1(\omega,\tau)\right], \quad n \in \mathbb{Z}. \tag{21}$$

For the above equation, if zero exists, it can be denoted as $\hat{\tau}^{i\pm}(\omega), i = 1,2,\cdots$. So, the corresponding value of 2τ. When ω takes the values throughout the interval I_ω, then we get the curve

$$\mathcal{C} := \left\{\left(\omega, \hat{\tau}^{i\pm}(\omega)\right) : \omega \in I_\omega, S_n^{\pm}\left(\omega, \hat{\tau}^{i\pm}(\omega)\right) = 0\right\}$$

on Ω, which will later determine the shape of the crossing curves

$$\mathcal{T} = \left\{\left(\hat{\tau}^{i\pm}(\omega), 2\hat{\tau}^{i\pm}(\omega)\right) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \omega \in I_\omega\right\}$$

on $(\tau, 2\tau)$-plane.

Lemma 12. The characteristic Equation (19) admits a pair of conjugate roots $\pm i\omega_*$, for $(\tau,2\tau) = (2\tau_*, 2\tau_*) \in \mathcal{T}$. Denote by $\lambda^{\pm}(\tau) = \alpha(\tau,2\tau) \pm i\omega(\tau,2\tau)$ the pair of conjugate complex roots of (19) in some neighborhood of $(\tau_*, 2\tau_*)$, such that $\alpha(\tau_*,2\tau_*) = 0$ and $\omega(\tau_*,2\tau_*) = \omega_*$. If $\delta(\tau_*,2\tau_*) > 0$, then $\lambda_+(\tau,2\tau)$ cross the imaginary axis from left to right, as $(\tau,2\tau)$ passes through the crossing curve to the region on the right. While if $\delta(\tau_*,2\tau_*) < 0$, then $\lambda_-(\tau,2\tau)$ cross the imaginary axis from left to right, as $(\tau,2\tau)$ passes through the crossing curve to the region on the left, where

$$\delta(\tau_*,2\tau_*) = -Re\left\{\left[P_{0\tau}^* + P_{1\tau}^* e^{-i\omega_*\tau_*} + \left(P_{2\tau}^* - i\omega P_2^*\right)e^{-i\omega_* 2\tau_*}\right]\left[\overline{P_1^*} e^{i\omega_*\tau_*} + \overline{P_2^*} e^{i\omega_* 2\tau_*}\right]\right\}$$

with $P_l^* = P_l(i\omega_*, \tau_*)$, $P_{l\tau}^* = \dfrac{\partial P_l}{\partial \tau}(i\omega_*, \tau_*)$, $l = 0,1,2$.

Comprehensive analysis of the above, then we have the following result.

Theorem 13. Suppose that $\mathcal{R}_0(\tau,\tau) > 1$ and $M_3(\tau,\tau) > 0$ holds. If $\tau \in [0, \tau_*)$, then the computer virus equilibrium P_* of the system (6) is locally asymptotically stable, and the system (6) undergoes a Hopf bifurcation at the computer virus equilibrium P_* when $\tau = \tau_*$.

6. The Numerical Simulations and Conclusions

A computer virus epidemic model with age structure and two delays effects are constructed and studied in this paper. In contrast to traditional computer virus models, we take into account not only the spreading ability of latent computers, but also the healing ability of infected computers. That is, the models considered in this paper are more realistic. There are fewer studies on computer virus models for time lag and age systems, and in contrast to the existing ones, this paper proposes specific defenses through basic regenerative number analysis.

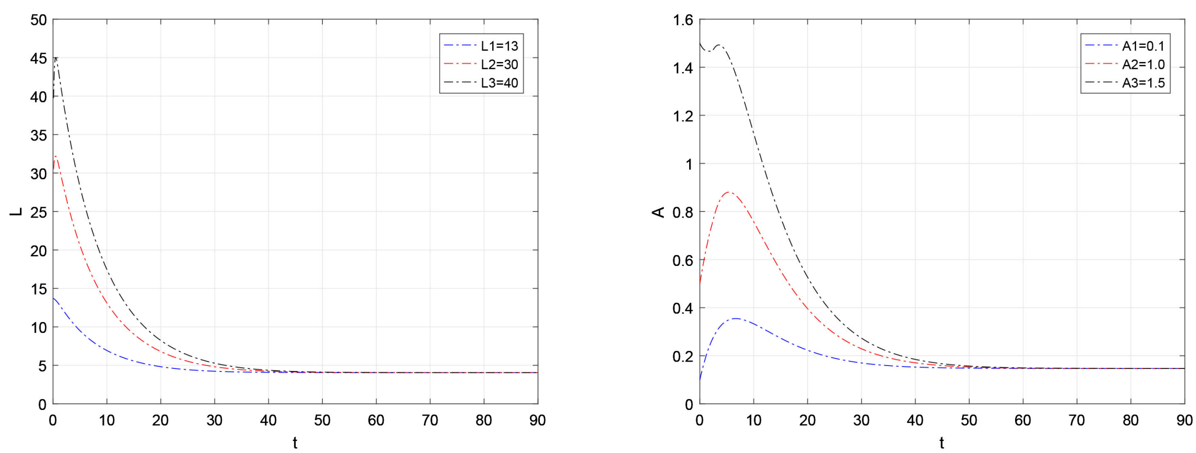
Theoretical analysis shows that the solution of the system (6) is non negative and bounded, the system only has the virus-free equilibrium P_0 when $\mathcal{R}_0(\tau_1,\tau_2) < 1$, which is locally asymptotically stable, and except the virus-free equilibrium, there is also an the computer virus equilibrium P_* when $\mathcal{R}_0(\tau_1,\tau_2) > 1$ and

$M_3(\tau_1,\tau_2)>0$, which is locally asymptotically stable for $\tau_1=\tau_2=0$. We also have discussed the existence of Hopf bifurcation under three different cases, which are 1) $\tau_1>0$ and $\tau_2=0$, 2) $\tau_1=0$ and $\tau_2>0$, and 3) $\tau_1=\tau_2=\tau>0$, respectively. In the following, we will use Matlab to verify the dynamic behaviors of the system (6), which includes the stability of the computer virus equilibrium P_* and the Hopf bifurcations under some cases.

In the cases where $\tau_1>0$ and $\tau_2=0$. Firstly, let $\Lambda=0.5$, $\beta_1=0.1$, $\beta_2=0.1$, $\gamma_1=0.0001$, $\gamma_2=0.1$, $\mu_1=0.0001$, $\mu_2=0.12$, $\mu_3=0.001$, $\alpha_1=0.01$, $\alpha_2=0.1$, and $\tau_1=2$. then we can obtain $\mathcal{R}_0(\tau_1,0)=41.8524>1$, $M_3(\tau_1,0)=0.9416>0$, $q_2^2(\tau_1)-3q_1(\tau_1)=-0.0038<0$. Figure 2 displays that the solution of the system (6) will converge to the computer virus equilibrium P_* as t tends to infinity for different initial value conditions.

And then, we take $\Lambda=0.5$, $\beta_1=0.008$, $\beta_2=0.1$, $\gamma_1=0.1$, $\gamma_2=0.0001$, $\mu_1=0.01$, $\mu_2=0.15$, $\mu_3=0.01$, $\alpha_1=0.01$, $\alpha_2=0.001$, and $\tau_1=2$. then $\mathcal{R}_0(\tau_1,0)=12.1031>1$, $M_3(\tau_1,0)=0.6053>0$, $q_2^2(\tau_1)-3q_1(\tau_1)=0.0132>0$, $\Theta_2=-0.0054<0$. Figure 3(a) displays the computer virus equilibrium P_* gradually tends to stabilize as t tends to infinity. However, when we take $\Lambda=0.2$, $\beta_1=0.008$, $\beta_2=0.1$, $\gamma_1=0.1$, $\gamma_2=0.0001$, $\mu_1=0.001$, $\mu_2=0.15$, $\mu_3=0.001$, $\alpha_1=0.01$, $\alpha_2=0.01$, and $\tau_1=2$, then $\mathcal{R}_0(\tau_1,0)=8.7446>1$, $M_3(\tau_1,0)=0.5848>0$, $q_2^2(\tau_1)-3q_1(\tau_1)=0.0002>0$, $\Theta_2=0.0007>0$, and $Q(\Theta_2)=5.1673\times10^{-8}>0$. Similarly, we can see the computer equilibrium P_* keeps stability in Figure 3(b).

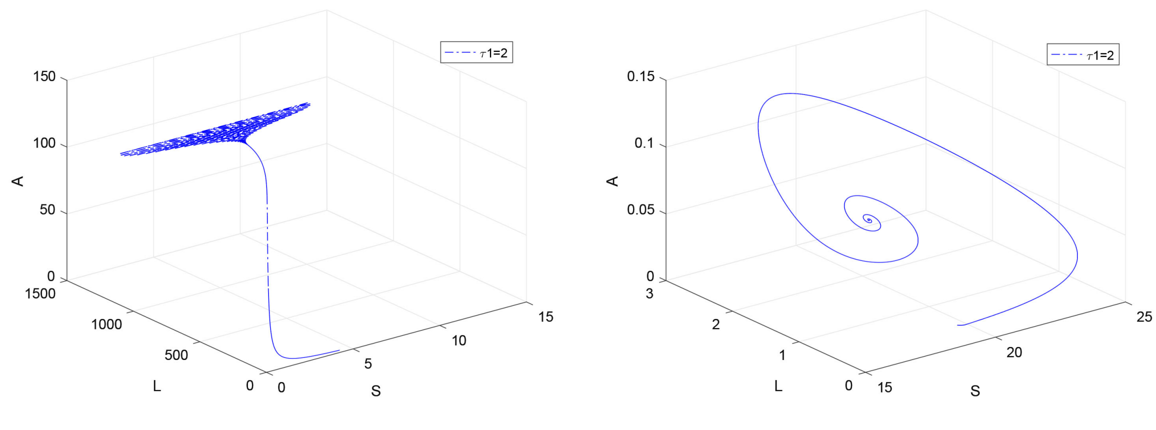
Therefore, under these conditions of Theorem 6, the system also does not have periodic behavior regardless of the change in the delay τ_1. Further, the distribution of infected individuals at the computer virus equilibrium P_*, $L_*(a)$ is shown in Figure 4(a), corresponding to the situation in Figure 3(b), and the distribution of immunization age and time, and the distribution of infection age and infection time $L(t,a)$ is shown in Figure 4(b).



(a) The latent computers

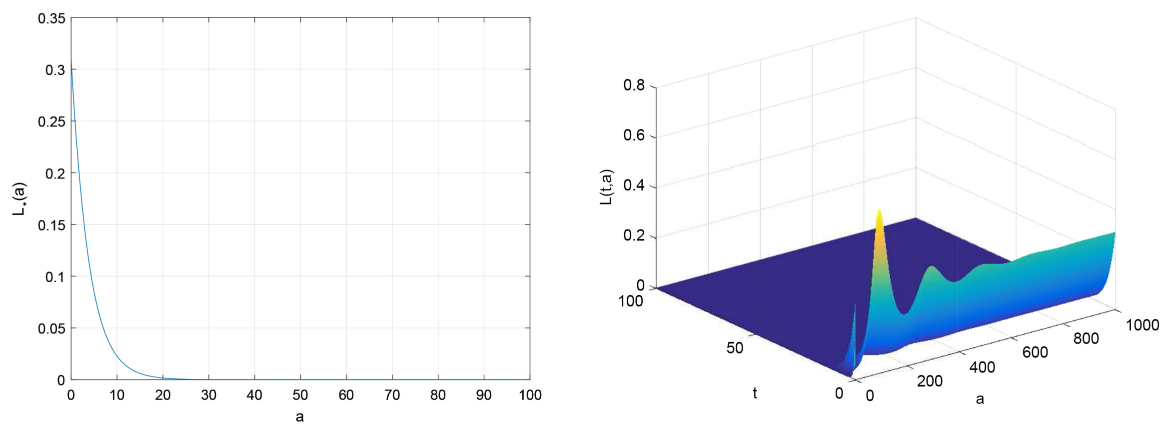
(b) The infected computers in which viruses are located in memory

Figure 2. The stability of the computer virus equilibrium P_* when $q_2^2(\tau_1)-3q_1(\tau_1)<0$.



(a) $\Theta_2 \le 0$ (b) $\Theta_2 > 0$ and $Q(\Theta_2) > 0$

Figure 3. The stability of the computer virus equilibrium P_* when $q_2^2(\tau_1) - 3q_1(\tau_1) \ge 0$.



(a) The distributions $L_*(a)$ at P_* (b) The distributions $L(t,a)$

Figure 4. The distributions of latent individuals when P_* is asymptotically stable under $\mathcal{R}_0(\tau_1, 0) > 1$.

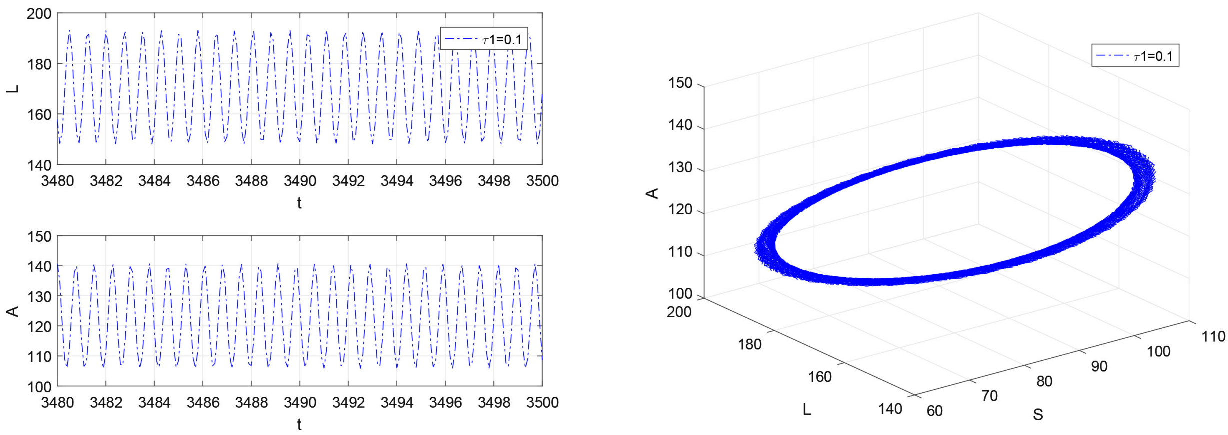
Finally, choosing $\Lambda = 16.8$, $\beta_1 = 0.04$, $\beta_2 = 0.0001$, $\gamma_1 = 0.1$, $\gamma_2 = 8.1$, $\mu_1 = 0.0003$, $\mu_2 = 0.09$, $\mu_3 = 0.012$, $\alpha_1 = 10$, $\alpha_2 = 0.001$, and $\tau_1 = 0.1$, we can get $\mathcal{R}_0(\tau_1, 0) = 598.2678 > 1$, $M_3(\tau_1, 0) = 0.0245 > 0$, $q_2^2(\tau_1) - 3q_1(\tau_1) = 3327.2594 > 0$, $\Theta_2 = 27.1586 > 0$, and $Q(\Theta_2) = -4861.8883 < 0$. **Figure 5** shows that the system (6) experiences the Hopf bifurcation.

In the cases where $\tau_1 = 0$ and $\tau_2 > 0$. Firstly, let $\Lambda = 0.5$, $\beta_1 = 0.1$, $\beta_2 = 0.1$, $\gamma_1 = 0.0001$, $\gamma_2 = 0.1$, $\mu_1 = 0.0001$, $\mu_2 = 0.12$, $\mu_3 = 0.001$, $\alpha_1 = 0.01$, $\alpha_2 = 0.1$, and $\tau_2 = 2$. then we can obtain $\mathcal{R}_0(0, \tau_2) = 42.5704 > 1$, $M_3(0, \tau_2) = 0.9252 > 0$, $q_2^2(\tau_2) - 3q_1(\tau_2) = -0.0012$. **Figure 6** displays that the solution of the system (6) will converge to the computer virus equilibrium P_* as t tends to infinity for different initial value conditions.

And then, we take $\Lambda = 0.5$, $\beta_1 = 0.008$, $\beta_2 = 0.1$, $\gamma_1 = 0.1$, $\gamma_2 = 0.0001$, $\mu_1 = 0.01$, $\mu_2 = 0.15$, $\mu_3 = 0.01$, $\alpha_1 = 0.01$, $\alpha_2 = 0.001$, and $\tau_2 = 2$. then $\mathcal{R}_0(0, \tau_2) = 18.9326 > 1$, $M_3(0, \tau_2) = 0.6116 > 0$, $q_2^2(\tau_2) - 3q_1(\tau_2) = 0.0334 > 0$, $\Theta_2 = -0.0057 < 0$. **Figure 7(a)** displays the computer virus equilibrium P_* gradually tends to stabilize as t tends to infinity. However, when we take $\Lambda = 0.2$,

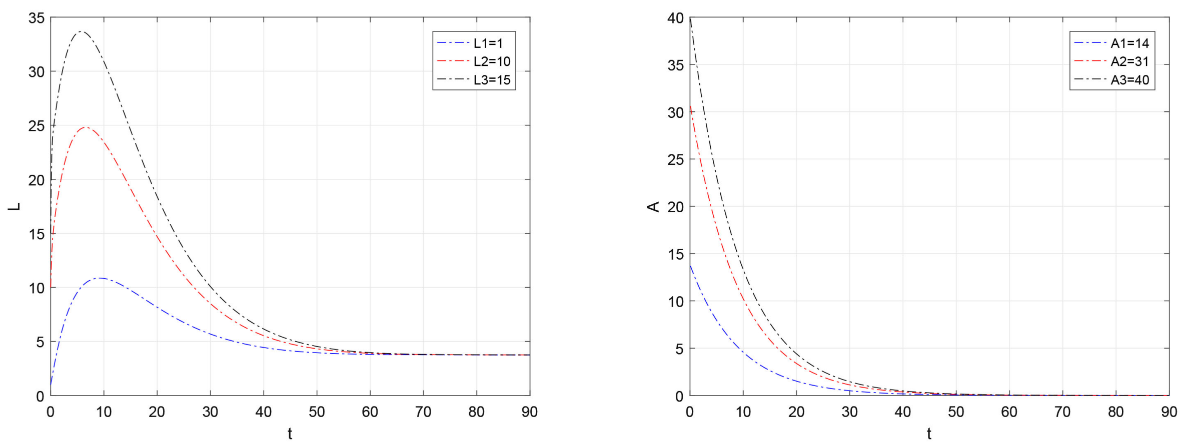
$\beta_1 = 0.008$, $\beta_2 = 0.1$, $\gamma_1 = 0.1$, $\gamma_2 = 0.0001$, $\mu_1 = 0.001$, $\mu_2 = 0.15$, $\mu_3 = 0.001$, $\alpha_1 = 0.01$, $\alpha_2 = 0.01$, and $\tau_2 = 2$. then $\mathcal{R}_0(0,\tau_2) = 14.3084 > 1$, $M_3(0,\tau_2) = 0.5773 > 0$, $q_2^2(\tau_2) - 3q_1(\tau_2) = 0.0043 > 0$, $\Theta_2 = 0.0011 > 0$, $Q(\Theta_2) = 2.8030\times10^{-8} > 0$. Similarly, we can see the computer equilibrium P_* keeps stability in **Figure 7(b)**.

Next, choosing $\Lambda = 6.8$, $\beta_1 = 0.2$, $\beta_2 = 0.4$, $\gamma_1 = 3$, $\gamma_2 = 2.1$, $\mu_1 = 0.007395$, $\mu_2 = 0.5$, $\mu_3 = 0.5$, $\alpha_1 = 10$, $\alpha_2 = 9.4$, and $\tau_2 = 0.1$. we can get $\mathcal{R}_0(0,\tau_2) = 146.5118072 > 1$, $M_3(0,\tau_2) = 0.0138 > 0$, and $q_0(\tau_2) = -79605.2596 < 0$. **Figure 8** shows that the system (6) experiences the Hopf bifurcation. Finally, choosing $\Lambda = 9.9$, $\beta_1 = 0.045$, $\beta_2 = 0.0001$, $\gamma_1 = 0.1$, $\gamma_2 = 10.1$, $\mu_1 = 0.0003$, $\mu_2 = 0.1$, $\mu_3 = 0.002$, $\alpha_1 = 9.2$, $\alpha_2 = 0.001$, and $\tau_2 = 0.1$, we can get $\mathcal{R}_0(0,\tau_2) = 158.8810 > 1$, $M_3(0,\tau_2) = 0.0486 > 0$, $q_2^2(\tau_2) - 3q_1(\tau_2) = 13375.6150 > 0$, $\Theta_2 = 8.5059 > 0$, and $Q(\Theta_2) = -80.0660 < 0$. **Figure 9** shows that the system (6) experiences the Hopf bifurcation.



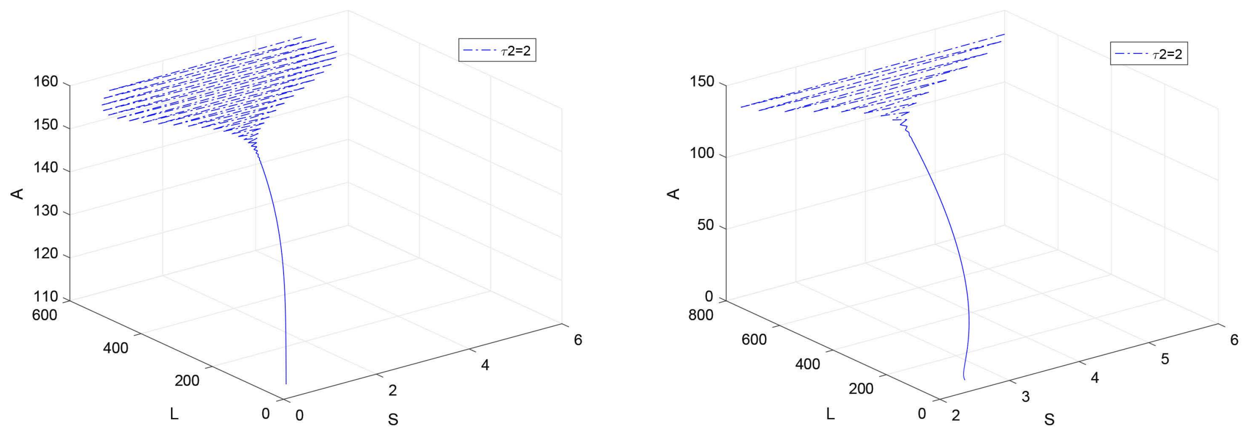
(a) The periodic solution (b) The phase diagram

Figure 5. The Hopf bifurcation around the computer virus equilibrium P_* for $\tau_1 > 0$ and $\tau_2 = 0$.



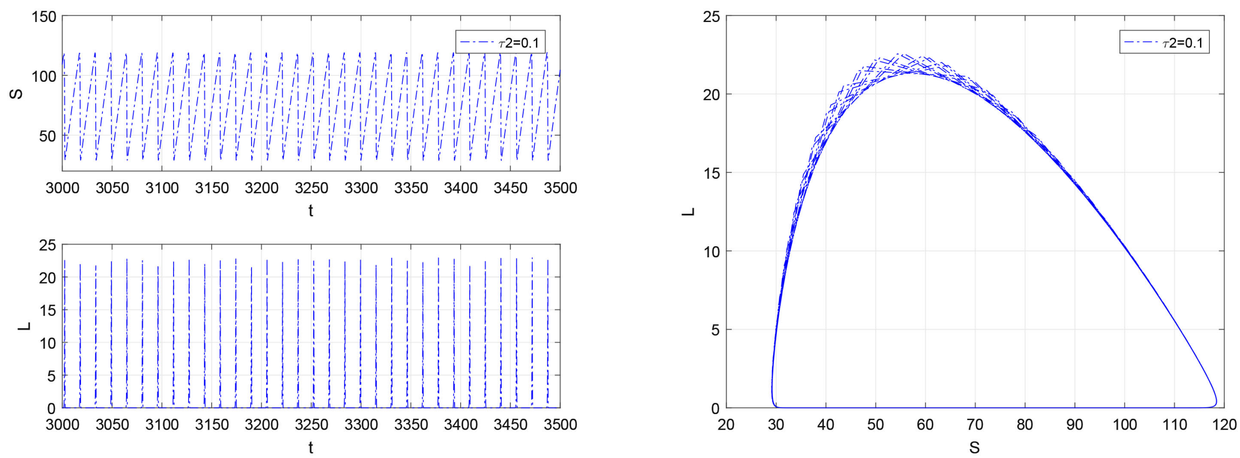
(a) The latent computers (b) The infected computers in which viruses are located in memory

Figure 6. The stability of the computer virus equilibrium P_* when $q_2^2(\tau_2) - 3q_1(\tau_2) < 0$.



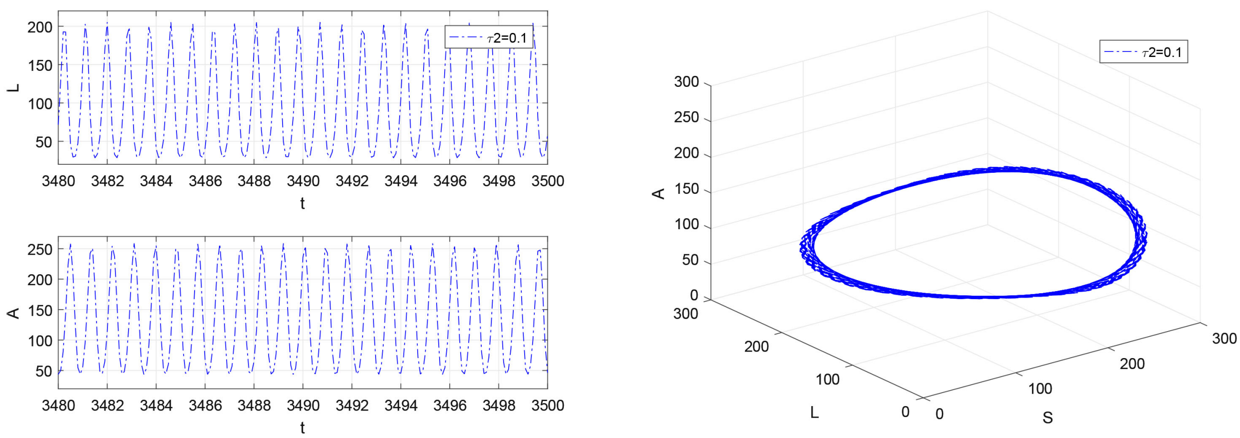
(a) $\Theta_2 \leq 0$ (b) $\Theta_2 > 0$ and $Q(\Theta_2) > 0$

Figure 7. The stability of the computer virus equilibrium P_* when $q_2^2(\tau_2) - 3q_1(\tau_2) \geq 0$.



(a) The periodic solution (b) The phase diagram

Figure 8. The Hopf bifurcation around the computer virus equilibrium P_* for $q_0(\tau_2) < 0$.



(a) The periodic solution (b) The phase diagram

Figure 9. The Hopf bifurcation around the computer virus equilibrium P_* for $q_0(\tau_2) > 0$.

In the cases where $\tau_1 = \tau_2 = \tau > 0$. Firstly, we illustrate the stability of the computer virus equilibrium P_* when $\omega \notin I_\omega$. Let $\Lambda = 9.8$, $\beta_1 = 0.05$, $\beta_2 = 0.0001$, $\gamma_1 = 0.1$, $\gamma_2 = 12.1$, $\mu_1 = 0.0003$, $\mu_2 = 0.09$, $\mu_3 = 0.01$, $\alpha_1 = 10$, and $\alpha_2 = 0.001$. Then we can obtain $\mathcal{R}_0(\tau,\tau) = 436.6949 > 1$, $M_3(\tau,\tau) = 0.0121 > 0$ when $\tau = 2$. **Figure 10** illustrates the computer virus equilibrium P_* gradually tends to stabilize as t increases. Secondly, we discuss the stability of the network virus equilibrium P_* when $\omega \in I_\omega$. Let $\Lambda = 2.8$, $\beta_1 = 0.05$, $\beta_2 = 0.0001$, $\gamma_1 = 0.04$, $\gamma_2 = 2.1$, $\mu_1 = 0.0003$, $\mu_2 = 0.2$, $\mu_3 = 0.01$, $\alpha_1 = 10$, $\alpha_2 = 0.5$, and $\tau = 0.1$, then we can obtain $\mathcal{R}_0(\tau,\tau) = 160.4551 > 1$, and $M_3(\tau,\tau) = 0.0351 > 0$. **Figure 11** illustrates the system (6) experiences the Hopf bifurcation.

In the case where $\tau_1 > 0$, $\tau_2 > 0$, and $\tau_1 \neq \tau_2$. Let $\Lambda = 8.5$, and $\mu_1 = 0.0001$. Taking $\tau_1 = 0.1$, and $\tau_2 = 0.2$, then we obtain $\mathcal{R}_0(\tau_1,\tau_2) = 1142.3502 > 1$, and $M_3(\tau_1,\tau_2) = 0.0125 > 0$. **Figure 12** displays that the system (6) still undergoes the Hopf bifurcation.

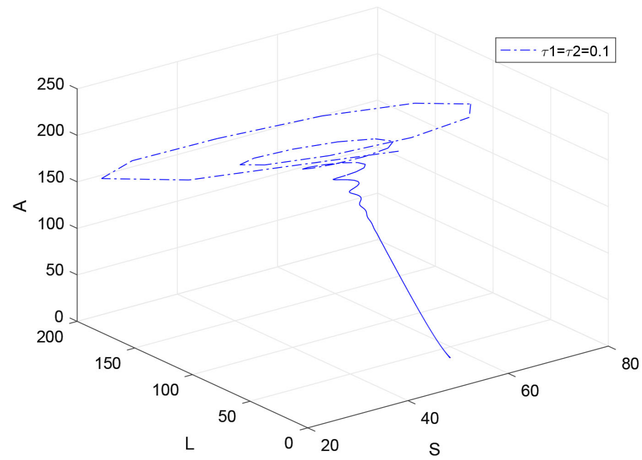
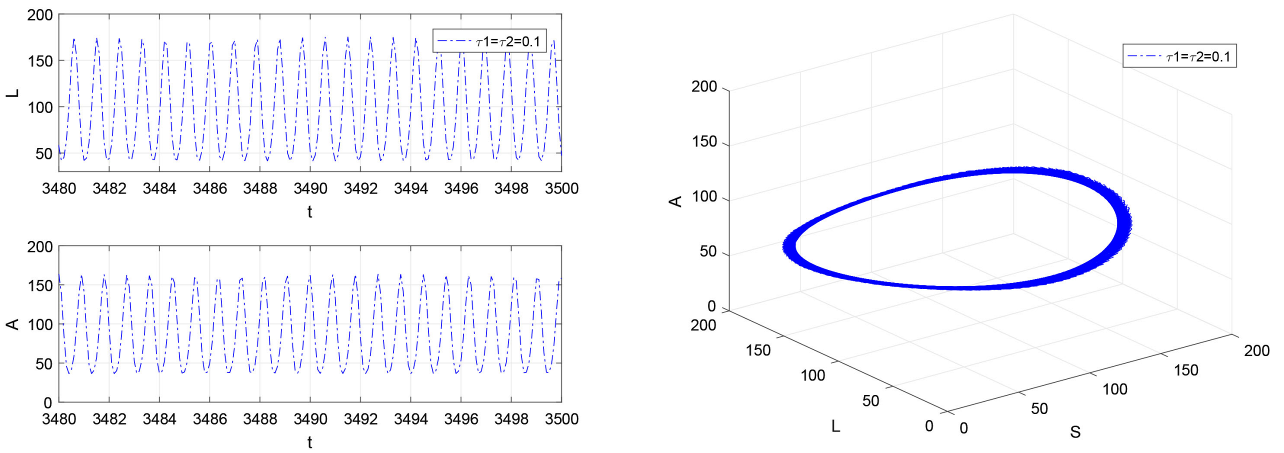
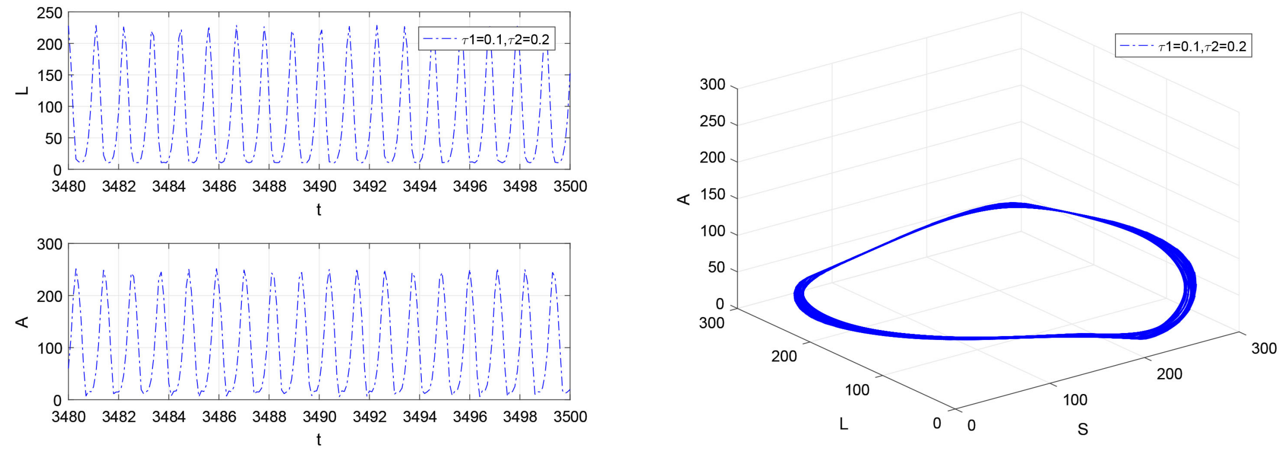


Figure 10. The stability of the computer virus equilibrium P_* for $\omega \notin I_\omega$.



(a) The periodic solution (b) The phase diagram

Figure 11. The Hopf bifurcation around the computer virus equilibrium P_* for $\tau_1 = \tau_2$.



(a) The periodic solution (b) The phase diagram

Figure 12. The Hopf bifurcation around the computer virus equilibrium P_* for $\tau_1 \neq \tau_2$.

In order to completely control the spread of computer viruses in the network, it is necessary to control the basic regeneration number $\mathcal{R}_0(\tau_1, \tau_2)$ when it is less than 1, which means that the spread of computer viruses can be controlled when virus-free equilibrium exists. Based on the basic regeneration number $\mathcal{R}_0(\tau_1, \tau_2)$ in the previous theoretical analysis, it is known that it is related to both delays τ_1 and τ_2. That $\mathcal{R}_0(\tau_1, \tau_2)$ is a strictly monotonically increasing with respect to τ_2. This indicates that the increase of τ_2 will have an impact on the stability of the system, so in order to control the spread of computer viruses, the range of values of τ_2 should be as small as possible, which means that the impact on the system is small at the same time. Therefore, we can effectively control the spread of computer viruses by reducing the recovery time of computer systems after infection. By adding age structure and two delays to the system, it is more suitable for the actual virus propagation in the computer in reality, which also provides effective measures for computer virus prevention.

The emergence of the Hopf bifurcation of the system (6) implies that when both factors, age structure and delay, are introduced together into the nonresident computer virus SLAS model, it leads to destabilization of the system, followed by the phenomenon of stability switching in the system. In other words, this would disrupt the threshold of dynamic behavior, while allowing the spread of the virus to get out of control.

The limitation of this paper is that it does not discuss the global stability of virus-free equilibrium P_* when $\mathcal{R}_0 > 1$. In addition, this paper does not determine the dynamical behavior of the system as the two delays τ_1 and τ_2 increases. We will continue to discuss these aspects in the future. At the same time, the spread of viruses in the computer and the infection process is more detailed, which is conducive to the subsequent study of computer virus propagation model, in which case the proposed control strategy will be more effective.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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