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Relative Ding Projective Modules over Formal Triangular Matrix Rings

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Abstract

Let U be a (B, A)-bimodule, A and B be rings, and $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ be a formal triangular matrix ring. In this paper, we characterize the structure of relative Ding projective modules over T under some conditions. Furthermore, using the left global relative Ding projective dimensions of A and B, we estimate the relative Ding projective dimension of a left T-module.

Keywords

Formal Triangular Matrix Ring, Relative Ding Projective Module, Relative Ding Projective Dimension

1. Introduction

Let A and B be rings an U a (B, A)-bimodule, $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ is called a formal triangular matrix ring with usual matrix addition and multiplication. This kind of ring is useful in the representation theory of algebras and ring theory. It is typically used to create examples and counterexamples, which add more examples and concreteness to the theory of rings and modules. Many authors have studied T in several directions. For example, Zhang [1] specifically described the Artin triangular matrix algebra with Gorenstein projective modules. Enochs, Izurdiaga and Torrecillas [2] characterized Gorenstein projective and injective modules over a triangular matrix ring. Mao [3] studied Gorenstein flat modules over T and provided a left global Gorenstein flat dimension estimate of T. Besides, he [4] studied cotorsion pairs and approximation classes over T.

This paper aims at investigating relative Ding projective modules and relative

Ding projective dimension over T. Following is the organization of this paper.

In Section 2, we present some terminology as well as preliminary results.

In Section 3, we describe relative Ding projective modules over T. Assume that ${}_AC_1$ and ${}_BC_2$ are semidualizing. Let $M=\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, $C=\mathbf{p}(C_1,C_2)\in T\text{-Mod}$ and U be Ding C-compatible. Then a left T-module $M=\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is D_C-projective if and only if M_1 is D_{C_1}-projective, Coker φ^M is D_{C_2}-projective, and $\varphi^M: U\otimes_A M_1\to M_2$ is injective.

In Section 4, we estimate the D_C-projective dimension of a left T-module and the left global D_C-projective dimension of T. It is proved that, given a left T-module $M=\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, if $C=\mathbf{p}(C_1,C_2)$, U is Ding C-compatible, ${}_AC_1$ and ${}_BC_2$ are semidualizing, and $SD_{C_2}\text{-}PD(B)=\sup\left\{D_{C_2}\text{-}pd_B(U\otimes_A D)\mid D\in D_{C_1}P(A)\right\}<\infty$, then:

$$\max\left\{D_{C_1}\text{-}pd(M_1),\left(D_{C_2}\text{-}pd(M_2)-SD_{C_2}\text{-}PD(B)\right)\right\}\le D_C\text{-}pd(M)$$
$$\le\max\left\{\left(D_{C_1}\text{-}pd(M_1)\right)+\left(SD_{C_2}\text{-}PD(B)\right)+1,D_{C_2}\text{-}pd(M_2)\right\}.$$

Consequently, we prove that,

$$\max\left\{D_{C_1}\text{-}PD(A),D_{C_2}\text{-}PD(B)\right\}\le D_C\text{-}PD(T)$$
$$\le\max\left\{D_{C_1}\text{-}PD(A)+SD_{C_2}\text{-}PD(B)+1,D_{C_2}\text{-}PD(B)\right\}.$$

So we establish a relationship between the relative Ding projective dimension of modules over T and modules over A and B.

All rings for this article are nonzero associative rings with identity, and all modules are unitary. Unless stated explicitly, all modules will serve as unital left R-modules. For a ring R, we write R-Mod (resp. Mod-R) for the category of left (resp. right) R-modules. For a left R-module C, we use $\text{Add}_R(C)$ (resp. $\text{add}_R(C)$) to represent the class that contains all left R-modules that are isomorphic to direct summands of (resp. finite) direct sums of copies of C, and we use $\text{Prod}_R(C)$ to represent the class that contains all left R-modules that are isomorphic to direct summands of direct products of copies of C. $\mathcal{P}(R)$ and $\mathcal{F}(R)$ denote the classes of projective and flat left R-modules respectively. The character module $\text{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ of a module M is signed by M^+.

Next, we will review some concepts and facts about formal triangular matrix rings. By [[5], Theorem 1.5], T-Mod corresponds to the category Ω, whose objects are triples $M=\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, where $M_1\in A\text{-Mod}$, $M_2\in B\text{-Mod}$ and $\varphi^M: U\otimes_A M_1\to M_2$ is a B-morphism and whose morphisms from $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$

to $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ are pairs $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$ satisfying that the following diagram

$$\begin{CD} U \otimes_A M_1 @>{1\otimes f_1}>> U \otimes_A N_1 \\ @V{\varphi^M}VV @VV{\varphi^N}V \\ M_2 @>{f_2}>> N_2 \end{CD}$$

is commutative. Given a triple $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ in Ω, there is an A-morphism $\widetilde{\varphi^M} : M_1 \to \mathrm{Hom}_B(U, M_2)$ given by $\widetilde{\varphi^M}(x)(u) = \varphi^M(u \otimes x)$ for each $u \in U$, and $x \in M_1$.

It is worth noting that a sequence $0 \to \begin{pmatrix} M_1' \\ M_2' \end{pmatrix}_{\varphi^{M'}} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} M_1'' \\ M_2'' \end{pmatrix}_{\varphi^{M''}} \to 0$ of left T-modules is exact if and only if both the sequences $0 \to M_1' \to M_1 \to M_1'' \to 0$ and $0 \to M_2' \to M_2 \to M_2'' \to 0$ are exact.

Throughout this article, $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ is a formal triangular matrix ring. Given a left T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, the B-module $\mathrm{Coker}\,\varphi^M$ is denoted as $\bar{M}_2$ and the A-module $\ker \widetilde{\varphi^M}$ is denoted as $\underline{M_1}$.

Analogously, Mod-T is equivalent to the category Γ whose objects are triples $W = (W_1, W_2)_{\varphi_W}$, where $W_1 \in$ Mod-A, $W_2 \in$ Mod-B and $\varphi_W : W_2 \otimes_B U \to W_1$ is an A-morphism, and whose morphisms from $(W_1, W_2)_{\varphi_W}$ to $(X_1, X_2)_{\varphi_X}$ are pairs (g_1, g_2) such that $g_1 \in \mathrm{Hom}_A(W_1, X_1)$, $g_2 \in \mathrm{Hom}_B(W_2, X_2)$ satisfying that the following diagram

$$\begin{CD} W_2 \otimes_B U @>{g_2\otimes 1}>> X_2 \otimes_B U \\ @V{\varphi_W}VV @VV{\varphi_X}V \\ W_1 @>{g_1}>> X_1 \end{CD}$$

is commutative.

Given such a triple $W = (W_1, W_2)_{\varphi_W}$ in Γ, there is the B-morphism $\widetilde{\varphi_W} : W_2 \to \mathrm{Hom}_A(U, W_1)$ given by $\widetilde{\varphi_W}(y)(u) = \varphi_W(y \otimes u)$ for each $u \in U$, and $y \in W_2$.

In the remaining sections of the paper, we will identify T-Mod (resp. Mod-T) with the category Ω (resp. Γ)

According to [2], the following functors exist between the category T-Mod and the product category A-Mod × B-Mod:

1) $\mathbf{p}$: A-Mod × B-Mod → T-Mod is defined as follows: for each object (M_1, M_2) of A-Mod × B-Mod, let $\mathbf{p}(M_1, M_2) = \begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}$ with the obvious map and for any morphism (f_1, f_2) in A-Mod × B-Mod, let

$\mathbf{p}(f_1, f_2) = \begin{pmatrix} f_1 \\ (1 \otimes_A f_1) \oplus f_2 \end{pmatrix}$.

2) $\mathbf{h}: A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod}$ is defined as follows: for each object (M_1, M_2) of $A\text{-Mod} \times B\text{-Mod}$, let $\mathbf{h}(M_1, M_2) = \begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix}$ with the obvious map and for any morphism (f_1, f_2) in $A\text{-Mod} \times B\text{-Mod}$, let $\mathbf{h}(f_1, f_2) = \begin{pmatrix} f_1 \oplus \mathrm{Hom}_B(U, f_2) \\ f_2 \end{pmatrix}$.

3) $\mathbf{q}: T\text{-Mod} \to A\text{-Mod} \times B\text{-Mod}$ is defined as follows: for each left T-module $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ as $\mathbf{q}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = (M_1, M_2)$, and for each morphism $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ in T-Mod as $\mathbf{q}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = (f_1, f_2)$.

Note that $\mathbf{p}$ is a left adjoint of $\mathbf{q}$ and $\mathbf{h}$ is a right adjoint of $\mathbf{q}$. It is clear that $\mathbf{q}$ is exact. $\mathbf{p}$, in particular, preserves projective objects, while $\mathbf{h}$ preserves injective objects.

Between the category Mod-T and the product category $\text{Mod-}A \times \text{Mod-}B$, there are similar functors $\mathbf{p}, \mathbf{q}, \mathbf{h}$.

Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \in T\text{-Mod}$. By [6], $M^+ = (M_1^+, M_2^+)_{\varphi_{M^+}}$ is the character right T-module of M, where $\varphi_{M^+}: M_2^+ \otimes_B U \to M_1^+$ is defined by $\varphi_{M^+}(f \otimes u)(x) = f(\varphi^M(u \otimes x))$ for any $f \in M_2^+$, $u \in U$ and $x \in M_1$.

2. Preliminaries

Definition 2.1. ([[7], Definition 2.1]) A (R, S)-bimodule C is called semidualizing if the following conditions are satisfied:

1) ${}_RC$ and C_S permit a degreewise finite projective resolution in the corresponding module categories.

2) The natural homothety morphisms $R \to \mathrm{Hom}_S(C, C)$ and $S \to \mathrm{Hom}_R(C, C)$ are ring isomorphisms.

3) $\mathrm{Ext}_R^{\geq 1}(C, C) = \mathrm{Ext}_S^{\geq 1}(C, C) = 0$.

Definition 2.2. ([[8], Section 3]) A Wakamatsu tilting module is a left R-module ${}_RC$ satisfying the following properties:

1) ${}_RC$ permits a degreewise finite projective resolution.

2) $\mathrm{Ext}_R^{\geq 1}(C, C) = 0$.

3) There exists a $\mathrm{Hom}_R(-, C)$-exact exact sequence of R-modules

$$\boldsymbol{X}: 0 \to R \to C^0 \to C^1 \to \cdots,$$

where $C^i \in \mathrm{add}_R(C)$ for every $i \in \mathbb{N}$.

By [[8], Corollary 3.2], ${}_RC_S$ is semidualizing if and only if ${}_RC$ is a Wakamatsu tilting module with $S \cong \mathrm{End}_R(C)$ if and only if C_S is a Wakamatsu tilting module with $R \cong \mathrm{End}_S(C)$.

Definition 2.3. ([[9], Definition 3.1]) Let $C, M \in R$-Mod, M is said to be $\mathcal{F}_C$-flat if M^+ belongs to the class $\mathrm{Prod}_{R^{op}}(C^+)$, and we will denote the class of all $\mathcal{F}_C$-flat modules as $\mathcal{F}_C(R)$.

When $C = R$, $\mathcal{F}_C(R) = \mathcal{F}(R)$. Thus $\mathcal{F}(R)$ is a special case of $\mathcal{F}_C(R)$.

Remark 2.4. If ${}_RC_S$ is semidualizing, then $\mathcal{F}_C(R) = C \otimes_S \mathcal{F}(S)$ by [[9], Proposition 3.3].

Lemma 2.5. ([[10], Lemma 4]) Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \in T$-Mod and $(C_1, C_2) \in A$-Mod $\times B$-Mod.

$X \in \mathrm{Add}_T(\mathbf{p}(C_1, C_2))$ if and only if

1) $X \cong \mathbf{p}(X_1, \overline{X_2})$;

2) $X_1 \in \mathrm{Add}_A(C_1)$ and $\overline{X_2} \in \mathrm{Add}_B(C_2)$.

In this instance, φ^X is injective.

Lemma 2.6. ([[11], Theorem 3.1]) Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \in T$-Mod. $M \in \mathcal{P}(T)$ if and only if $M_1 \in \mathcal{P}(A)$, $\overline{M_2} \in \mathcal{P}(B)$ and φ^M is injective.

Lemma 2.7. Let $X = (X_1, X_2)_{\varphi_X} \in$ Mod-T and $(C_1, C_2) \in A$-Mod $\times B$-Mod.

$X \in \mathrm{Prod}_{T^{op}}(\mathbf{p}^+(C_1, C_2))$ if and only if

1) $X \cong \mathbf{h}(X_1, \ker(\widetilde{\varphi_X}))$;

2) $X_1 \in \mathrm{Prod}_{A^{op}}(C_1^+)$ and $\ker(\widetilde{\varphi_X}) \in \mathrm{Prod}_{B^{op}}(C_2^+)$.

In this instance, $\widetilde{\varphi_X}$ is surjective.

Proof. "$\Leftarrow$" If $X_1 \in \mathrm{Prod}_{A^{op}}(C_1^+)$ and $\ker(\widetilde{\varphi_X}) \in \mathrm{Prod}_{B^{op}}(C_2^+)$, then $X_1 \oplus \Upsilon_1 = (C_1^+)^{I_1}$ and $\ker(\widetilde{\varphi_X}) \oplus \Upsilon_2 = (C_2^+)^{I_2}$ for some $(\Upsilon_1, \Upsilon_2) \in$ Mod-$A \times$ Mod-B and some sets I_1 and I_2. Without loss of generality, we can assume that $I = I_1 = I_2$. Then:

$$\begin{aligned}
X \oplus \mathbf{h}(\Upsilon_1, \Upsilon_2) &\cong \mathbf{h}(X_1, \ker(\widetilde{\varphi_X})) \oplus \mathbf{h}(\Upsilon_1, \Upsilon_2) \\
&= (X_1, \mathrm{Hom}_{A^{op}}(U, X_1) \oplus \ker(\widetilde{\varphi_X})) \oplus (\Upsilon_1, \mathrm{Hom}_{A^{op}}(U, \Upsilon_1) \oplus \Upsilon_2) \\
&= \left((C_1^+)^I, \mathrm{Hom}_{A^{op}}\left(U, (C_1^+)^I\right) \oplus (C_2^+)^I\right) \\
&\cong \left((C_1^+)^I, \mathrm{Hom}_{A^{op}}(U, C_1^+)^I \oplus (C_2^+)^I\right) \\
&\cong \left((C_1^+)^I, ((U \otimes_A C_1)^+)^I \oplus (C_2^+)^I\right) \\
&= (\mathbf{p}^+(C_1, C_2))^I.
\end{aligned}$$

Hence, $X \in \mathrm{Prod}_{T^{op}}(\mathbf{p}^+(C_1, C_2))$.

"$\Rightarrow$" Let $X \in \mathrm{Prod}_{T^{op}}(\mathbf{p}^+(C_1, C_2))$ and $\Upsilon = (\Upsilon_1, \Upsilon_2)_{\varphi_\Upsilon} \in$ Mod-T such that $X \oplus \Upsilon = (\mathbf{p}^+(C_1, C_2))^I$ for some set *I*. Then $\widetilde{\varphi_X}$ is surjective as X is a submodule of $(\mathbf{p}^+(C_1, C_2))^I$ and $\widetilde{\varphi_{C^+}}$ is surjective. Now, let $C := \mathbf{p}(C_1, C_2)$, there is an exact split sequence:

$$0 \to \Upsilon \xrightarrow{(\lambda_1,\lambda_2)} \left(C^+\right)^I \xrightarrow{(p_1,p_2)} X \to 0,$$

which induces the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\widetilde{\varphi_\Upsilon}) & \xrightarrow{f} & (C_2^+)^I & \xrightarrow{g} & \ker(\widetilde{\varphi_X}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & \Upsilon_2 & \xrightarrow{\lambda_2} & ((U\otimes_A C_1)^+)^I \oplus (C_2^+)^I & \xrightarrow{p_2} & X_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{\varphi_\Upsilon}} & & \downarrow{\scriptstyle \widetilde{\varphi_{C^+}}} & & \downarrow{\scriptstyle \widetilde{\varphi_X}} & & \\
0 & \longrightarrow & \mathrm{Hom}_A(U,\Upsilon_1) & \xrightarrow{\lambda_{1*}} & \mathrm{Hom}_{A^{op}}(U,(C_1^+)^I) & \xrightarrow{p_{1*}} & \mathrm{Hom}_A(U,X_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where h,j,k are the canonical injections. Clearly, p_1 and p_{1*} are split epimorphisms. Thus, $X_1 \in \mathrm{Prod}_{A^{op}}\left(C_1^+\right)$. Next, we prove that the short exact sequence:

$$0 \to \ker\left(\widetilde{\varphi_X}\right) \xrightarrow{k} X_2 \xrightarrow{\widetilde{\varphi_X}} \mathrm{Hom}_A\left(U, X_1\right) \to 0$$

splits. Let r be the retraction of p_1^*. If $i: \mathrm{Hom}_{A^{op}}\left(U,\left(C_1^+\right)^I\right) \to \left(\left(U\otimes_A C_1\right)^+\right)^I \oplus \left(C_2^+\right)^I$ denotes the canonical injection by $\mathrm{Hom}_{A^{op}}\left(U,\left(C_1^+\right)\right) \cong \left(U\otimes_A C_1\right)^+$, then $\widetilde{\varphi_X} p_2 ir = p_{1*}\widetilde{\varphi_{C^+}} ir = p_{1*}r = 1_{\mathrm{Hom}_A(U,X_1)}$. Thus $X_2 \cong \mathrm{Hom}_A\left(U,X_1\right) \oplus \ker\left(\widetilde{\varphi_X}\right)$ and the first row is a split exact sequence too. So $\ker\left(\widetilde{\varphi_X}\right) \in \mathrm{Prod}_{B^{op}}\left(C_2^+\right)$ and $X \cong \mathbf{h}\left(X_1, \ker\left(\widetilde{\varphi_X}\right)\right)$.□

Corollary 2.8. Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \in T\text{-Mod}$ and $\left(C_1, C_2\right) \in A\text{-Mod} \times B\text{-Mod}$.

If $C = \mathbf{p}\left(C_1, C_2\right)$, then $X \in \mathcal{F}_C\left(T\right)$ if and only if

1) $X^+ \cong \mathbf{h}\left(X_1^+, \overline{X}_2^+\right)$;

2) $X_1 \in \mathcal{F}_{C_1}\left(A\right)$ and $\overline{X}_2 \in \mathcal{F}_{C_2}\left(B\right)$.

In this instance, φ^X is injective.

Proof. $X \in \mathcal{F}_C\left(T\right)$ if and only if $X^+ = \left(X_1^+, X_2^+\right)_{\varphi_{X^+}} \in \mathrm{Prod}_{T^{op}}\left(C^+\right)$ if and only if $X^+ \cong \mathbf{h}\left(X_1^+, \ker\left(\widetilde{\varphi_{X^+}}\right)\right)$, $X_1^+ \in \mathrm{Prod}_{A^{op}}\left(C_1^+\right)$, $\ker\left(\widetilde{\varphi_{X^+}}\right) \in \mathrm{Prod}_{B^{op}}\left(C_2^+\right)$ by Lemma 2.7. Note that $\widetilde{\varphi_{X^+}}$ is surjective. Hence, φ^X is injective. Then we get an exact sequence

$$0 \to U \otimes_A X_1 \xrightarrow{\varphi^X} X_2 \to \overline{X}_2 \to 0.$$

Consider the commutative diagram with exact rows shown below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{X}_2^+ & \longrightarrow & X_2^+ & \longrightarrow & (U\otimes_A X_1)^+ & \longrightarrow & 0 \\ & & \downarrow \cong & & \| & & \downarrow \cong & & \\ 0 & \longrightarrow & \ker(\widetilde{\varphi_{X^+}}) & \longrightarrow & X_2^+ & \xrightarrow{\widetilde{\varphi_{X^+}}} & \mathrm{Hom}_A(U, X_1^+) & \longrightarrow & 0 \end{array}$$

Thus $\overline{X}_2^+ \cong \ker\left(\widetilde{\varphi_{X^+}}\right) \in \mathrm{Prod}_{B^{op}}\left(C_2^+\right)$. So $X \in \mathcal{F}_C\left(T\right)$ if and only if $X^+ \cong \mathbf{h}\left(X_1^+, \overline{X}_2^+\right)$, $X_1 \in \mathcal{F}_{C_1}\left(A\right)$ and $\overline{X}_2 \in \mathcal{F}_{C_2}\left(B\right)$, and the proof is finished.

3. Relative Ding Projective Modules

This section will characterize relative Ding projective modules over a formal triangular matrix ring.

Definition 3.1 ([[12], Definition 1.1]) Let ${}_RC_S$ be a semidualizing bimodule. A left R-module M is said to be D_C-projective if there exists a $\mathrm{Hom}_R\left(-, C\otimes_S F\right)$-exact exact sequence in R-Mod:

$$\cdots \to P_1 \to P_0 \to A^0 \to A^1 \to \cdots$$

with $A^i \in \mathrm{Add}_R\left(C\right)$, $P_i \in \mathcal{P}\left(R\right)$ for every $i \in \mathbb{N}$ and $F \in \mathcal{F}\left(S\right)$, such that $M \cong \mathrm{Im}\left(P_0 \to A^0\right)$.

The class of all D_C-projective R-modules is denoted by $D_CP\left(R\right)$.

Note that if $C = R$, then D_C-projective R-modules are Ding projective R-modules.

We introduce the following concept, which is critical to the rest of this study, inspired by the definition of C-compatible bimodule in [[10], Definition 4].

Definition 3.2. Let $\left(C_1, C_2\right) \in A\text{-Mod} \times B\text{-Mod}$ and $C = \mathbf{p}\left(C_1, C_2\right)$. A bimodule ${}_BU_A$ is said to be Ding C-compatible if the following two conditions hold:

(a) The complex $U \otimes_A \boldsymbol{X}_1$ is exact for every exact sequence in A-Mod:

$$\boldsymbol{X}_1 : \cdots \to P_1^1 \to P_1^0 \to A_1^0 \to A_1^1 \to \cdots$$

with $P_1^i \in \mathcal{P}\left(A\right)$ and $A_1^i \in \mathrm{Add}_A\left(C_1\right)$ for every $i \in \mathbb{N}$.

(b) The complex $\mathrm{Hom}_B\left(\boldsymbol{X}_2, U \otimes_A \mathcal{F}_{C_1}\left(A\right)\right)$ is exact for every $\mathrm{Hom}_B\left(-, \mathcal{F}_{C_2}\left(B\right)\right)$-exact exact sequence in B-Mod:

$$\boldsymbol{X}_2 : \cdots \to P_2^1 \to P_2^0 \to A_2^0 \to A_2^1 \to \cdots$$

with $P_2^i \in \mathcal{P}\left(B\right)$ and $A_2^i \in \mathrm{Add}_B\left(C_2\right)$ for every $i \in \mathbb{N}$.

Furthermore, U is said to be weakly Ding C-compatible if it meets (b) and the following condition:

(a') The complex $U \otimes_A \boldsymbol{X}_1$ is exact for every $\mathrm{Hom}_A\left(-, \mathcal{F}_{C_1}\left(A\right)\right)$-exact exact sequence in A-Mod:

$$\boldsymbol{X}_1 : \cdots \to P_1^1 \to P_1^0 \to A_1^0 \to A_1^1 \to \cdots$$

with $P_1^i \in \mathcal{P}\left(A\right)$ and $A_1^i \in \mathrm{Add}_A\left(C_1\right)$ for every $i \in \mathbb{N}$.

Proposition 3.3. Suppose that $C = \mathbf{p}\left(C_1, C_2\right)$ be a left T-module and U be weakly Ding C-compatible. If ${}_AC_1$ and ${}_BC_2$ are semidualizing, then $\mathbf{p}\left(C_1, C_2\right)$ is semidualizing.

Proof. Assume that ${}_AC_1$ and ${}_BC_2$ are semidualizing. By [[8], Corollary 3.2], ${}_AC_1$ and ${}_BC_2$ are tilting. To prove C is tilting, the functor $\mathbf{p}$ preserves finitely generated modules by [13]. Then $\mathrm{Ext}_A^{i\geq 1}(C_1,C_1)=0$ and $\mathrm{Ext}_B^{i\geq 1}(C_2,C_2)=0$. Observe that $C_1\in D_{C_1}P(A)$ and $C_2\in D_{C_2}P(B)$ by [[12], Proposition 1.8]. Since U satisfies (a'), $\mathrm{Tor}_{i\geq 1}^A(U,C_1)=0$. And, as U satisfies (b), $\mathrm{Ext}_B^{i\geq 1}(C_2,U\otimes_A C_1)=0$. For every $n\geq 1$, by [[10], Lemma 3], we get that:

$$\begin{aligned}\mathrm{Ext}_T^n(C,C)&=\mathrm{Ext}_T^n(\mathbf{p}(C_1,C_2),\mathbf{p}(C_1,C_2))\\&\cong \mathrm{Ext}_A^n(C_1,C_1)\oplus \mathrm{Ext}_B^n(C_2,U\otimes_A C_1)\oplus \mathrm{Ext}_B^n(C_2,C_2)\\&=0.\end{aligned}$$

Furthermore, there exist exact sequences:

$$\boldsymbol{X}_1:0\to A\to C_1^0\to C_1^1\to\cdots,$$

and:

$$\boldsymbol{X}_2:0\to B\to C_2^0\to C_2^1\to\cdots$$

which are $\mathrm{Hom}_A(-,\mathrm{Add}_A(C_1))$-exact and $\mathrm{Hom}_B(-,\mathrm{Add}_B(C_1))$-exact, respectively, and $C_1^i\in\mathrm{add}_A(C_1)$, $C_2^i\in\mathrm{add}_B(C_2)$, $\forall i\in\mathbb{N}$. Note that every cokernel in $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are finitely presented. Thus, $\mathrm{Hom}_A(\boldsymbol{X}_1,\mathcal{F}_{C_1}(A))$ and $\mathrm{Hom}_A(\boldsymbol{X}_2,\mathcal{F}_{C_2}(B))$ are exact. Since U is weakly Ding C-compatible, the complex $U\otimes_A\boldsymbol{X}_1$ is exact. As a result, we get the following exaxt sequence

$$\mathbf{p}(\boldsymbol{X}_1,\boldsymbol{X}_2):0\to T\to\mathbf{p}(C_1^0,C_2^0)\to\mathbf{p}(C_1^1,C_2^1)\to\cdots,$$

with $\mathbf{p}(C_1^i,C_2^i)=\begin{pmatrix}C_1^i\\U\otimes_A C_1^i\oplus C_2^i\end{pmatrix}\in\mathrm{add}_T(\mathbf{p}(C_1,C_2))$, $\forall i\in\mathbb{N}$, by Lemma 2.5.

Let $X\in\mathrm{Add}_T(C)$, by Lemma 2.5, $X\cong\mathbf{p}(X_1,X_2)$ where $X_1\in\mathrm{Add}_A(C_1)$ and $X_2\in\mathrm{Add}_B(C_2)$. There is a complex isomorphism using adjointness ($\mathbf{p},\mathbf{q}$):

$$\mathrm{Hom}_T(\mathbf{p}(\boldsymbol{X}_1,\boldsymbol{X}_2),X)\cong\mathrm{Hom}_A(\boldsymbol{X}_1,X_1)\oplus\mathrm{Hom}_B(\boldsymbol{X}_2,U\otimes_A X_1)\oplus\mathrm{Hom}_B(\boldsymbol{X}_2,X_2).$$

It should be noted that the complexes $\mathrm{Hom}_A(\boldsymbol{X}_1,X_1)$ and $\mathrm{Hom}_B(\boldsymbol{X}_2,X_2)$, as well as the complex $\mathrm{Hom}_B(\boldsymbol{X}_2,U\otimes_A X_1)$ are exact since U is weakly Ding C-compatible. Then $\mathrm{Hom}_T(\mathbf{p}(\boldsymbol{X}_1,\boldsymbol{X}_2),X)$ is exact. So $\mathbf{p}(C_1,C_2)$ is semidualizing by [[8], Corollary 3.2].□

Lemma 3.4. Assume that ${}_AC_1$ and ${}_BC_2$ are semidualizing. Let $C=\mathbf{p}(C_1,C_2)$ be a left T-module and U be weakly Ding C-compatible.

1) If $M_1\in D_{C_1}P(A)$, then $\mathbf{p}(M_1,0)\in D_CP(T)$.

2) If $M_2\in D_{C_2}P(B)$, then $\mathbf{p}(0,M_2)\in D_CP(T)$.

Proof. By Proposition 3.3, the functor $\mathbf{p}$ preservers semidualizing. Thus $C\otimes_S F\cong\mathcal{F}_C(T)$ by Remark 2.4.

1) Assume that $M_1\in D_{C_1}P(A)$. There exists a $\mathrm{Hom}_A(-,\mathcal{F}_{C_1}(A))$-exact exact sequence in A-Mod:

$$\boldsymbol{X}_1:\cdots\to P_1^1\to P_1^0\to C_1^0\to C_1^1\to\cdots,$$

where $P_1^i\in\mathcal{P}(A)$ and $C_1^i\in\mathrm{Add}_A(C_1)$ $\forall i\in\mathbb{N}$ and $M_1\cong\mathrm{Im}(P_1^0\to C_1^0)$.

Since U is weakly Ding C-compatible, we have the complex $U\otimes_A \boldsymbol{X}_1$ is exact in B-Mod. So we get an exact sequence

$$\mathbf{p}(\boldsymbol{X}_1,0):\cdots\to\begin{pmatrix}P_1^1\\ U\otimes_A P_1^1\end{pmatrix}\to\begin{pmatrix}P_1^0\\ U\otimes_A P_1^0\end{pmatrix}\to\begin{pmatrix}C_1^0\\ U\otimes_A C_1^0\end{pmatrix}\to\begin{pmatrix}C_1^1\\ U\otimes_A C_1^1\end{pmatrix}\to\cdots$$

with

$$\begin{pmatrix}M_1\\ U\otimes_A M_1\end{pmatrix}\cong \operatorname{Im}\left(\begin{pmatrix}P_1^0\\ U\otimes_A P_1^0\end{pmatrix}\to\begin{pmatrix}C_1^0\\ U\otimes_A C_1^0\end{pmatrix}\right).$$

Clearly, $\mathbf{p}(P_1^i,0)=\begin{pmatrix}P_1^i\\ U\otimes_A P_1^i\end{pmatrix}\in\mathcal{P}(T)$ and $\mathbf{p}(C_1^i,0)=\begin{pmatrix}C_1^i\\ U\otimes_A C_1^i\end{pmatrix}\in \operatorname{Add}_T(C)$ for every $i\in\mathbb{N}$ by Lemmas 2.6 and 2.5.

If $N=\begin{pmatrix}N_1\\ N_2\end{pmatrix}_{\varphi^N}\in\mathcal{F}_C(T)$, then $N_1\in\mathcal{F}_{C_1}(A)$ by Corollary 2.8. Then using the adjointness, we get that $\operatorname{Hom}_T(\mathbf{p}(\boldsymbol{X}_1,0),N)\cong\operatorname{Hom}_A(\boldsymbol{X}_1,N_1)$ is exact. Thus $\begin{pmatrix}M_1\\ U\otimes_A M_1\end{pmatrix}$ is D_C-projective.

2) Assume that $M_2\in D_{C_2}P(B)$. There exists a $\operatorname{Hom}_B(-,\mathcal{F}_{C_2}(B))$-exact exact sequence in B-Mod:

$$\boldsymbol{X}_2:\cdots\to P_2^1\to P_2^0\to C_2^0\to C_2^1\to\cdots,$$

where $P_2^i\in\mathcal{P}(B)$ and $C_2^i\in\operatorname{Add}_B(C_2)$ $\forall i\in\mathbb{N}$ and $M_2\cong\operatorname{Im}(P_2^0\to C_2^0)$. As a result, we have an exact sequence

$$\mathbf{p}(0,\boldsymbol{X}_2):\cdots\to\begin{pmatrix}0\\ P_2^1\end{pmatrix}\to\begin{pmatrix}0\\ P_2^0\end{pmatrix}\to\begin{pmatrix}0\\ C_2^0\end{pmatrix}\to\begin{pmatrix}0\\ C_2^1\end{pmatrix}\to\cdots$$

with $\begin{pmatrix}0\\ M_2\end{pmatrix}\cong\operatorname{Im}\left(\begin{pmatrix}0\\ P_2^0\end{pmatrix}\to\begin{pmatrix}0\\ C_2^0\end{pmatrix}\right)$, $\mathbf{p}(0,P_2^i)=\begin{pmatrix}0\\ P_2^i\end{pmatrix}\in\mathcal{P}(T)$ and $\mathbf{p}(0,C_2^i)=\begin{pmatrix}0\\ C_2^i\end{pmatrix}\in\operatorname{Add}_T(C)$ for every $i\in\mathbb{N}$ by Lemmas 2.6 and 2.5 respectively. Let $N=\begin{pmatrix}N_1\\ N_2\end{pmatrix}_{\varphi^N}\in\mathcal{F}_C(T)$, then $N_1\in\mathcal{F}_{C_1}(A)$, $\overline{N}_2\in\mathcal{F}_{C_2}(B)$ and φ^N is injective by Corollary 2.8. Thus we obtain a short exact sequence:

$$0\to U\otimes_A N_1\to N_2\to\overline{N}_2\to 0.$$

Because U is weakly Ding C-compatible, $\cdots\to P_2^1\to P_2^0\to M_2\to 0$ is a $\operatorname{Hom}_B(-,U\otimes_A N_1)$-exact exact sequence. Then $\operatorname{Ext}_B^1(M_2,U\otimes_A N_1)=0$. Consider a short exact sequence $0\to M_2\to C_2^0\to L\to 0$ with $L\cong\operatorname{Im}(M_2\to C_2^0)$ is D_{C_2}-projective by [[12], Proposition 1.13]. Thus $\operatorname{Ext}_B^1(L,U\otimes_A N_1)=0$, and then $\operatorname{Ext}_B^1(C_2^0,U\otimes_A N_1)=0$. Consequently, $\operatorname{Ext}_B^1(C_2^i,U\otimes_A N_1)=0$. Then we obtain the exact sequence of complexes shown below.

$$0\to\operatorname{Hom}_B(\boldsymbol{X}_2,U\otimes_A N_1)\to\operatorname{Hom}_B(\boldsymbol{X}_2,N_2)\to\operatorname{Hom}_B(\boldsymbol{X}_2,\overline{N}_2)\to 0$$

As U is weakly Ding C-compatible, $\operatorname{Hom}_B(\boldsymbol{X}_2,U\otimes_A N_1)$ is exact and $\operatorname{Hom}_B(\boldsymbol{X}_2,\overline{N}_2)$ is exact. Thus $\operatorname{Hom}_B(\boldsymbol{X}_2,N_2)$ is exact. Then

$\mathrm{Hom}_T\left(\mathbf{p}\left(0,\boldsymbol{X}_2\right),N\right)\cong \mathrm{Hom}_B\left(\boldsymbol{X}_2,N_2\right)$ is exact. Above all, $\mathbf{p}\left(0,M_2\right)\in D_C P(T)$.

Theorem 3.5. Assume that ${}_AC_1$ and ${}_BC_2$ are semidualizing. Let $M=\begin{pmatrix}M_1\\M_2\end{pmatrix}_{\varphi^M}$, $C=\mathbf{p}\left(C_1,C_2\right)\in T$-Mod and U be Ding C-compatible. Then the following statements are equivalent:

1) M is D_C-projective.

2) φ^M is injective, M_1 is D_{C_1}-projective and $\bar{M}_2 := \mathrm{Coker}\varphi^M$ is D_{C_2}-projective.

In this instance, $U\otimes_A M_1$ is D_{C_2}-projective if and only if M_2 is D_{C_2}-projective.

Proof. (1) $\Rightarrow$ (2) There exists a $\mathrm{Hom}_T\left(-,\mathcal{F}_C(T)\right)$-exact exact sequence in T-Mod:

$$\boldsymbol{X}=\cdots\to\begin{pmatrix}P_1^1\\P_2^1\end{pmatrix}_{\varphi^{P^1}}\to\begin{pmatrix}P_1^0\\P_2^0\end{pmatrix}_{\varphi^{P^0}}\to\begin{pmatrix}C_1^0\\C_2^0\end{pmatrix}_{\varphi^{C^0}}\to\begin{pmatrix}C_1^1\\C_2^1\end{pmatrix}_{\varphi^{C^1}}\to\cdots,$$

where $P^i=\begin{pmatrix}P_1^i\\P_2^i\end{pmatrix}_{\varphi^{P^i}}\in\mathcal{P}(T)$ and $C^i=\begin{pmatrix}C_1^i\\C_2^i\end{pmatrix}_{\varphi^{C^i}}\in\mathrm{Add}_T(C)$ $\forall i\in\mathbb{N}$, and such that $M\cong\mathrm{Im}\left(P^0\to C^0\right)$. Then we get an exact sequence in A-Mod:

$$\boldsymbol{X}_1:\cdots\to P_1^1\to P_1^0\to C_1^0\to C_1^1\to\cdots,$$

where $P_1^i\in\mathcal{P}(A)$ and $C_1^i\in\mathrm{Add}_A\left(C_1\right)$ $\forall i\in\mathbb{N}$ by Lemmas 2.6 and 2.5 and such that $M_1\cong\mathrm{Im}\left(P_1^0\to C_1^0\right)$. As U is Ding C-compatible, the complex $U\otimes_A\boldsymbol{X}_1$ is exact with $U\otimes_A M_1\cong\mathrm{Im}\left(U\otimes_A P_1^0\to U\otimes_A C_1^0\right)$. Let $l_1:M_1\to C_1^0$ and $l_2:M_2\to C_2^0$ be the inclusions, then $1_U\otimes l_1$ is injective. Consequently, the commutative diagram is as follows:

$$\begin{array}{ccc}
U\otimes_A M_1 & \xrightarrow{1_u\otimes l_1} & U\otimes_A C_1^0\\
\downarrow{\scriptstyle\varphi^M} & & \downarrow{\scriptstyle\varphi^{C^0}}\\
M_2 & \xrightarrow{l_2} & C_2^0.
\end{array}$$

According to Lemma 2.5, φ^{C^0} is injective, then φ^M will be as well. Furthermore, for every $i\in\mathbb{N}$, φ^{C^i} and φ^{P^i} are injective by Lemmas 2.5 and 2.6. The result is the commutative diagram with exact columns shown below.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & U\otimes_A P_1^1 & \to & U\otimes_A P_1^0 & \to & U\otimes_A C_1^0 & \to & U\otimes_A C_1^1 & \to & \cdots\\
 & & \downarrow{\scriptstyle\varphi^{P^1}} & & \downarrow{\scriptstyle\varphi^{P^0}} & & \downarrow{\scriptstyle\varphi^{C^0}} & & \downarrow{\scriptstyle\varphi^{C^1}} & & \\
\cdots & \to & P_2^1 & \to & P_2^0 & \to & C_2^0 & \to & C_2^1 & \to & \cdots\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \overline{P_2^1} & \to & \overline{P_2^0} & \to & \overline{C_2^0} & \to & \overline{C_2^1} & \to & \cdots\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the first and the second rows are exact in the above diagram, we get an exact sequence in B-Mod:

$$\overline{X}_2 : \cdots \to \overline{P_2^1} \to \overline{P_2^0} \to \overline{C_2^0} \to \overline{C_2^1} \to \cdots,$$

where $\overline{P_2^i} \in \mathcal{P}(B)$ and $\overline{C_2^i} \in \mathrm{Add}_B(C_2)$ for every $i \in \mathbb{N}$ by Lemmas 2.6 and 2.5, and such that $\overline{M}_2 \cong \mathrm{Im}\left(\overline{P_2^0} \to \overline{C_2^0}\right)$. Let $N_1 \in \mathcal{F}_{C_1}(A)$ and $N_2 \in \mathcal{F}_{C_2}(B)$, then $\mathbf{p}(N_1,0) \in \mathcal{F}_C(T)$ and $\mathbf{p}(0,N_2) \in \mathcal{F}_C(T)$ by Corollary 2.8. Then by using adjointness, $\mathrm{Hom}_T(\boldsymbol{X},\mathbf{p}(0,N_2)) \cong \mathrm{Hom}_B(\overline{\boldsymbol{X}}_2,N_2)$ is exact. Thus, $\overline{M}_2$ is D_{C_2}-projective. Note that $C^i \cong \mathbf{p}\left(C_1^i,\overline{C_2^i}\right)$ by Lemma 2.5. Then

$\mathrm{Ext}_T^1\left(C_i,\begin{pmatrix}0\\ U\otimes_A N_1\end{pmatrix}\right) \cong \mathrm{Ext}_B^1\left(\overline{C_2^i},U\otimes_A N_1\right)=0$ by [[10], Lemma 3] and U is Ding C-compatible. As a result, when we apply the functor $\mathrm{Hom}_T(\boldsymbol{X},-)$ to the sequence:

$$0\to\begin{pmatrix}0\\ U\otimes_A N_1\end{pmatrix}\to\begin{pmatrix}N_1\\ U\otimes_A N_1\end{pmatrix}\to\begin{pmatrix}N_1\\ 0\end{pmatrix}\to 0,$$

we get the exact sequence of complexes:

$$0\to\mathrm{Hom}_T\left(\boldsymbol{X},\begin{pmatrix}0\\ U\otimes_A N_1\end{pmatrix}\right)\to\mathrm{Hom}_T\left(\boldsymbol{X},\begin{pmatrix}N_1\\ U\otimes_A N_1\end{pmatrix}\right)\to\mathrm{Hom}_T\left(\boldsymbol{X},\begin{pmatrix}N_1\\ 0\end{pmatrix}\right)\to 0.$$

By applying adjointness, we obtain that

$$\mathrm{Hom}_T\left(\boldsymbol{X},\begin{pmatrix}0\\ U\otimes_A N_1\end{pmatrix}\right)\cong\mathrm{Hom}_B\left(\overline{\boldsymbol{X}}_2,U\otimes_A N_1\right)$$

and

$$\mathrm{Hom}_T\left(\boldsymbol{X},\begin{pmatrix}N_1\\ 0\end{pmatrix}\right)\cong\mathrm{Hom}_A\left(\boldsymbol{X}_1,N_1\right).$$

Note that $\mathrm{Hom}_T\left(\boldsymbol{X},\begin{pmatrix}N_1\\ U\otimes_A N_1\end{pmatrix}\right)$ is exact, and since U is Ding C-compatible, $\mathrm{Hom}_B\left(\overline{\boldsymbol{X}}_2,U\otimes_A N_1\right)$ is exact too. It implies that $\mathrm{Hom}_A(\boldsymbol{X}_1,N_1)$ is exact. So M_1 is D_{C_1}-projective.

2) $\Rightarrow$ 1) Because φ^M is injective, an exact sequence exists in T-Mod:

$$0\to\begin{pmatrix}M_1\\ U\otimes_A M_1\end{pmatrix}\to M\to\begin{pmatrix}0\\ \overline{M}_2\end{pmatrix}\to 0.$$

By Theorem 3.5, $\begin{pmatrix}M_1\\ U\otimes_A M_1\end{pmatrix}$ and $\begin{pmatrix}0\\ \overline{M}_2\end{pmatrix}$ are D_C-projective T-modules. Hence, M is D_C-projective according to [[12], Theorem 1.12]. Finally, there exists an exact sequence

$$0\to U\otimes_A M_1\xrightarrow{\varphi^M}M_2\to\overline{M}_2\to 0.$$

Since $\overline{M}_2$ is D_{C_2}-projective, $U\otimes_A M_1$ is D_{C_2}-projective if and only if

M_2 is D_{C_2}-projective by [[12], Theorem 2.12].

Corollary 3.6. Assume that $_RC_1$ is semidualizing. Let R be a ring, $T(R)=\begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$, $C=\mathbf{p}(C_1,C_1)$ and $M=\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left $T(R)$-module, then the following conditions are equivalent:

1) M is a D_C-projective left T(R)-module.

2) M_1 and $\bar{M}_2$ is D_{C_1}-projective, and φ^M is injective.

3) M_1 and M_2 is D_{C_1}-projective, and φ^M is injective.

Proof. It is an immediate consequence of Theorem 3.5.□

4. Relative Ding Projective Dimension

This section aims to search in the D_C-projective dimension of *T*-modules as well as the left D_C-projective global dimension of *T*. We now recall [12] that the concept of relative Ding projective dimenion. The D_C-projective dimension D_C-pd(*M*) of a left *R*-module *M* is defined as inf{*n*| there there is an exact sequence

$$0\to D_n\to\cdots\to D_1\to D_0\to M\to 0$$

with $D_i\in D_CP(R)$ for every $i\in\{0,\cdots,n\}$. The left global D_C-projective dimension of *R* is defined as: $D_C\text{-}PD(R)=\sup\{D_C\text{-}pd(M)\mid M\in R\text{-Mod}\}$.

Lemma 4.1. Assume that $_AC_1$ and $_BC_2$ are semidualizing. Let $C=\mathbf{p}(C_1,C_2)$ and *U* Ding *C*-compatible. Then the following statements hold.

1) $D_{C_2}\text{-}pd(M_2)=D_C\text{-}pd\left(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}\right)$.

2) $D_{C_1}\text{-}pd(M_2)\le D_C\text{-}pd(\mathbf{p}(M_1,0))$, and the equality is true if $\mathrm{Tor}_{i\ge1}^A(U,M_1)=0$.

Proof. 1) Consider the following exact sequence

$$0\to K_2^n\to D_2^{n-1}\to\cdots\to D_2^0\to M_2\to 0$$

with D_2^i is D_{C_2}-projective. As a result, we have an exact sequence in *T*-Mod:

$$0\to\begin{pmatrix} 0 \\ K_2^n \end{pmatrix}\to\begin{pmatrix} 0 \\ D_2^{n-1} \end{pmatrix}\to\cdots\to\begin{pmatrix} 0 \\ D_2^0 \end{pmatrix}\to\begin{pmatrix} 0 \\ M_2 \end{pmatrix}\to 0$$

with $\begin{pmatrix} 0 \\ D_2^i \end{pmatrix}$ D_C-projective by Theorem 3.5. Furthermore, by Theorem 3.5, $\begin{pmatrix} 0 \\ K_2^n \end{pmatrix}$ is D_C-projective if and only if K_2^n is D_{C_1}-projective. This means that $D_C\text{-}pd\left(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}\right)\le n$ if and only if $D_{C_2}\text{-}pd(M_2)\le n$ by [[12], Theorem 2.4].

2) We may assume that $D_C\text{-}pd\left(\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}\right)=m<\infty$. There exists an exact sequence in *T*-Mod:

$$0\to D^m\to D^{m-1}\to\cdots\to D^0\to\begin{pmatrix} M_1 \\ U\otimes_A M_1 \end{pmatrix}\to 0$$

with $D^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}_{\varphi^{D^i}} \in D_C P(T)$. Then there is an exact sequence

$$0 \to D_1^m \to D_1^{m-1} \to \cdots \to D_1^0 \to M_1 \to 0$$

with $D_1^i \in D_{C_1} P(A)$ by Theorem 3.5. Thus $D_{C_1}\text{-}pd(M_1) \le m$.

In contrast, we demonstrate that $D_C\text{-}pd\left(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}\right) \le D_{C_1}\text{-}pd(M_1)$ when $\operatorname{Tor}_{i\ge1}^A(U, M_1) = 0$. We may assume that $D_{C_1}\text{-}pd(M_1) = m < \infty$. So there is an exact sequence

$$\boldsymbol{X}_1 : 0 \to K_1^m \to P_1^{m-1} \to \cdots \to P_1^0 \to M_1 \to 0$$

with $P_1^i \in \mathcal{P}(A)$. As a result, the complex $U \otimes \boldsymbol{X}_1$ is exact and each P_1^i is D_{C_1}-projective by [[12], Proposition 1.8], and then, K_1^m is D_{C_1}-projective by [[12], Theorem 2.4]. So there is an exact sequence

$$0 \to \begin{pmatrix} K_1^m \\ U \otimes_A K_1^m \end{pmatrix} \to \begin{pmatrix} P_1^{m-1} \\ U \otimes_A P_1^{m-1} \end{pmatrix} \to \cdots \to \begin{pmatrix} P_1^0 \\ U \otimes_A P_1^0 \end{pmatrix} \to \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \to 0.$$

We obtain that $\begin{pmatrix} K_1^m \\ U \otimes_A K_1^m \end{pmatrix}$ and all $\begin{pmatrix} P_1^i \\ U \otimes_A P_1^i \end{pmatrix}$ are D_C-projective by Theorem 3.5. Thus we get $D_C\text{-}pd\left(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}\right) \le m = D_{C_1}\text{-}pd(M_1)$.

Inspired by the strong notion of the G_{C_2}-projective global dimension of B in [10] for estimating the G_C-projective dimension of a T-module and the left global G_C-projective dimension of T, we give the strong notion of the D_{C_2}-projective global dimension of B. Set:

$SD_{C_2}\text{-}PD(B) = \sup\{D_{C_2}\text{-}pd_B(U \otimes_A D) \mid D \in D_{C_1} P(A)\}$.

Theorem 4.2. Assume that ${}_A C_1$ and ${}_B C_2$ are semidualizing. Let $C = \mathbf{p}(C_1, C_2)$ and U Ding C-compatible. If $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left T-module and $SD_{C_2}\text{-}PD(B) < \infty$, then:

$$\max\{D_{C_1}\text{-}pd(M_1), (D_{C_2}\text{-}pd(M_2) - SD_{C_2}\text{-}PD(B))\} \le D_C\text{-}pd(M)$$
$$\le \max\{(D_{C_1}\text{-}pd(M_1)) + (SD_{C_2}\text{-}PD(B)) + 1, D_{C_2}\text{-}pd(M_2)\}.$$

Proof. Let $k := SD_{C_2}\text{-}PD(B)$. Firstly, we prove that

$$\max\{D_{C_1}\text{-}pd(M_1), (D_{C_2}\text{-}pd(M_2) - k)\} \le D_C\text{-}pd(M).$$

We may assume that $n := D_C\text{-}pd(M) < \infty$. Then there is an exact sequence

$$0 \to D^n \to \cdots \to D^1 \to D^0 \to M \to 0$$

with $D^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}_{\varphi^{D^i}} \in D_C P(T)$. Thus we achieve an exact sequence.

$$0 \to D_1^n \to D_1^{n-1} \to \cdots \to D_1^0 \to M_1 \to 0$$

with $D_1^i \in D_{C_1}P(A)$ by Theorem 3.5. Thus, $D_{C_1}\text{-}pd(M_1) \le n$.

Furthermore, according to Theorem 3.5, there is an exact sequence in B-Mod for each i

$$0 \to U \otimes_A D_1^i \to D_2^i \to \overline{D_2^i} \to 0$$

with $\overline{D_2^i} \in D_{C_2}P(B)$. Then $D_{C_2}\text{-}pd(D_2^i) = D_{C_2}\text{-}pd(U \otimes_A D_1^i) \le k$ by [[14], Theorem 3.2]. There exists an exact sequence in B-Mod:

$$0 \to D_2^n \to D_2^{n-1} \to \cdots \to D_2^0 \to M_2 \to 0.$$

By [[14], Theorem 3.2], $D_{C_2}\text{-}pd(M_2) \le n+k$.

Next, we prove that $D_C\text{-}pd(M) \le \max\{(D_{C_1}\text{-}pd(M_1))+k+1, D_{C_2}\text{-}pd(M_2)\}$. We may assume that: $m := \max\{(D_{C_1}\text{-}pd(M_1))+k+1, D_{C_2}\text{-}pd(M_2)\} < \infty$, $n_1 := D_{C_1}\text{-}pd(M_1) < \infty$ and $n_2 := D_{C_2}\text{-}pd(M_2) < \infty$. Since $D_{C_1}\text{-}pd(M_1) = n_1 \le m-k-1$, we have an exact sequence in A-Mod:

$$0 \to D_1^{m-k-1} \to \cdots \to D_1^{n_2-k} \to \cdots \xrightarrow{f_1^1} D_1^0 \xrightarrow{f_1^0} M_1 \to 0$$

with $D_1^i \in D_{C_1}P(A)$. There exists an epimorphism $D_2^0 \xrightarrow{g_2^0} M_2 \to 0$ with $D_2^0 \in D_{C_2}P(B)$ by [[12], Proposition 1.8]. Let $K_1^i = \ker f_1^i$ and define the map $f_2^0 : U \otimes_A D_1^0 \oplus D_2^0$ to be $(\varphi^M(1_u \otimes f_1^0)) \oplus g_2^0$. Then we get an exact sequence

$$0 \to \begin{pmatrix} K_1^1 \\ K_2^1 \end{pmatrix}_{\varphi^{K^1}} \to \begin{pmatrix} D_1^0 \\ (U \otimes_A D_1^0) \oplus D_2^0 \end{pmatrix} \xrightarrow{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}} M \to 0.$$

In a similar way, there exists an exact sequence of B-modules $D_2^1 \xrightarrow{g_2^1} K_2^1 \to 0$ with $D_2^1 \in D_{C_2}P(B)$. So we obtain an exact sequence

$$0 \to \begin{pmatrix} K_1^2 \\ K_2^2 \end{pmatrix}_{\varphi^{K^2}} \to \begin{pmatrix} D_1^1 \\ (U \otimes_A D_1^1) \oplus D_2^1 \end{pmatrix} \to \begin{pmatrix} K_1^1 \\ K_2^1 \end{pmatrix}_{\varphi^{K^1}} \to 0.$$

Repeating this process, we obtain an exact sequence

$$0 \to \begin{pmatrix} 0 \\ K_2^{m-k} \end{pmatrix} \to \begin{pmatrix} D_1^{m-k-1} \\ (U \otimes_A D_1^{m-k-1}) \oplus D_2^{m-k-1} \end{pmatrix} \xrightarrow{\begin{pmatrix} f_1^{m-k-1} \\ f_2^{m-k-1} \end{pmatrix}} \cdots$$

$$\to \begin{pmatrix} D_1^1 \\ (U \otimes_A D_1^1) \oplus D_2^1 \end{pmatrix} \xrightarrow{\begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix}} \begin{pmatrix} D_1^0 \\ (U \otimes_A D_1^0) \oplus D_2^0 \end{pmatrix} \xrightarrow{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \to 0.$$

Note that $D_{C_2}\text{-}pd((U \otimes_A D_1^i) \oplus D_2^i) = D_{C_2}\text{-}pd(U \otimes_A D_1^i) \le k$, $i \in \{0, \cdots, m-k-1\}$. By [[14], Theorem 3.2], the exact sequence $0 \to K_2^{m-k} \to (U \otimes_A D_1^{m-k-1}) \oplus D_2^{m-k-1} \to \cdots \to (U \otimes_A D_1^1) \oplus D_2^1 \to (U \otimes_A D_1^0) \oplus G_2^0 \to M_2 \to 0$ gives that $D_{C_2}\text{-}pd(K_2^{m-k}) \le \max\{k, n_2 - m + k\} = k$. As a result, we have an exact sequence in B-Mod

$$0 \to D_2^m \to \cdots \to D_2^{m-k+1} \to D_2^{m-k} \to K_2^{m-k} \to 0,$$

which induces an exact sequence in T-Mod:

$$0 \to \begin{pmatrix} 0 \\ D_2^m \end{pmatrix} \to \cdots \to \begin{pmatrix} 0 \\ D_2^{m-k+1} \end{pmatrix} \to \begin{pmatrix} 0 \\ D_2^{m-k} \end{pmatrix}$$
$$\to \begin{pmatrix} D_1^{m-k-1} \\ \left(U \otimes_A D_1^{m-k-1}\right) \oplus D_2^{m-k-1} \end{pmatrix} \xrightarrow{\begin{pmatrix} f_1^{m-k-1} \\ f_2^{m-k-1} \end{pmatrix}} \cdots$$
$$\to \begin{pmatrix} D_1^1 \\ \left(U \otimes_A D_1^1\right) \oplus D_2^1 \end{pmatrix} \xrightarrow{\begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix}} \begin{pmatrix} D_1^0 \\ \left(U \otimes_A D_1^0\right) \oplus D_2^0 \end{pmatrix} \xrightarrow{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \to 0.$$

Since all $\begin{pmatrix} D_1^i \\ \left(U \otimes_A D_1^0\right) \oplus D_2^i \end{pmatrix}$ and $\begin{pmatrix} 0 \\ D_2^j \end{pmatrix}$ are D_C-projective by Theorem 3.5, $D_C\text{-}pd(M) \le m$.□

Corollary 4.3. Assume that ${}_A C_1$ and ${}_B C_2$ are semidualizing. Let $C = \mathbf{p}(C_1, C_2)$ and U Ding C-compatible. If $SD_{C_2}\text{-}PD(B) < \infty$, then $D_C\text{-}pd(M) < \infty$ if and only if $D_{C_1}\text{-}pd(M_1) < \infty$ and $D_{C_2}\text{-}pd(M_2) < \infty$.

Theorem 4.4. Assume that ${}_A C_1$ and ${}_B C_2$ are semidualizing. Let $C = \mathbf{p}(C_1, C_2)$ and U Ding C-compatible. Then

$$\max\left\{D_{C_1}\text{-}PD(A), D_{C_2}\text{-}PD(B)\right\} \le D_C\text{-}PD(T)$$
$$\le \max\left\{D_{C_1}\text{-}PD(A) + SD_{C_2}\text{-}PD(B) + 1, D_{C_2}\text{-}PD(B)\right\}.$$

Proof. Firstly, we show that the left side of the inequality. Assume that $n := D_C\text{-}PD(T) < \infty$. Let $M_1 \in A\text{-Mod}$ and $M_2 \in B\text{-Mod}$. Because $D_C\text{-}pd\left(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}\right) \le n$ and $D_C\text{-}pd\left(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}\right) \le n$, $D_{C_1}\text{-}pd(M_1) \le n$ and $D_{C_2}\text{-}pd(M_2) \le n$ by Lemma 4.1. Consequently, $D_{C_1}\text{-}PD(A) \le n$ and $D_{C_2}\text{-}PD(B) \le n$.

Secondly, we show that the right side of the inequality. Assume that:

$$m := \max\left\{D_{C_1}\text{-}PD(A) + SD_{C_2}\text{-}PD(B) + 1, D_{C_2}\text{-}PD(B)\right\} < \infty.$$

Then $D_{C_1}\text{-}PD(A) < \infty$ and $SD_{C_2}\text{-}PD(B) \le D_{C_2}\text{-}PD(B) < \infty$. Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left T-module. According to Theorem 4.2,

$$D_C\text{-}pd(M) \le \max\left\{D_{C_1}\text{-}PD(A) + SD_{C_2}\text{-}PD(B) + 1, D_{C_2}\text{-}PD(B)\right\}.$$

Corollary 4.5. Assume that ${}_A C_1$ and ${}_B C_2$ are semidualizing. Let $C = \mathbf{p}(C_1, C_2)$ and U Ding C-compatible. Then $D_C\text{-}PD(T) < \infty$ if and only if $D_{C_1}\text{-}PD(A) < \infty$ and $D_{C_2}\text{-}PD(B) < \infty$.

Corollary 4.6. Assume that ${}_R C_1$ is semidualizing. Let $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ and $C = \mathbf{p}(C_1, C_2)$. Then $D_C\text{-}PD(T(R)) = D_{C_1}\text{-}PD(R) + 1$.

Proof. We know that R is Ding C-compatible and $SD_{C_1}\text{-}PD(R) = 0$. Therefore by Theorem 4.4,

$$D_{C_1}\text{-}PD(R) \le D_C\text{-}PD(T(R)) \le D_{C_1}\text{-}PD(R)+1.$$

It is obvious in the case $D_{C_1}\text{-}PD(R)=\infty$. We may assume that $n := D_{C_1}\text{-}PD(R)<\infty$. Then there exists a left R-module M with $D_{C_1}\text{-}pd(M)=n$ and $\mathrm{Ext}_R^n(M,X)\ne 0$ for some $X\in\mathcal{F}_{C_1}(R)$ by [[12], Theorem 2.4]. Now we consider an exact sequence in $T(R)$-Mod:

$$0\to\begin{pmatrix}0\\M\end{pmatrix}\to\begin{pmatrix}M\\M\end{pmatrix}_{1_M}\to\begin{pmatrix}M\\0\end{pmatrix}\to 0.$$

By applying the long exact sequence theorem to the preceding exact sequence, we obtain that

$$\cdots\to\mathrm{Ext}_{T(R)}^n\left(\begin{pmatrix}M\\M\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}\right)\to\mathrm{Ext}_{T(R)}^n\left(\begin{pmatrix}0\\M\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}\right)$$
$$\to\mathrm{Ext}_{T(R)}^{n+1}\left(\begin{pmatrix}M\\0\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}\right)\to\mathrm{Ext}_{T(R)}^{n+1}\left(\begin{pmatrix}M\\M\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}\right)\to\cdots.$$

By [[10], Lemma 3], we know that $\mathrm{Ext}_{T(R)}^{i\ge 1}\left(\begin{pmatrix}M\\M\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}\right)\cong\mathrm{Ext}_R^{i\ge 1}(M,0)=0$.

Thus by [[10], Lemma 3] and the above exact sequence,

$$\mathrm{Ext}_{T(R)}^n\left(\begin{pmatrix}0\\M\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}\right)\cong\mathrm{Ext}_{T(R)}^{n+1}\left(\begin{pmatrix}M\\0\end{pmatrix},\begin{pmatrix}0\\X\end{pmatrix}\right)\cong\mathrm{Ext}_R^n(M,X)\ne 0.$$

As $\begin{pmatrix}0\\X\end{pmatrix}\in\mathcal{F}_C(T(R))$ by Corollary 2.8, we have $D_C\text{-}pd\left(\begin{pmatrix}M\\0\end{pmatrix}\right)>n$ by [[12], Theorem 2.4]. Besides, $D_C\text{-}pd\left(\begin{pmatrix}M\\0\end{pmatrix}\right)\le D_C\text{-}PD(T(R))\le n+1$. Thus $D_C\text{-}pd\left(\begin{pmatrix}M\\0\end{pmatrix}\right)=n+1$, which implies that $D_C\text{-}PD(T(R))=n+1$.□

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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