Journal of Applied Mathematics and Physics, 2023, 11, 2093-2102
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Sign-Changing Solutions for Superlinear Kirchhoff Type Problem via the Nehari Method

Xiaohan Duan, Guanggang Liu*

School of Mathematical Sciences, Liaocheng University, Liaocheng, China
Email: *lgg112@163.com

How to cite this paper: Duan, X.H. and Liu, G.G. (2023) Sign-Changing Solutions for Superlinear Kirchhoff Type Problem via the Nehari Method. *Journal of Applied Mathematics and Physics*, **11**, 2093-2102.
https://doi.org/10.4236/jamp.2023.117132

Received: June 16, 2023
Accepted: July 24, 2023
Published: July 27, 2023

Abstract

In this paper, we consider a class of Kirchhoff type problem with superlinear nonlinearity. A sign-changing solution with exactly two nodal domains will be obtained by combining the Nehari method and an iterative technique.

Keywords

Kirchhoff Type Problem, Nehari Method, Iterative Technique

1. Introduction

In this paper, we consider the following Kirchhoff type problem

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^{2}\,\mathrm{d}x\right)\Delta u=f(u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^{N}, N=1,2,3$ is a bounded smooth domain. The problem (1.1) is related to the stationary analogue of the equation

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^{2}\,\mathrm{d}x\right)\Delta u=f(x,u) \tag{1.2}$$

proposed by Kirchhoff as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. After Lions [1] introduced an abstract framework to the problem, the Equation (1.2) began to receive much attention. In recent years, the existence and multiplicity of nontrivial solutions for the Kirchhoff type problem on a bounded domain $\Omega\subset\mathbb{R}^{N}$ or on $\mathbb{R}^{N}$ has been studied by many authors, see [2]-[20] and references therein. To obtain the existence of nontrivial solutions for (1.1), various growth conditions with the nonlinearity f for problem (1.1) are always needed. For example, the subcritical growth case was considered in [2] [4] [6] [9], the critical growth case was considered in [5] [8], the superlinear case was considered in [7] [10] [11] [13] [14] [15] [17], the asymptotically linear case was considered in [3] [12] [14]. In [12], by using the Yang index and Morse theory, Perera and Zhang established the existence of nontrivial solutions for (1.1) when the nonlinearity f is asymptotically linear near zero and asymptotically 4-linear at infinity. In [14], Sun and Liu obtained the existence of nontrivial solutions via the Morse theory when the nonlinearity is superlinear near zero but asymptotically 4-linear at infinity, and the nonlinearity is asymptotically linear near zero but 4-superlinear at infinity. In [16], by applying the mountain pass theorem, the local linking theorem, and the fountain theorem, Sun and Tang obtained the existence and multiplicity of nontrivial solutions for (1.1) when the nonlinearity f is 4-superlinear at infinity.

In the previous existence and multiplicity results, the additional properties about the solutions are not be considered. Recently, there has been increasing interest to obtain additional information on the solutions of (1.1). The existence of sign-changing solutions for (1.1) has attracted a lot of attention. In [19], Zhang and Perera studied the existence of sign-changing solutions for a class of Kirchhoff type problems by using variational method. In [13], Shuai proved the problem (1.1) possesses one least energy sign-changing solution via the Nehari method when the nonlinearity f is 4-superlinear at infinity by combining constraint variational method and quantitative deformation lemma.

In this paper, motivated by [13], we will study the existence of sign-changing solution for (1.1) when the nonlinearity f is 4-superlinear at infinity. Our result has somewhat improved the result of [13]. In [13], Shuai obtained the existence of sign-changing solution for (1.1) under the following conditions:

(f_1) $f \in C^1(\mathbb{R},\mathbb{R})$, $f(s)=o(|s|)$ as $s \to 0$;

(f_2) For some constant $p \in (4,2^*)$, $\lim\limits_{s\to\infty}\frac{f(s)}{s^{p-1}}=0$, where $2^*=+\infty$ for $N=1,2$ and $2^*=6$ for $N=3$;

(f_3) $\lim\limits_{s\to\infty}\frac{F(s)}{s^4}=+\infty$, where $F(s)=\int_0^s f(t)\mathrm{d}t$;

(f_4) $\frac{f(s)}{|s|^3}$ is an increasing function of $s \in \mathbb{R}\setminus\{0\}$.

Here we replace the condition (f_4) with the following conditions (f_5) and (f_6):

(f_5) There exist constant $\theta > 2$ and $s_0 > 0$ such that $0<\theta F(s)<sf(s), \forall |s|\ge s_0$,

(f_6) $f'(s)>\frac{f(s)}{s}$ for all $s \neq 0$.

If $F>0$, we can see that our conditions (f_5) and (f_6) are weaker than (f_4). In fact, if (f_4) holds, then for $s \neq 0$, one has that

$$f'(s)s^2-3f(s)s>0 \quad \text{and} \quad f(s)s>4F(s),$$

which implies that (f_5) and (f_6) hold when $F>0$. On the other hand, let $f(s)=\frac{7}{3}|s|^{\frac{1}{3}}s+4s^3\ln(1+s^2)+\frac{2s^5}{1+s^2}$, then $F(s)=|s|^{\frac{7}{3}}+s^4\ln(1+s^2)$. By calculation, we see that $f(s)$ satisfies (f_1) - (f_3), (f_5), and (f_6). But notice that

$$f(s)s - 4F(s) = -\frac{5}{3}|s|^{\frac{7}{3}} + \frac{2s^6}{1+s^2} < 0$$

when $0 < |s| \leq 1$, thus the condition (f_4) is not satisfied. Hence our result is new and we have partially extended the result in [13] when $F > 0$.

We will use a method different from [13]. The existence of a sign-changing solution with exactly two nodal domains will be proved by combining the Nehari method and an iterative technique proposed in [21]. The main idea is to fix the nonlocal term first and to consider the corresponding usual second order elliptic problem. The sign-changing solution for this usual second order elliptic problem will be obtained by the Nehari method. Then we use the iterative technique to get a sequence of approximate solutions, and the sign-changing solution for (1.1) will be obtained through a limit argument. The key point is to obtain the boundedness of this sequence of approximate solutions.

Our main result is the following theorem.

Theorem 1.1. Assume that (f_1) - (f_3), (f_5), (f_6) hold, then the problem (1.1) has at least one sign-changing solution which has exactly two nodal domains.

Remark 1.2. In fact, under the conditions of Theorem 1.1, we can also obtain the positive and negative solutions of (1.1) by combining the mountain pass theorem and a similar iterative process.

The paper is organized as follows. In Section 2, we fix the nonlocal term of (1.1) and consider the corresponding usual second order elliptic problem. We apply the Nehari method to obtain the sign-changing solution for this usual second order elliptic problem. In Section 3, we give the proof of our main result by using an iterative technique.

2. Preliminaries

Let $E = H_0^1(\Omega)$ be the usual Sobolev space with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. For any fixed $\omega \in E$, we consider the following problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla \omega|^2 \mathrm{d}x\right)\Delta u = f(u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

The associated functional corresponding to (2.1) is $I_\omega : E \to \mathbb{R}$,

$$I_\omega(u) = \frac{1}{2}\left(a + b\int_\Omega |\nabla \omega|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(u)\mathrm{d}x.$$

By (f_1) and (f_2), $I_\omega \in C^2(E, \mathbb{R})$ is weakly lower semi-continuous and the weak solution of the problem (2.1) corresponds to the critical point of the functional I_ω.

Define

$$G_\omega(u) = \langle I'(u), u\rangle = \left(a + b\int_\Omega |\nabla \omega|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega f(x,u)u\mathrm{d}x,$$

$$N_\omega = \left\{u \in H_0^1(u) \setminus \{0\} \mid G_\omega(u) = 0\right\}, S_\omega = \left\{u \in N_\omega \mid u^+ \in N_\omega, u^- \in N_\omega\right\},$$

where $u^+ = \max\{u, 0\}, u^- = \min\{u, 0\}$. The set N_ω is called the Nehari manifold.

Obviously, any sign-changing solutions of (2.1) must be on S_ω. Note that for any $u \in E$,

$$I_\omega(u) = I_\omega(u^+) + I_\omega(u^-),$$

$$\left\langle I'_\omega(u), u^+\right\rangle = \left\langle I'_\omega(u^+), u^+\right\rangle,\ \left\langle I'_\omega(u), u^-\right\rangle = \left\langle I'_\omega(u^-), u^-\right\rangle.$$

Then if $u \in E$ satisfies $u^+ \in N_\omega$, and $u^- \in N_\omega$, we have that $u \in N_\omega$ and thus $u \in S_\omega$.

Now we give a detailed explanation of our proof. Firstly, for every $\omega \in E$, we prove that I_ω is bounded on N_ω and so also bounded on S_ω. Then we can find a minimizer u_ω of I_ω on S_w, which is proved to be a sign-changing solution of (2.1). Secondly, we prove that there exists a constant R_1 such that if $\|\omega\| \le R_1$ then $\|u_\omega\| \le R_1$. Using this conclusion again and again we can obtain a sequence $\{u_n\}$ such that u_n is a sign-changing critical point of $I_{u_{n-1}}$ and $\|u_n\| \le R_1$. Thirdly, let $n \to \infty$, we can prove that $u_n \to \tilde{u}$ for some $\tilde{u} \in E$ and $\tilde{u}$ is a sign-changing solution of the original problem (1.1). Finally, we show that $\tilde{u}$ is the minimizer of $I_{\tilde{u}}$ on $S_{\tilde{u}}$, and using this fact we prove that $\tilde{u}$ has exactly two nodal domains.

In order to prove the main result, we need the following lemmas. However, the proofs of them are standard and similar to Lemmas 3.1 - 3.4 of our recent paper [10], so we omit their proofs. Note that in [10], we only proved the existence of sign-changing solution for (1.1) when b is sufficiently small, and the number of nodal domains is not obtained there. By contrast, here for any $b > 0$, a sign-changing solution is obtained, and it has exactly two nodal domains.

Lemma 2.1. Assume that (f_1), (f_2), (f_5), (f_6) hold, then for each $u \in E \setminus \{0\}$ there exists unique $t = t(u) > 0$ such that $t(u)u \in N_\omega$.

Lemma 2.2. Assume that (f_1), (f_2), (f_5), (f_6) hold, there exists constants $\alpha > 0$ and $c > 0$ independent of ω such that $I_\omega(u) \ge \alpha$ and $\|u\| \ge c$ for all $u \in N_\omega$.

Define $m_1 = \inf_{S_\omega} I_\omega$, then it is clearly that $m_1 \ge \alpha \ge 0$.

Lemma 2.3. m_1 is achieved at some $u_\omega \in S_\omega$, and u_ω is a critical point of I_ω.

Remark 2.1. In [10], we assumed that $\Omega \in \mathbb{R}^N, N = 3$. But it is not difficult to see that the above lemmas still true for $\Omega \in \mathbb{R}^N, N = 1, 2$, from the proofs there.

3. Proof of the Main Result

In this section, we prove our main result.

Proof of Theorem 1.1.

Step 1. We construct a bounded sign-changing functions sequence $\{u_n\}$ in E such that $I'_{u_{n-1}}(u_n) = 0$ for any $n \ge 2$.

For any $\omega \in E$, by Lemma 2.3, there exists a minimizer u_ω of I_ω on S_ω and $I'_\omega(u_\omega) = 0$. We fix a function $v_0 \in E$ with $v_0^+ \neq 0$ and $v_0^- \neq 0$. By Lemma 2.1, there exist $t(v_0^+) > 0$ and $t(v_0^-) > 0$ such that $t(v_0^+)v_0^+ \in N_\omega$ and $t(v_0^-)v_0^- \in N_\omega$. Then it is clear that

$$t(v_0^+)v_0^+ + t(v_0^-)v_0^- \in S_\omega. \tag{3.1}$$

By (f3), for any $\varepsilon > 0$ there exists $C'_\varepsilon > 0$ such that

$$F(s) \geq \frac{1}{\varepsilon}s^4 - C'_\varepsilon. \tag{3.2}$$

Then by (3.1), (3.2), and notice that u_w is a minimizer of I_ω on S_ω, we have

$$\begin{aligned}
I_\omega(u_\omega) &\leq I_\omega\left(t(v_0^+)v_0^+ + t(v_0^-)v_0^-\right) \\
&= I_\omega\left(t(v_0^+)v_0^+\right) + I_\omega\left(t(v_0^-)v_0^-\right) \\
&\leq \sup_{t>0} I_\omega(tv_0^+) + \sup_{t>0} I_\omega(tv_0^-) \\
&\leq \sup_{t>0}\left(\frac{t^2}{2}\left(a + b\int_\Omega |\nabla\omega|^2\,\mathrm{d}x\right)\int_\Omega |\nabla v_0^+|^2\,\mathrm{d}x - \frac{1}{\varepsilon}t^4\int_\Omega |v_0^+|^4\,\mathrm{d}x + C'_\varepsilon|\Omega|\right) \\
&\quad + \sup_{t>0}\left(\frac{t^2}{2}\left(a + b\int_\Omega |\nabla\omega|^2\,\mathrm{d}x\right)\int_\Omega |\nabla v_0^-|^2\,\mathrm{d}x - \frac{1}{\varepsilon}t^4\int_\Omega |v_0^-|^4\,\mathrm{d}x + C'_\varepsilon|\Omega|\right) \\
&= C_1\varepsilon\left(a + b\int_\Omega |\nabla\omega|^2\,\mathrm{d}x\right)^2 + C''_\varepsilon,
\end{aligned} \tag{3.3}$$

where $|\Omega|$ is the Lebesgue measure of Ω,

$$C_1 = \frac{1}{16}\left(\frac{\left(\int_\Omega |v_0^+|^2\,\mathrm{d}x\right)^2}{\left(\int_\Omega |v_0^+|^4\,\mathrm{d}x\right)} + \frac{\left(\int_\Omega |v_0^-|^2\,\mathrm{d}x\right)^2}{\left(\int_\Omega |v_0^-|^4\,\mathrm{d}x\right)}\right), C''_\varepsilon = 2C'_\varepsilon|\Omega|.$$

By (f5), there exists a constant C_0 such that

$$F(s) \leq \frac{1}{\theta}f(s)s + C_0, s \in \mathbb{R}. \tag{3.4}$$

Since u_ω is a critical point of I_ω, we have

$$\left(a + b\int_\Omega |\nabla\omega|^2\,\mathrm{d}x\right)\int_\Omega |\nabla u_\omega|^2\,\mathrm{d}x = \int_\Omega f(x, u_\omega)u_\omega\mathrm{d}x. \tag{3.5}$$

Then by (3.3), (3.4) and (3.5), we have

$$\begin{aligned}
&\frac{1}{2}\left(a + b\int_\Omega |\nabla\omega|^2\,\mathrm{d}x\right)\int_\Omega |\nabla u_\omega|^2\,\mathrm{d}x \\
&= I_\omega(u_\omega) + \int_\Omega F(u_\omega)\mathrm{d}x \\
&\leq I_\omega(u_\omega) + \frac{1}{\theta}\int_\Omega f(u_\omega)u_\omega\mathrm{d}x + C_0|\Omega| \\
&\leq C_1\varepsilon\left(a + b\int_\Omega |\nabla\omega|^2\,\mathrm{d}x\right)^2 + \frac{1}{\theta}\left(a + b\int_\Omega |\nabla\omega|^2\,\mathrm{d}x\right)\int_\Omega |\nabla u_\omega|^2\,\mathrm{d}x + C_\varepsilon,
\end{aligned}$$

where $C_\varepsilon = C''_\varepsilon + C_0|\Omega|$. Note that $\theta > 2$, thus

$$\int_\Omega |\nabla u_\omega|^2 \,\mathrm{d}x \le \frac{C_1\varepsilon\left(a+b\int_\Omega|\nabla\omega|^2\,\mathrm{d}x\right)^2 + C_\varepsilon}{\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(a+b\int_\Omega|\nabla\omega|^2\,\mathrm{d}x\right)}$$

$$= \frac{2\theta}{\theta-2}C_1\varepsilon\left(a+b\int_\Omega|\nabla\omega|^2\,\mathrm{d}x\right)+\frac{2\theta}{\theta-2}\left(\frac{C_\varepsilon}{a+b\int_\Omega|\nabla\omega|^2\,\mathrm{d}x}\right) \tag{3.6}$$

$$\le \frac{2\theta}{\theta-2}C_1\varepsilon\left(a+b\int_\Omega|\nabla\omega|^2\,\mathrm{d}x\right)+\frac{2\theta}{\theta-2}\frac{C_\varepsilon}{a}.$$

Take $\varepsilon=\varepsilon_0$ sufficiently small such that

$$\frac{2\theta}{\theta-2}C_1\varepsilon_0 b \le \frac{1}{2},$$

then from (3.6), we have

$$\int_\Omega |\nabla u_\omega|^2\,\mathrm{d}x \le \frac{1}{2}\int_\Omega|\nabla\omega|^2\,\mathrm{d}x + \frac{2\theta}{\theta-2}\left(C_1\varepsilon_0 a+\frac{C_{\varepsilon_0}}{a}\right). \tag{3.7}$$

Choose a sufficiently large constant $R_1>0$ such that

$$\frac{2\theta}{\theta-2}\left(C_1\varepsilon_0 a+\frac{C_{\varepsilon_0}}{a}\right)\le\frac{1}{2}R_1^2. \tag{3.8}$$

Notice that the constants $\varepsilon_0, C_{\varepsilon_0}$ and C_1 are all independent of ω, then R_1 is also independent of ω. By (3.7) and (3.8), for any $\omega\in E$ with $\|\omega\|\le R_1$, we have

$$\int_\Omega |\nabla u_\omega|^2\,\mathrm{d}x \le \frac{1}{2}R_1^2 + \frac{2\theta}{\theta-2}\left(C_1\varepsilon_0 a+\frac{C_{\varepsilon_0}}{a}\right)\le R_1^2. \tag{3.9}$$

Now let $\omega=u_1$ for some $u_1\in E$ with $\|u_1\|\le R_1$, then by Lemma 2.3 and (3.9), I_{u_1} has a critical point u_2 with $u_2\in S_{u_1}$ and $\|u_2\|\le R_1$. Again, let $\omega=u_2$, then similarly I_{u_2} has a critical point u_3 with $u_3\in S_{u_2}$ and $\|u_3\|\le R_1$. By induction, we get a sequence $\{u_n\}$ with $I'_{u_{n-1}}(u_n)=0, u_n\in S_{u_{n-1}}$ and $\|u_n\|\le R_1$.

Step 2. We prove that $u_n\to\tilde u$ in E for some $\tilde u\in E$ up to a subsequence and $\tilde u$ is a sign-changing solution of (1.1).

Since $\|u_n\|\le R_1$. We can get a subsequence of $\{u_n\}$ (for simplicity still denoted by $\{u_n\}$) such that $u_n\rightharpoonup\tilde u$ in E and $u_n\to\tilde u$ in $L^p(\Omega)$ for some $\tilde u\in E$. By $\|u_n\|\le R_1$ and (f_2), we have

$$\begin{aligned}&\lim_{x\to\infty}\left\langle I'_{u_{n-1}}(\tilde u),(u_n-\tilde u)\right\rangle\\&=\lim_{x\to\infty}\left[\left(a+b\int_\Omega|\nabla u_{n-1}|^2\,\mathrm{d}x\right)\int_\Omega\nabla\tilde u\cdot\nabla(u_n-\tilde u)\,\mathrm{d}x-\int_\Omega f(\tilde u)(u_n-\tilde u)\,\mathrm{d}x\right]\\&=0.\end{aligned}$$

Hence

$$\begin{aligned}0&=\lim_{x\to\infty}\left[\left\langle I'_{u_{n-1}}(u_n),(u_n-\tilde u)\right\rangle-\left\langle I'_{u_{n-1}}(\tilde u),(u_n-\tilde u)\right\rangle\right]\\&=\lim_{x\to\infty}\left[\left(a+b\int_\Omega|\nabla u_{n-1}|^2\,\mathrm{d}x\right)\int_\Omega|\nabla(u_n-\tilde u)|^2\,\mathrm{d}x-\int_\Omega\left(f(u_n)-f(\tilde u)\right)(u_n-\tilde u)\,\mathrm{d}x\right]\\&=\lim_{x\to\infty}\left(a+b\int_\Omega|\nabla u_{n-1}|^2\,\mathrm{d}x\right)\|u_n-\tilde u\|^2,\end{aligned}$$

which implies that $u_n \to \tilde{u}$ in E as $n \to \infty$. Thus for any $\varphi \in E$, we have

$$\begin{aligned}
0 &= \lim_{x\to\infty} \left\langle I'_{u_{n-1}}(u_n), \varphi \right\rangle \\
&= \lim_{x\to\infty} \left[\left(a + b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x \right) \int_\Omega \nabla u_n \cdot \nabla \varphi \mathrm{d}x - \int_\Omega f(u_n)\varphi \mathrm{d}x \right] \\
&= \left(a + b\int_\Omega |\nabla \tilde{u}|^2 \mathrm{d}x \right) \int_\Omega \nabla \tilde{u} \cdot \nabla \varphi \mathrm{d}x - \int_\Omega f(\tilde{u})\varphi \mathrm{d}x \\
&= \left\langle I'_{\tilde{u}}(\tilde{u}), \varphi \right\rangle.
\end{aligned}$$

Therefore, $\tilde{u}$ is a critical point of $I_{\tilde{u}}$, and $\tilde{u}$ satisfies (1.1). By $u_n \in S_{u_{n-1}}$, we have $u_n^+ \in N_{u_{n-1}}$ and $u_n^- \in N_{u_{n-1}}$ for $n \geq 2$. Then from Lemma 2.2, we have $\|u_n^+\| \geq c$ and $\|u_n^-\| \geq c$ for $n \geq 2$, so $\|\tilde{u}^+\| \geq c$ and $\|\tilde{u}^-\| \geq c$. Hence $\tilde{u}$ is a sign-changing solution of (1.1).

Step 3. We prove that

$$I_{\tilde{u}}(\tilde{u}) = \inf_{v \in S_{\tilde{u}}} I_{\tilde{u}}(v). \tag{3.10}$$

For any $v \in S_{\tilde{u}}$, we have that $v^+ \in N_{\tilde{u}}$ and $v^- \in N_{\tilde{u}}$. Since $v^+ \neq 0$ and $v^+ \neq 0$, by Lemma 2.1, there exists $t_n > 0$ and $s_n > 0$ such that $t_n v^+ \in N_{u_{n-1}}$ and $t_n v^- \in N_{u_{n-1}}, n = 2,3,4,\cdots$.

By Lemma 2.2, we have that

$$t_n \|v^+\| \geq c, s_n \|v^-\| \geq c.$$

Let $T_1 = \dfrac{c}{\|v^+\|}$ and $S_1 = \dfrac{c}{\|v^-\|}$, then for any $n \geq 2$, $t_n \geq T_1$ and $s_n \geq S_1$. On the other hand, by (f_5), there exist constants $C_1' > 0$ and $C_2' > 0$ such that

$$F(s) \geq C_1' s^\theta - C_2'. \tag{3.11}$$

Since $t_n v^+ \in N_{u_{n-1}}$, by Lemma 2.2 and (3.11), we have that

$$\begin{aligned}
\alpha &\leq I_{u_{n-1}}(t_n v^+) \\
&= \frac{t_n^2}{2}\left(a + b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x \right) \int_\Omega |\nabla v^+|^2 \mathrm{d}x - \int_\Omega F(t_n v^+)\mathrm{d}x \\
&\leq \frac{t_n^2}{2}\left(a + b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x \right) \int_\Omega |\nabla v^+|^2 \mathrm{d}x - C_1' t_n^\theta \int_\Omega |v^+|^\theta \mathrm{d}x + C_2'|\Omega|.
\end{aligned} \tag{3.12}$$

Note that $\|u_{n-1}\| \to \|\tilde{u}\|$ and $\theta > 2$, by (3.12) we can conclude that there must exist $T_2 > 0$ such that $t_n \leq T_2$ for any $n \geq 2$. Similarly, there exists $S_2 > 0$ such that $s_n \leq S_2$ for any $n \geq 2$. Then the sequence $\{t_n\}$ has a subsequence still denoted by $\{t_n\}$ such that $t_n \to t_0$ and the sequence s_n has a subsequence still denoted by s_n such that $s_n \to s_0$.

We show that $t_0 = 1$ and $s_0 = 1$. In fact, since $t_n v^+ \in N_{u_{n-1}}$, we have that

$$\left(a + b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x \right) t_n^2 \int_\Omega |\nabla v^+|^2 \mathrm{d}x - \int_\Omega f(t_n v^+) t_n v^+ \mathrm{d}x = 0,$$

letting $n \to \infty$, we get

$$\left(a + b\int_\Omega |\nabla \tilde{u}|^2 \mathrm{d}x \right) t_0^2 \int_\Omega |\nabla v^+|^2 \mathrm{d}x - \int_\Omega f(t_0 v^+) t_0 v^+ \mathrm{d}x = 0. \tag{3.13}$$

which implies that $t_0 v^+ \in N_{\tilde{u}}$. Recall that $v^+ \in N_{\tilde{u}}$, then by Lemma 2.1, we have

$t_0 = 1$. Similarly, we also have $s_0 = 1$.

Since $I_{u_{n-1}}(u_n) = \inf_{S_{u_{n-1}}} I_{u_{n-1}}$ and $t_n v^+ + s_n v^- \in S_{u_{n-1}}$, we have that

$$I_{u_{n-1}}(u_n) \le I_{u_{n-1}}\left(t_n v^+ + s_n v^-\right),$$

letting $n \to \infty$, we get

$$I_{\tilde{u}}(\tilde{u}) \le I_{\tilde{u}}(v).$$

This implies (3.10).

Step 4. We prove that $\tilde{u}$ has exactly two nodal domains.

Suppose in contradiction that $\tilde{u}$ has at least three nodal domains. We choose nodal domains Ω_1, Ω_2, such that $\tilde{u}_1 \ge 0$ and $\tilde{u}_2 \le 0$, where $\tilde{u}_i \ne 0, i = 1, 2$, are defined by

$$\tilde{u}_i(x) = \begin{cases} \tilde{u}(x), & \text{if } x \in \Omega_i, \\ 0, & \text{if } x \in \Omega \setminus \Omega_i. \end{cases}$$

Let $\tilde{u}_3 = \tilde{u} - \tilde{u}_1 - \tilde{u}_2$, then $\tilde{u}_3 \ne 0$. Since

$$\begin{aligned} 0 &= \left\langle I'_{\tilde{u}}(\tilde{u}), \tilde{u}_i \right\rangle \\ &= \left(a + b\int_\Omega |\nabla \tilde{u}|^2 \,\mathrm{d}x\right)\int_\Omega \nabla \tilde{u} \cdot \nabla \tilde{u}_i \mathrm{d}x - \int_\Omega f(\tilde{u})\tilde{u}_i \mathrm{d}x \\ &= \left(a + b\int_\Omega |\nabla \tilde{u}|^2 \,\mathrm{d}x\right)\int_\Omega |\nabla \tilde{u}_i|^2 \,\mathrm{d}x - \int_\Omega f(\tilde{u}_i)\tilde{u}_i \mathrm{d}x \\ &= \left\langle I'_{\tilde{u}}(\tilde{u}_i), \tilde{u}_i \right\rangle \end{aligned}$$

for $i = 1, 2, 3$, we have that $\tilde{u}_i \in N_{\tilde{u}}, i = 1, 2, 3$. Then $\tilde{u}_1 + \tilde{u}_2 \in S_{\tilde{u}}$, and by Lemma 2.1, $I_{\tilde{u}}(\tilde{u}_3) \ge \alpha > 0$. Hence, by (3.10),

$$I_{\tilde{u}}(\tilde{u}) \le I_{\tilde{u}}(\tilde{u}_1 + \tilde{u}_2) < I_{\tilde{u}}(\tilde{u}_1 + \tilde{u}_2) + I_{\tilde{u}}(\tilde{u}_3) = I_{\tilde{u}}(\tilde{u}),$$

we get a contradiction. Therefore, $\tilde{u}$ has exactly two nodal domains.

Acknowledgements

The authors would like to thank the reviewer for the valuable comments, which have helped to improve the quality of this paper.

Funding

This research was supported by National Natural Science Foundation of China (11901270) and Shandong Provincial Natural Science Foundation (ZR2019BA019).

Authors' Contributions

Xiaohan Duan: Conception and design of study, writing original draft, Writing review & editing.

Guanggang Liu: Conception and design of study, writing original draft, Writing review & editing.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Lions, J.L. (1978) On Some Questions in Boundary Value Problems of Mathematical Physics. *North-Holland Mathematics Studies*, **30**, 284-346. https://doi.org/10.1016/S0304-0208(08)70870-3

[2] Alves, C.O., Correa, F.J.S.A. and Ma, T.F. (2005) Positive Solutions for a Quasilinear Elliptic Equationof Kirchhoff Type. *Computers and Mathematics with Applications*, **49**, 85-93. https://doi.org/10.1016/j.camwa.2005.01.008

[3] Chen, B. and Ou, Z.Q. (2020) Sign-Changing and Nontrivial Solutions for a Class of Kirchhoff-Type Problems. *Journal of Mathematical Analysis and Applications*, **481**, Article ID: 123478. https://doi.org/10.1016/j.jmaa.2019.123476

[4] Cheng, B. (2012) New Existence and Multiplicity of Nontrivial Solutions for Nonlocal Elliptic Kirchhoff Type Problems. *Journal of Mathematical Analysis & Applications*, **394**, 488-495. https://doi.org/10.1016/j.jmaa.2012.04.025

[5] Figueiredo, G.M. (2013) Existence of a Positive Solution for a Kirchhoff Problem Type with Critical Growth via Truncation Argument. *Journal of Mathematical Analysis and Applications*, **401**, 706-713. https://doi.org/10.1016/j.jmaa.2012.12.053

[6] He, X. and Zou, W. (2012) Existence and Concentration Behavior of Positive Solutions for a Kirchhoff Equation in R3. *Journal of Differential Equations*, **252**, 1813-1834. https://doi.org/10.1016/j.jde.2011.08.035

[7] Li, Y., Li, F. and Shi, J. (2012) Existence of a Positive Solution to Kirchhoff Type Problems without Compactness Conditions. *Journal of Differential Equations*, **253**, 2285-2294. https://doi.org/10.1016/j.jde.2012.05.017

[8] Liang,S. and Shi, S. (2013) Soliton Solutions to Kirchhoff Type Problems Involving the Critical Growth in $\mathbb{R}^N$. *Nonlinear Analysis*, **81**, 31-41. https://doi.org/10.1016/j.na.2012.12.003

[9] Liang, S. and Shi, S. (2013) Existence of Multi-Bump Solutions for a Class of Kirchhoff Type Problems in $\mathbb{R}^3\mathbb{R}^3$. *Journal of Mathematical Physics*, **54**, Article ID: 121510. https://doi.org/10.1063/1.4850835

[10] Liu, G., Shi, S. and Wei, Y. (2019) Multiplicity of Solutions for Kirchhoff-Type Problem with Two Superlinear Potentials. *Bulletin of the Malaysian Mathematical Sciences Society*, **42**, 1657-1673. https://doi.org/10.1007/s40840-017-0571-z

[11] Mao, A.M. and Zhang, Z.T. (2009) Sign-Changing and Multiple Solutions of Kirchhoff Type Problems without the P.S. Condition. *Nonlinear Analysis*: *Theory*, *Methods & Applications*, **70**, 1275-1287. https://doi.org/10.1016/j.na.2008.02.011

[12] Perera, K. and Zhang, Z. (2006) Nontrivial Solutions of Kirchhoff-Type Problems via the Yang Index. *Journal of Differential Equations*, **221**, 246-255. https://doi.org/10.1016/j.jde.2005.03.006

[13] Shuai, W. (2015) Sign-Changing Solutions for a Class of Kirchhoff-Type Problem in Bounded Domains. *Journal of Differential Equations*, **259**, 1256-1274. https://doi.org/10.1016/j.jde.2015.02.040

[14] Sun, J. and Liu, S. (2012) Nontrivial Solutions of Kirchhoff Type Problems. *Applied Mathematics Letters*, **25**, 500-504. https://doi.org/10.1016/j.aml.2011.09.045

[15] Sun, J.T. and Wu, T.F. (2016) Existence and Multiplicity of Solutions for an Indefinite Kirchhoff-Type Equation in Bounded Domains. *Proceedings of the Royal Society of Edinburgh Section A*: *Mathematics*, **146**, 435-448. https://doi.org/10.1017/S0308210515000475

[16] Sun, J.J. and Tang, C.L. (2011) Existence and Multiplicity of Solutions for Kirchhoff Type Equations. *Nonlinear Analysis Theory Methods & Applications*, **74**, 1212-

1222. https://doi.org/10.1016/j.na.2010.09.061

[17] Tang, X. and Cheng, B. (2016) Ground State Sign-Changing Solutions for Kirchhoff Type Problems in Bounded Domains. *Journal of Differential Equations*, **261**, 2384-2402. https://doi.org/10.1016/j.jde.2016.04.032

[18] Wu, X. (2012) Existence of Nontrivial Solutions and High Energy Solutions for Schrödinger-Kirchhoff-Type Equations with Radial Potential. *Nonlinear Analysis Real World Applications*, **75**, 3470-3479. https://doi.org/10.1016/j.na.2012.01.004

[19] Zhang, Z. and Perera, K. (2006) Sign Changing Solutions of Kirchhoff Type Problems via Invariant Sets of Descent Flow. *Journal of Mathematical Analysis and Applications*, **317**, 456-463. https://doi.org/10.1016/j.jmaa.2005.06.102

[20] Zhang, Q.G., Sun, H.R. and Nieto, J.J. (2014) Positive Solution for a Superlinear Kirchhoff Type Problem with a Parameter. *Nonlinear Analysis: Theory, Methods & Applications*, **95**, 333-338. https://doi.org/10.1016/j.na.2013.09.019

[21] Figuereido, D.G., de, Girardi, M. and Matzeu, M. (2004) Semilinear Ellptic Equations with Dependence on the Gradient via Mountain-Pass Techniques. *Differential and Integral Equations*, **17**, 119-126. https://doi.org/10.57262/die/1356060475