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Uniform Hölder Bounds for Competition Systems with Strong Interaction on a Subdomain

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Abstract

We prove the uniform Hölder bounds of solutions to a singularly perturbed elliptic system arising in competing models in population dynamics. In this system, two species compete to some extent throughout the whole domain but compete strongly on a subdomain. The proof relies upon the blow up technique and the monotonicity formula by Alt, Caffarelli and Friedman.

Keywords

Singularly Perturbation, Subdomain, Free Boundary Problem, Liouville Type Theorem

1. Introduction

A central problem in population ecology is the understanding of spatial behavior of interacting species, in particular in the case when the interactions are large and of competitive type. Spatial segregation may occur when two or more species interact in a highly competitive way. Such phenomenon has been studied using competition models (or its parabolic case) with positive parameter $k \to +\infty$:

$$-\Delta u_i = f_i(u_i) - k u_i \sum_{j \neq i} b_{ij} u_j \quad \text{in } \Omega. \tag{1.1}$$

Here Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$, u_i denotes the density of the i-th population, whose internal dynamics is prescribed by $f_i(u_i)$, $i = 1, 2, \cdots, M$, and $M \geq 2$ is the number of the species. The positive constant kb_{ij} is the interspecific competition rate between the population u_i and u_j, which is possibly symmetric.

In the study of system (1.1), we are mostly concerned with the asymptotic behavior of the solutions as the parameter $k \to +\infty$. It turns out that uniformly bounded solutions $u_k = (u_{1,k}, u_{2,k}, \cdots, u_{M,k})$ of system (1.1) converge, as $k \to \infty$, to a limiting configuration in some weak sense, $u = (u_1, u_2, \cdots, u_M)$, the limit satisfies $u_i u_j = 0$ for $i \neq j$, which is called the spatial segregation (cf. [1]). Segregation systems arise in different applicative contests, from biological models for competing species to the phase segregation phenomenon in Bose-Einstein condensation of the form:

$$-\Delta u_i = f_i(u_i) - k u_i \sum_{j \neq i} b_{ij} u_j^2 \quad \text{in } \Omega. \tag{1.2}$$

In recent years, people show a lot of interests in segregation phenomenon, and abroad literature is present: starting from [2]-[9], in a series of recent papers [10]-[22], also in the fractional diffusion case [23] [24] [25] [26]. Among the others, the following results are known: the uniform Hölder bounds [7] [12] [15] [24]) and the optimal Lipschitz bound [16], the Lipschitz regularity of the limiting profiles and the regularity of the free boundaries, which is defined as the nodal set $\Gamma(u) = \{u = 0\}$ of the singular limit. It is proved that the free boundary consists of two parts: a regular set, which is $C^{1,\alpha}$ locally smooth hypersurface, and a singular set of Hausdorff dimension less then $n-2$, see [11] [17], for the nondivergence system, [10] [14] [17] for the variational one. Further information about the structure of the singular set has been provided in [27].

Among the models proposed so far, the species compete strongly on the whole of Ω. However, in some heterogeneous environment, species may compete to some extent in the whole of a region Ω, but compete strongly on a subdomain *A*. To analysis the corresponding spatial segregation phenomenon governed by strong competition on *A*, Crooks and Dancer [28] proposed the following *k*-dependent system:

$$\begin{cases} -\Delta u = f(u) - suv - k\chi_A uv, & \text{in } \Omega, \\ -\Delta v = g(v) - ruv - k\chi_A uv, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where k is again a positive competition parameter, u and v denote the densities of two species, the self-interaction functions f and g are assumed to be continuously differentiable and such that $f(0) = g(0) = 0$ and $f(y) < 0, g(y) < 0$ for large y. A is a nonempty open subset of Ω with smooth boundary such that $\overline{A} \subset \Omega$. The parameters r and s are assumed to nonnegative, and χ_A is the characteristic function on A.

Due to the presence of the characteristic function χ_A in (1.3), we cannot expect classical solutions in general. By a *k*-dependent solution of (1.3), we will mean a pair of functions (u_k, v_k) such that $u_k, v_k \in W^{2,p}(\Omega)$, $p > n$, and satisfy (1.3) almost everywhere. The asymptotic behavior of solutions to system (1.3) has been investigated in [28], where it is proved uniform convergence of (u_k, v_k) to a limiting profile (u, v), u and v segregate on $\overline{A}$ but not necessarily on $\Omega \backslash A$. The limit problem is a system on $\Omega \backslash A$ and a scalar equation on A. The objective of this paper is to improve the convergence result of [28], we shall establish the uniform Hölder bounds for solutions to system (1.3). To begin with, we define

$$\alpha^* = \begin{cases} 2/3, & \text{when } n = 2, \\ 1/2, & \text{when } n \geq 3. \end{cases} \tag{1.4}$$

Due to the apparent of subdomain, we can not expect boundedness for every Hölder exponent. In fact we have the following.

Theorem 1.1. *Let* (u_k, v_k) *be nonnegative solutions of (1.3), and* α^* *be defined in (4). Assume that for every k, there exists* $M > 0$*, independent of k, such that*

$$\|(u_k, v_k)\|_{L^\infty(\Omega)} \leq M.$$

Then for every $\alpha \in (0, \alpha^*)$, there exists $C > 0$, independent of *k*, such that

$$\|(u_k, v_k)\|_{C^{0,\alpha}(\bar{A})} \leq C.$$

Notations Throughout the paper, we denote by $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ the open ball with center x_0 and radius $R > 0$. If $x_0 = 0$, we simply denote by $B_R := B_R(0)$. We assume that any $x \in \mathbb{R}^n$ can be written as $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. In this way, we denote by $\mathbb{R}^n_+ := \mathbb{R}^n \cap \{x_n > 0\}$. For any $D \subset \mathbb{R}^n$ we write $D^+ := D \cap \{x_n > 0\}$ and $\partial^+ D := \partial D \cap \{x_n > 0\}$. We also denote by $\nabla_\theta u$ and $\partial_\nu u$ the tangential derivative and the radial derivative of *u*, respectively.

The proof of Theorem 1.1 mainly follows the blow up method, developed by Terracini and her coauthors in [7] [15]. This method is a blow up analysis and need us to establish some Liouville type results, which can be achieved by some monotonicity formulas of Alt-Caffarelli-Friedman type. Compared with [7] [15], the segregation occurs only in the subdomain $\bar{A}$, and we lack the essential information both of the location of *A* and the boundary conditions on ∂A. In the blow up procedure, the entire solutions may segregate only on the half space. Thus the Liouville type theorems established in [7] are no longer valid in the current situation. To attack this problem, new ACF type monotonicity formulas and corresponding Liouville type theorems are needed.

The rest of this paper is organized as follows: in Section 2, we establish a monotonicity formula of ACF type, and by utilizing this monotonicity formula, we prove a Liouville type theorem for entire solutions to a semilinear system. In Section 3, we perform the blow up procedure and complete the proof of Theorem 1.1.

2. Liouville-Type Results

In this section, we prove some nonexistence result in $\mathbb{R}^n$. The main tools will be the monotonicity formula by Alt, Caffarelli, Friedman originally stated in [29], as well as some generalizations made by Conti, Terracini, Verzini [7], Dancer, Wang, Zhang [12], and Terracini, Verzini, Zilio [23] [24]. The validity of ACF type formula depends on optimal partition problems involving spectral properties of the domain. In the current situation, the spectral problem we consider involves a pair of functions defined on $\partial B_1(0)$ with disjoint support on $\partial^+ B_1(0)$. In this way we are lead to consider the following optimal partition problem. Let E be an open subset of $\partial B_1(0)$, and we define the first eigenvalue associated to E as

$$\lambda_1(E) := \inf_{u \in H_0^1(E)} \frac{\int_E |\nabla_\theta u|^2}{\int_E u^2}.$$

Here $\nabla_\theta u$ stands for the tangential gradient of u on E.

Lemma 2.1. *Let* α^* *be as in (1.4). We define the nondecreasing function* $\gamma(x)$ *as*

$$\gamma(x) = \sqrt{\left(\frac{n-2}{2}\right)^2 + x} - \frac{n-2}{2}, \ \forall x \in \mathbb{R}^+,$$

and the admissible set $\mathcal{A}$ by

$$\mathcal{A} := \left\{(E_1, E_2) : E_i \subset \partial B_1(0), E_1^+ \cap E_2^+ = \varnothing\right\}.$$

Then we have

$$\inf_{(E_1,E_2)\in\mathcal{A}} \sum_{i=1}^{2} \gamma(\lambda_1(E_i)) \geq 2\alpha^*.$$

Proof If $n = 2$, it is obviously that

$$\gamma(\lambda(E_1)) + \gamma(\lambda(E_2)) = \sqrt{\lambda(E_1)} + \sqrt{\lambda(E_2)}.$$

A symmetrization argument gives that the optimal domain is a connected arc. Moreover, the longer the arc is, the smaller the first eigenvalue is. Thus the sum $\sum_{i=1}^{2} \gamma(\lambda_1(E_i))$ takes its minimum for two arcs E_1, E_2 with

$$E_1^+ \cup E_2^+ = \partial^+ B_1(0) \text{ and } \partial B_1(0) \setminus \partial^+ B_1(0) \subset E_i, i = 1, 2.$$

If we assume that the length of E_1 is $(1+\tau)\pi$, then the length of E_2 is $(2-\tau)\pi$, and the corresponding eigenfunctions are

$$\sin\frac{\theta}{1+\tau} \text{ and } \sin\frac{\theta}{2-\tau}, 0 < \tau < 1.$$

Thus we have

$$\gamma(\lambda(E_1)) + \gamma(\lambda(E_2)) \geq \frac{1}{1+\tau} + \frac{1}{2-\tau} \geq \frac{4}{3},$$

and the equality holds if and only if $\tau = 1/2$.

If $n > 2$, according to the argument in [30], we have

$$\gamma(\lambda(E)) \geq \psi(s),$$

where $E \subset \partial B_1(0)$, $s = meas(E)/meas(\partial B_1(0))$, and $\psi(s)$ is convex and decreasing:

$$\psi(s) = \begin{cases} \frac{1}{2}\log\frac{1}{4s} + \frac{3}{2}, & s < \frac{1}{4}, \\ 2(1-s), & \frac{1}{4} < s < 1. \end{cases}$$

Setting $s_i = meas(E_i)/meas(\partial B_1(0))$, $i=1,2$, we then have

$$\gamma(\lambda(E_1))+\gamma(\lambda(E_2)) \ge \psi(s_1)+\psi(s_2) \ge 2\psi((s_1+s_2)/2) \ge 2\psi(3/4)=1.$$

This completes the proof of Lemma 2.1. □

In the following, we shall prove an ACF type monotonicity formula associated with the following system

$$\begin{cases} -\Delta u = -\chi_T uv, & \text{in } \mathbb{R}^n, \\ -\Delta v = -\chi_T uv, & \text{in } \mathbb{R}^n. \end{cases} \tag{2.1}$$

where $T = \mathbb{R}^n_+$, χ_T is the characteristic function on *T*. As in [15], we introduced an auxiliary function:

$$f(r) = \begin{cases} \frac{2-n}{2} r^2 + \frac{n}{2}, & r \le 1, \\ \frac{1}{r^{n-2}}, & r > 1, \end{cases}$$

and denote $m(|x|) = -\frac{\Delta f(|x|)}{2}$. In this setting, we note that $m(|x|)$ is bounded in $\mathbb{R}^n$, vanishes in $\mathbb{R}^n \setminus B_1(0)$ and $m(|x|) \ge 0$ for a.e. *x*.

Under the previous notations, we can prove the following monotonicity formula.

Theorem 2.2. *Let* $u, v \in H^1_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ *be positive solutions of (2.1) and let* $\varepsilon > 0$ *be fixed. Then there exists* $\overline{r} > 1$ *such that the function*

$$J(r) \triangleq \frac{1}{r^{4\alpha^*-\varepsilon}} \int_{B_r(0)} \left(f(|x|)\left(|\nabla u|^2 + \chi_T u^2 v\right) + m(|x|) u^2 \right) \times \int_{B_r(0)} \left(f(|x|)\left(|\nabla v|^2 + \chi_T u v^2\right) + m(|x|) v^2 \right).$$

is increasing for $r \in (\overline{r}, +\infty)$.

Proof The proof is inspired by [15]. In order to simplify notations we shall denote

$$J_1(r) \triangleq \int_{B_r(0)} \left(f(|x|)\left(|\nabla u|^2 + \chi_T u^2 v\right) + m(|x|) u^2 \right),$$

$$J_2(r) \triangleq \int_{B_r(0)} \left(f(|x|)\left(|\nabla v|^2 + \chi_T u v^2\right) + m(|x|) v^2 \right).$$

Then $J(r) = r^{\varepsilon - 4\alpha^*} J_1(r) J_2(r)$. Let us first evaluate the derivative of $J(r)$ for $r > 1$. A straightforward calculation leads to

$$\frac{J'(r)}{J(r)} = -\frac{4\alpha^* - \varepsilon}{r} + \frac{\int_{\partial B_r(0)} \left(f(|x|)\left(|\nabla u|^2 + \chi_T u^2 v\right) + m(|x|) u^2 \right)}{J_1(r)} + \frac{\int_{\partial B_r(0)} \left(f(|x|)\left(|\nabla v|^2 + \chi_T u v^2\right) + m(|x|) v^2 \right)}{J_2(r)}. \tag{2.2}$$

By testing the equation for *u* in (2.1) with $f(|x|)u$ on $B_r(0)$, we obtain

$$
\begin{aligned}
&\int_{B_r(0)} f(|x|)\left(|\nabla u|^2+\chi_T u^2 v\right) \\
&=-\int_{B_r(0)} \nabla\left(\frac{u^2}{2}\right)\cdot\nabla f(|x|)+\int_{\partial B_r(0)} f(|x|)u\partial_\nu u \\
&=-\int_{B_r(0)} m(|x|)u^2+\int_{\partial B_r(0)}\left(f(|x|)u\partial_\nu u-\frac{u^2}{2}\partial_\nu f(|x|)\right).
\end{aligned}
$$

Thus we can rewrite the term $J_1(r)$ in a different way

$$
J_1(r)=\int_{\partial B_r(0)}\left(f(|x|)u\partial_\nu u-\frac{u^2}{2}\partial_\nu f(|x|)\right)=\frac{1}{r^{n-2}}\int_{\partial B_r(0)} u\partial_\nu u+\frac{n-2}{r^{n-1}}\int_{\partial B_r(0)}\frac{u^2}{2}. \quad (2.3)
$$

Now we define

$$
\Lambda_1(r)=\frac{r^2\int_{\partial B_r(0)}\left(|\nabla_\theta u|^2+\chi_T u^2 v\right)}{\int_{\partial B_r(0)} u^2},\ \Lambda_2(r)=\frac{r^2\int_{\partial B_r(0)}\left(|\nabla_\theta v|^2+\chi_T u v^2\right)}{\int_{\partial B_r(0)} v^2},
$$

where $|\nabla_\theta u|^2=|\nabla u|^2-|\partial_\nu u|^2$. Then for every $\delta\in\mathbb{R}$, by Hölder inequality and Young's inequality, there holds

$$
\begin{aligned}
\left|\int_{\partial B_r(0)} u\partial_\nu u\right| &\le\left(\int_{\partial B_r(0)} u^2\right)^{\frac12}\left(\int_{\partial B_r(0)}(\partial_\nu u)^2\right)^{\frac12} \\
&\le\frac{\sqrt{\Lambda_1(r)}}{2\delta^2 r}\int_{\partial B_r(0)} u^2+\frac{\delta^2 r}{2\sqrt{\Lambda_1(r)}}\int_{\partial B_r(0)}(\partial_\nu u)^2 \\
&\le\frac12\left(\frac{1}{\delta^2}\int_{\partial B_r(0)}\left(|\nabla_\theta u|^2+\chi_T u^2 v\right)+\delta^2\int_{\partial B_r(0)}(\partial_\nu u)^2\frac{r}{\sqrt{\Lambda_1(r)}}\right).
\end{aligned}
$$

Substituting in (2.3) we obtain

$$
J_1(r)\le\frac{1}{2r^{n-3}}\left[\left(\frac{1}{\delta^2\sqrt{\Lambda_1(r)}}+\frac{n-2}{\sqrt{\Lambda_1(r)}}\right)\int_{\partial B_r(0)}\left(|\nabla_\theta u|^2+\chi_T u^2 v\right)+\frac{\delta^2}{\sqrt{\Lambda_1(r)}}\int_{\partial B_r(0)}(\partial_\nu u)^2\right].
$$

Now we choose δ in such a way that

$$
\frac{1}{\delta^2\sqrt{\Lambda_1(r)}}+\frac{n-2}{\sqrt{\Lambda_1(r)}}=\frac{\delta^2}{\sqrt{\Lambda_1(r)}}.
$$

After some calculation, we obtain

$$
\frac{\sqrt{\Lambda_1(r)}}{\delta^2}=\gamma\left(\Lambda_1(r)\right),
$$

where $\gamma:\mathbb{R}^+\to\mathbb{R}$ is defined as

$$
\gamma(x)=\sqrt{\left(\frac{n-2}{2}\right)^2+x}-\frac{n-2}{2}.
$$

With this choice of δ we have

$$
J_1(r)\le\frac{1}{2\gamma\left(\sqrt{\Lambda_1(r)}\right)}\int_{\partial B_r(0)}\left(f(|x|)\left(|\nabla u|^2+\chi_T u^2 v\right)\right).
$$

Similarly, we also have

$$J_2(r) \le \frac{1}{2\gamma\left(\sqrt{\Lambda_2(r)}\right)} \int_{\partial B_r(0)} \left(f(|x|) \left(|\nabla v|^2 + \chi_T u v^2 \right) \right).$$

Substituting in (2.2) we obtain

$$\frac{J'(r)}{J(r)} \ge -\frac{4\alpha^* - \varepsilon}{r} + \frac{2\gamma\left(\sqrt{\Lambda_1(r)}\right)}{r} + \frac{2\gamma\left(\sqrt{\Lambda_2(r)}\right)}{r}.$$

Therefore it only remains to prove that there exists a $\bar{r} > 1$ such that for every $r \ge \bar{r}$ there holds

$$\gamma\left(\sqrt{\Lambda_1(r)}\right) + \gamma\left(\sqrt{\Lambda_2(r)}\right) > \frac{4\alpha^* - \varepsilon}{2}. \tag{2.4}$$

To this aim we define the functions $u_{(r)}(\theta), v_{(r)}(\theta) : \partial B_1(0) \to \mathbb{R}$ as

$$u_{(r)}(\theta) \triangleq u(r\theta), v_{(r)}(\theta) \triangleq v(r\theta).$$

Then a change of variables gives

$$\Lambda_1(r) = \frac{\int_{\partial B_1(0)} \left(\left|\nabla u_{(r)}\right|^2 + r^2 \chi_T u_{(r)}^2 v_{(r)} \right)}{\int_{\partial B_1(0)} u_{(r)}^2}, \Lambda_2(r) = \frac{\int_{\partial B_1(0)} \left(\left|\nabla v_{(r)}\right|^2 + r^2 \chi_T u_{(r)} v_{(r)}^2 \right)}{\int_{\partial B_1(0)} v_{(r)}^2}.$$

Notice first of all that there exists a constant $C > 0$ such that $\int_{\partial B_1(0)} u_{(r)}^2 \ge C$ for r sufficiently large. Indeed assume by contradiction this is not true, then $\lim_{r\to\infty} \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} u = 0$, which implies $u(0) = 0$ since u is subharmonic, and this contradicts the assumption $u > 0$. The same result clear holds also for $v_{(r)}$.

Assume (2.4) does not hold, then there exists $r_n \to \infty$ such that

$$\gamma\left(\sqrt{\Lambda_1(r_n)}\right) + \gamma\left(\sqrt{\Lambda_2(r_n)}\right) \le \frac{4\alpha^* - \varepsilon}{2} < 2\alpha^*, \tag{2.5}$$

in particular, $\Lambda_1(r_n), \Lambda_2(r_n)$ are bounded. We define

$$\tilde{u}_{(r_n)} \triangleq \frac{u_{(r)}}{\left\|u_{(r)}\right\|_{L^2(\partial B_1(0))}}, \tilde{v}_{(r_n)} \triangleq \frac{v_{(r)}}{\left\|v_{(r)}\right\|_{L^2(\partial B_1(0))}}.$$

Then there exists a constant $C > 0$ (independent of r_n) such that

$$C \ge \Lambda_1(r_n) \ge \int_{\partial B_1(0)} \left|\nabla \tilde{u}_{(r_n)}\right|^2, C \ge \Lambda_2(r_n) \ge \int_{\partial B_1(0)} \left|\nabla \tilde{v}_{(r_n)}\right|^2,$$

which ensure the existence of $\bar{u}, \bar{v} \ne 0$ such that up to a subsequence, we have $\tilde{u}_{(r_n)} \rightharpoonup \bar{u}, \tilde{v}_{(r_n)} \rightharpoonup \bar{v}$ in $H^1(\partial B_1(0))$. Moreover, since

$$C \ge \Lambda_1(r_n) = \int_{\partial B_1(0)} \left(\left|\nabla \tilde{u}_{(r_n)}\right|^2 + r_n^2 \chi_T \tilde{u}_{(r_n)}^2 v_{(r_n)} \right) \ge r_n^2 \int_{\partial B_1(0)} \chi_T \tilde{u}_{(r_n)}^2 \tilde{v}_{(r_n)}.$$

We infer that $\bar{u} \cdot \bar{v} \equiv 0$ on $\partial^+ B_1(0)$. Then Lemma 2.1 yields

$$\liminf_{n\to\infty} \left[\gamma\left(\sqrt{\Lambda_1(r_n)}\right) + \gamma\left(\sqrt{\Lambda_2(r_n)}\right) \right] \ge \gamma\left(\lambda\{\operatorname{supp}(\bar{u})\}\right) + \gamma\left(\lambda\{\operatorname{supp}(\bar{v})\}\right) \ge 2\alpha^*,$$

that is in contradiction with (2.5). □

As in [15], we have a suitable monotonicity formula we are ready to prove a Liouville type result for solutions to system (2.1). To begin with, we recall a Liouville type result for harmonic functions.

Lemma 2.3. ([15]) *Let u be a harmonic function in* $\mathbb{R}^n$ *such that for some* $\alpha \in (0,1)$ *there holds*

$$\sup_{x,y\in\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < \infty.$$

Then u is constant.

Theorem 2.4. *Let* $u, v \in H^1_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ *be nonnegative solutions of system (2.1). Assume that for some* $\alpha \in (0, \alpha^*)$ *there holds*

$$\sup_{x,y\in\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < \infty, \ \sup_{x,y\in\mathbb{R}^n} \frac{|v(x)-v(y)|}{|x-y|^{\alpha}} < \infty. \tag{2.6}$$

Then one of the functions is identically zero and the other is a constant.

Proof We first note that, by (2.6) and Lemma 2.3, if one of the functions is identically zero or a positive constant, then the other must be a constant or 0 respectively. Hence we may assume by contradiction that neither u nor v is constant. Then by the maximum principle u and v are positive, and Theorem 2.2 ensures the existence of a constant $C > 0$ such that

$$\int_{B_r} \left[f(|x|)\left(|\nabla u|^2 + \chi_T u^2 v\right) + mu^2 \right] \int_{B_r} \left[f(|x|)\left(|\nabla v|^2 + \chi_T uv^2\right) + mv^2 \right] \geq Cr^{4\alpha^*-\varepsilon}, \tag{2.7}$$

for r sufficiently large. Let $\eta = \eta_{r,2r} \in C_0^{\infty}(\mathbb{R}^n)$ be any smooth, radial, cut-off function with the following properties: $0 \leq \eta_{r,2r} \leq 1$, $\eta_{r,2r} = 1$ in B_r, $\eta_{r,2r} = 0$ in $\mathbb{R}^n \setminus B_{2r}$ and $|\nabla \eta_{r,2r}| \leq C/r$. Testing the equation $-\Delta u = -\chi_T uv$ with the function $\eta^2 fu$ on B_{2r}, we obtain

$$-\int_{B_{2r}} \left(2\eta fu \nabla \eta \cdot \nabla u + \eta^2 u \nabla f \cdot \nabla u + \eta^2 f |\nabla u|^2\right) = \int_{B_{2r}} \chi_T \eta^2 fu^2 v.$$

Consequently,

$$\begin{aligned}
&\int_{B_{2r}} \eta^2 f\left(|\nabla u|^2 + \chi_T u^2 v\right) = -\int_{B_{2r}} \left(2\eta fu \nabla \eta \cdot \nabla u + \eta^2 \nabla f \cdot \nabla \frac{u^2}{2}\right) \\
&\leq \int_{B_{2r}} \left(\frac{1}{2} f\eta^2 |\nabla u|^2 + 2fu^2 |\nabla \eta|^2 - \eta^2 \nabla \frac{u^2}{2} \cdot \nabla f\right) \\
&= \int_{B_{2r}} \left(\frac{1}{2} f\eta^2 |\nabla u|^2 + 2fu^2 |\nabla \eta|^2 - \nabla \left(\frac{\eta^2 u^2}{2}\right) \cdot \nabla f + u^2 \eta \nabla \eta \cdot \nabla f\right).
\end{aligned}$$

Testing the equality $\Delta f = -2m$ with the function $\eta^2 u^2/2$ on B_{2r}, we obtain

$$-\int_{B_{2r}} \nabla \left(\frac{\eta^2 u^2}{2}\right) \cdot \nabla f = \int_{B_{2r}} -2m \frac{\eta^2 u^2}{2},$$

which together with the previous inequality and the fact that $m \geq 0$, gives

$$\int_{B_{2r}} \eta^2 \left[f\left(|\nabla u|^2 + \chi_T u^2 v \right) + mu^2 \right] \le 2\int_{B_{2r}} \left(2 f u^2 |\nabla \eta|^2 + u^2 \eta \nabla \eta \cdot \nabla f \right).$$

Now, recalling the definition of η and f and using assumption (2.6), we obtain

$$\int_{B_r} \eta^2 \left[f\left(|\nabla u|^2 + \chi_T u^2 v \right) + mu^2 \right] \le C\int_{B_{2r}\setminus B_r} \frac{u^2}{|x|^n} \le C\int_0^{2r} \frac{\rho^{2\alpha}}{\rho^n} \rho^{n-1} \mathrm{d}\rho \le Cr^{2\alpha}.$$

Similarly,

$$\int_{B_r} \left[f\left(|\nabla v|^2 + \chi_T u v^2 \right) + mv^2 \right] \le Cr^{2\alpha}.$$

Thus we have

$$\int_{B_r} \left[f\left(|\nabla u|^2 + \chi_T u^2 v \right) + mu^2 \right] \int_{B_r} \left[f\left(|\nabla v|^2 + \chi_T u v^2 \right) + mv^2 \right] \le Cr^{4\alpha},$$

which contradicts with (2.7) for r large. □

Remark 2.5. *If $T = \mathbb{R}^n$, then u, v compete in the whole $\mathbb{R}^n$. In this case we have $\alpha^* = 1$, see [7] for detailed proof.*

A similar nonexistence result is true when studying 2-tuple of subharmonic functions on $\mathbb{R}^n$ having disjoint supports on T

Corollary 2.6. *Let $u, v \in H_0^1(\mathbb{R}^n) \cap C_{loc}(\mathbb{R}^n)$ such that*

$$\begin{cases} -\Delta u \le 0, & \text{in } \mathbb{R}^n, \\ -\Delta v \le 0, & \text{in } \mathbb{R}^n, \\ uv = 0, & \text{in } T, \end{cases}$$

and for some fixed $\alpha \in (0, \alpha^*)$, there exists a constant $C > 0$ such that

$$\sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \le C, \ \sup_{x,y\in\Omega} \frac{|v(x) - v(y)|}{|x - y|^\alpha} \le C,$$

then one of the functions is identically zero and the other is a constant.

3. The Uniform Hölder Bounds

In this section, we shall establish the uniform Hölder bounds for solutions to system (1.3). Note that the strong competition effect of the system only occurs in subdomain A, while in the other regions, the equation does not contain the strong competition parameter k, so the solutions u_k and v_k are uniformly bounded independently of k in $W^{2,p}(\Omega \setminus \overline{A})$, for each $p \in [1, +\infty)$, and in $C^{1,\lambda}(\Omega \setminus \overline{A})$, for each $\lambda \in (0,1)$. Therefore, up to a subsequence, u_k, v_k converge strongly in $C^{1,\lambda}(\Omega \setminus \overline{A})$. In order to improve the uniform convergence result obtained in [28], it suffices to establish the uniform C^α bounds on subdomain $\overline{A}$. We now state the main results in this section.

Theorem 3.1. *Let (u_k, v_k) be nonnegative solutions of system (1.3) uniformly bounded in $L^\infty(\Omega)$. Then for every $\alpha \in (0, \alpha^*)$ there exists $C > 0$, independent of k, such that*

$$\|(u_k, v_k)\|_{C^{0,\alpha}(\overline{A})} \le C,$$

for every $k>0$.

The proof of Theorem 3.1 is inspired from the work of [15]. We assume by contradiction that, for some $\alpha\in(0,\alpha^*)$, up to a subsequence, it holds

$$L_k=\max\left\{\sup_{x,y\in\bar{A},x\neq y}\frac{|u_k(x)-u_k(y)|}{|x-y|^\alpha},\sup_{x,y\in\bar{A},x\neq y}\frac{|v_k(x)-v_k(y)|}{|x-y|^\alpha}\right\}\to\infty.$$

We can assume that L_k is achieved, say, by u_k at the pair (u_k,v_k). That is

$$L_k=\frac{|u_k(x_k)-u_k(y_k)|}{|x_k-y_k|^\alpha}\to\infty.$$

Let us define the rescaled functions

$$\tilde{u}_k\triangleq\frac{u_k(x_k+r_kx)}{L_kr_k^\alpha},\tilde{v}_k\triangleq\frac{v_k(x_k+r_kx)}{L_kr_k^\alpha},\text{for } x\in\Omega_k=\frac{\Omega-x_k}{r_k},$$

where $r_k\to 0$ will be chosen later. By direct calculation, ($\tilde{u}_k$ and $\tilde{v}_k$) satisfy the following system

$$\begin{cases}-\Delta\tilde{u}_k=\dfrac{r_k^{2-\alpha}}{L_k}f\left(L_kr_k^\alpha\tilde{u}_k\right)-sL_kr_k^{2+\alpha}\tilde{u}_k\tilde{v}_k-kL_kr_k^{2+\alpha}\chi_{A_k}(x)\tilde{u}_k\tilde{v}_k, & \text{in } \Omega_k,\\ -\Delta\tilde{v}_k=\dfrac{r_k^{2-\alpha}}{L_k}g\left(L_kr_k^\alpha\tilde{v}_k\right)-rL_kr_k^{2+\alpha}\tilde{u}_k\tilde{v}_k-kL_kr_k^{2+\alpha}\chi_{A_k}(x)\tilde{u}_k\tilde{v}_k, & \text{in } \Omega_k,\end{cases}$$

where $A_k=\dfrac{A-x_k}{r_k}$. We note that $\Omega_k\to\mathbb{R}^n$ as $k\to+\infty$, and depending on the asymptotic behavior of the distance $d(x_k,\partial A)$, we have $A_k\to T$, where T is either $\mathbb{R}^n$ or an half-space (when $d(x_k,\partial A)\to\infty$ or the limit is finite, respectively). We also observe that

$$\max\left\{\sup_{x,y\in\bar{A}_k,x\neq y}\frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|}{|x-y|^\alpha},\sup_{x,y\in\bar{A}_k,x\neq y}\frac{|\tilde{v}_k(x)-\tilde{v}_k(y)|}{|x-y|^\alpha}\right\}$$
$$=\frac{\left|\tilde{u}_k(0)-\tilde{u}_k\left(\dfrac{y_k-x_k}{r_k}\right)\right|}{\left|\dfrac{y_k-x_k}{r_k}\right|^\alpha}=1.$$

Since u_k,v_k are uniformly bounded in $L^\infty(\Omega)\cap C^{1,\lambda}(\Omega\setminus\bar{A})$, $r_k\to 0$ and $L_k\to+\infty$, by diect calculations it is easy to see that

$$\frac{r_k^{2-\alpha}}{L_k}f\left(L_kr_k^\alpha\tilde{u}_k\right)-sL_kr_k^{2+\alpha}\tilde{u}_k\tilde{v}_k,\frac{r_k^{2-\alpha}}{L_k}g\left(L_kr_k^\alpha\tilde{v}_k\right)-rL_kr_k^{2+\alpha}\tilde{u}_k\tilde{v}_k\to 0 \text{ in } L^\infty(\Omega_k), \quad (3.1)$$

$$|\nabla\tilde{u}_k|,|\nabla\tilde{v}_k|\to 0 \text{ in } L^\infty\left(\Omega_k\setminus\bar{A}_k\right). \quad (3.2)$$

In the following, we need to make different choices of the sequence r_k. Once r_k is chosen, we will use Ascoli-Arzelà's Theorem to pass to the limit on compact sets. Now since the $\tilde{u}_k,\tilde{v}_k$'s are uniform α-Hölder continuous, it is suffices to show that $\tilde{u}_k(0)$ and $\tilde{v}_k(0)$ are bounded in k. To begin with, we need the following technical lemma, which is proved in [7].

Lemma 3.2. [7] *Let* $u \in H^1(B_{2R})$ *satisfy that*

$$\begin{cases} -\Delta u \le -Hu, & \text{in } B_{2R}, \\ u \ge 0, & \text{in } B_{2R}, \\ u \le A, & \text{on } \partial B_{2R}, \end{cases}$$

where H is a positive constant, then for every $\delta \in (0,1)$, it holds

$$\|u\|_{L^\infty(B_R)} \le CA\mathrm{e}^{-\delta R\sqrt{H}},$$

where $C > 0$ is a constant, and only dependent on δ, R.

Lemma 3.3. *Let* $r_k \to \infty$ *as* $k \to \infty$ *be such that*

(i) $d(x_k, y_k) \le R' r_k$ for some $R' > 0$.

(ii) $kL_k r_k^{2+\alpha} \nrightarrow 0$.

Then $\tilde{u}_k(0), \tilde{v}_k(0)$ are uniformly bounded in *k*.

Proof We prove the estimate for $\tilde{u}_k(0)$; that for $\tilde{v}_k(0)$ follows similarly. Assume by contradiction that $\{\tilde{u}_k(0)\}$ is unbounded. Let $R \ge R'$ and choose k sufficiently large such that $B_R(0) \subset \Omega_k$. Moreover since $[\tilde{u}_k]_{\alpha,\bar{A}_k} = 1$ and $kL_k r_k^{2+\alpha} \nrightarrow 0$, we have

$$I_k := \inf_{B_R(0)\cap A_k} kL_k r_k^{2+\alpha} \tilde{u}_k \to +\infty.$$

Claim. $\forall R \ge R'$, $\left\| kL_k r_k^{2+\alpha} \tilde{u}_k \tilde{v}_k \right\|_{L^\infty(B_R(0)\cap A_k)} \to 0$.

Indeed, note that

$$-\Delta \tilde{v}_k = \frac{r_k^{2-\alpha}}{L_k} g\left(L_k r_k^\alpha \tilde{v}_k\right) - rL_k r_k^{2+\alpha} \tilde{u}_k \tilde{v}_k - kL_k r_k^{2+\alpha} \chi_{A_k}(x) \tilde{u}_k \tilde{v}_k, \text{ in } \Omega_k,$$

and by (3.1), we have

$$-\Delta \tilde{v}_k \le -\frac{I_k}{2} \tilde{v}_k \text{ in } B_R(0) \cap A_k. \tag{3.3}$$

In order to simplify the notation, let $K \triangleq B_R(0) \cap A_k$ and for each compact set $K' \Subset K$, we choose a cut-off function $\eta \in C_0^\infty(K)$ such that $\eta \equiv 1$ on K', $\eta \equiv 0$ on $\mathbb{R}^n \setminus K$. Then by testing (3.3) with $\eta^2 \tilde{v}_k$ on *K*, we obtain

$$\frac{I_k}{2}\int_K \eta^2 \tilde{v}_k^2 + \int_K \eta^2 |\nabla \tilde{v}_k|^2 \le -2\int_K \eta \nabla \tilde{v}_k \cdot \nabla \eta \tilde{v}_k \le \int_K \eta^2 |\nabla \tilde{v}_k|^2 + \int_K |\nabla \eta|^2 \tilde{v}_k^2,$$

So, there exist two positive constants C_1, C_2 such that

$$C_2 \frac{I_k}{2} \inf_{x\in K} \tilde{v}_k^2 \le \frac{I_k}{2}\int_K \eta^2 \tilde{v}_k^2 \le \int_K |\nabla \eta|^2 \tilde{v}_k^2 \le C_1 \sup_{x\in K} \tilde{v}_k^2. \tag{3.4}$$

Since $[\tilde{v}_k]_{\alpha,K} \le 1$, we have

$$\inf_{x\in K} \tilde{v}_k \ge \sup_{x\in K} \tilde{v}_k - R^\alpha.$$

Then (3.4) implies that

$$C_2 \frac{I_k}{2}\left(\sup_{x\in K} \tilde{v}_k - R^\alpha\right)^2 \le C_1 \sup_{x\in K} \tilde{v}_k^2.$$

If we choose k sufficient large such that $\frac{C_1}{I_k} \le \frac{C_2}{8}$, then

$$\sup_{x\in K}\tilde{v}_k^2 \le \frac{2C_2(R)}{C_1(R)}R^{2\alpha},$$

which implies the boundedness of $\tilde{u}_k$ in K. Thus we can apply Lemma 3.1, which gives

$$\sup_{K'}\tilde{v}_k \le C\mathrm{e}^{-C'\sqrt{l_k}},$$

and the claim can be easily seen.

Define

$$\hat{u}(x) \triangleq \tilde{u}_k(x)-\tilde{u}_k(0).$$

We have $\hat{u}_k(0)=0$, and it is Hölder continuous, and

$$\frac{\left|\hat{u}_k(\overline{y}_k)-\hat{u}_k(0)\right|}{\left|\overline{y}_k\right|^{\alpha}}=1, \tag{3.5}$$

where $\overline{y}_k=\dfrac{y_k-x_k}{r_k}$. Moreover, by the claim

$$-\Delta\hat{u}_k(x)=\varepsilon_k,$$

where $\varepsilon_k \to 0$, as $k\to\infty$, uniformly on any $B_R(0)$.

From this equation, we can infer from L^p theory and Sobolev embeddig theory that $\hat{u}_k(x)$ is uniformly Lipschitz continuous, that is, there exists $L>0$ is a constant, independent with k such that

$$\sup_{x,y\in B_R(0)}\frac{\left|\hat{u}_k(x)-\hat{u}_k(y)\right|}{|x-y|}\le L.$$

We have that, up to a subsequence, $\overline{y}_k \to \overline{y}$, since $\overline{y}_k=\dfrac{y_k-x_k}{r_k}\le R'$. We claim that $\overline{y}\ne 0$.

In fact, if $\overline{y}=0$, then we obtain

$$1=\frac{\left|\hat{u}_k(\overline{y}_k)-\hat{u}_k(0)\right|}{\left|\overline{y}_k\right|^{\alpha}}=\frac{\left|\hat{u}_k(\overline{y}_k)-\hat{u}_k(0)\right|}{\left|\overline{y}_k\right|}\left|\overline{y}_k\right|^{1-\alpha}\le L\left|\overline{y}_k\right|^{1-\alpha}\to 0,$$

as $k\to\infty$, which is a contradiction. After passing to a subsequence, $\hat{u}_k$ converges to a continuous function $\hat{u}_\infty$ on compacts and satisfying

$$-\Delta\hat{u}_\infty(x)=0 \quad \text{in } \mathbb{R}^n.$$

Moreover, (3.5) can be passed to the limit, which is

$$\frac{\left|\hat{u}_\infty(\overline{y})-\hat{u}_\infty(0)\right|}{\left|\overline{y}\right|^{\alpha}}=1. \tag{3.6}$$

Thus we have $\hat{u}_\infty \equiv C$ by Lemma 2.3, this contradicts (3.6). So $\tilde{u}_k(0)$ is uniformly bounded. □

Lemma 3.4. *Up to a subsequence, we have* $kL_k\left|x_k-y_k\right|^{2+\alpha}\to+\infty$.

Proof We assume by contradiction that there exists $R'>0$ such that $kL_k\left|x_k-y_k\right|^{2+\alpha}\le R'$. Let

$$r_k = (kL_k)^{-\frac{1}{2+\alpha}}, N_k = L_k r_k^{2+\alpha},$$

then we obtain

$$kN_k = 1, \lim_{k\to\infty} r_k = 0, |x_k - y_k|^\alpha \le R'(kL_k)^{-1} = R' r_k^{2+\alpha} \le R' r_k,$$

while k is sufficient large, so we can use Lemma 3.3 to conclude that $\tilde{u}_k(0), \tilde{v}_k(0)$ are uniformly bounded.

On the other hand, by the uniform Hölder continuity and the Ascoli-Arzelà theorem we have that, up to a subsequence, there exist $\tilde{u}_\infty$ and $\tilde{v}_\infty$ such that $\tilde{u}_k \to \tilde{u}_\infty, \tilde{v}_k \to \tilde{v}_\infty$ uniformly on the compact set of $\mathbb{R}^n$. Moreover, the choice of r_k implies that the equations of $\tilde{u}_k$ and $\tilde{v}_k$ are

$$\begin{cases} -\Delta \tilde{u}_k = \dfrac{r_k^{2-\alpha}}{L_k} f\left(L_k r_k^\alpha \tilde{u}_k\right) - sL_k r_k^{2+\alpha} \tilde{u}_k \tilde{v}_k - \chi_{A_k}(x) \tilde{u}_k \tilde{v}_k, & \text{in } \Omega_k, \\ -\Delta \tilde{v}_k = \dfrac{r_k^{2-\alpha}}{L_k} g\left(L_k r_k^\alpha \tilde{v}_k\right) - rL_k r_k^{2+\alpha} \tilde{u}_k \tilde{v}_k - \chi_{A_k}(x) \tilde{u}_k \tilde{v}_k, & \text{in } \Omega_k. \end{cases} \tag{3.7}$$

From the first equation, we can also obtain a uniform Lipschitz estimate of $\tilde{u}_k$, and we also have

$$\frac{|\tilde{u}_\infty(\bar{y}) - \tilde{u}_\infty(0)|}{|\bar{y}|^\alpha} = 1. \tag{3.8}$$

Let $k \to \infty$ in (3.7), we can obtain, up to a subsequence,

$$\begin{cases} -\Delta \tilde{u}_\infty = -\chi_T \tilde{u}_\infty \tilde{v}_\infty, & \text{in } \mathbb{R}^n, \\ -\Delta \tilde{v}_\infty = -\chi_T \tilde{u}_\infty \tilde{v}_\infty, & \text{in } \mathbb{R}^n, \end{cases}$$

where $T = \mathbb{R}^n_+$, or $T = \mathbb{R}^n$. So we can use Theorem 2.4 and Remark 2.5 to conclude that one of $\tilde{u}_\infty$ and $\tilde{v}_\infty$ is identically zero and the other is a constant, which contradicts (3.8). □

Now we come to the proof of Theorem 3.1.

The proof of Theorem 3.1. From Lemma 3.4, we must have $kL_k |x_k - y_k|^{2+\alpha} \to +\infty$. Let $r_k = |x_k - y_k|$. With this choice, we know that all the assumptions of Lemma 3.3 are satisfied and hence $\tilde{u}_k(0)$ and $\tilde{v}_k(0)$ are uniformly bounded. Again by the uniform Hölder continuity and the Ascoli-Arzelà theorem we have that, up to a subsequence, there exist $\tilde{u}_\infty$ and $\tilde{v}_\infty$ such that $\tilde{u}_k \to \tilde{u}_\infty, \tilde{v}_k \to \tilde{v}_\infty$ uniformly on the compact set of $\mathbb{R}^n$. Note that $|\bar{y}_k| = |(y_k - x_k)/r_k| = 1$, (3.5) implies that

$$|\tilde{u}_\infty(\bar{y}) - \tilde{u}_\infty(0)| = 1 \tag{3.9}$$

Moreover $\tilde{u}_k$ and $\tilde{v}_k$ satisfy the following inequalities

$$-\Delta \tilde{u}_k \le \frac{r_k^{2-\alpha}}{L_k} f\left(L_k r_k^\alpha \tilde{u}_k\right) - sL_k r_k^{2+\alpha} \tilde{u}_k \tilde{v}_k \ \text{ in } \Omega_k, \tag{3.10}$$

$$-\Delta \tilde{v}_k \le \frac{r_k^{2-\alpha}}{L_k} g\left(L_k r_k^\alpha \tilde{v}_k\right) - rL_k r_k^{2+\alpha} \tilde{u}_k \tilde{v}_k \ \text{ in } \Omega_k. \tag{3.11}$$

Let $k \to \infty$ in (3.10) and (3.11), we obtain

$$\begin{cases} -\Delta \tilde{u}_\infty \le 0 & \text{in } \mathbb{R}^n, \\ -\Delta \tilde{v}_\infty \le 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Now let $K \subset\subset \mathbb{R}^n$ be a compact set, we can choose k sufficient large such that $K \subset\subset \Omega_k$. Let us choose a cut-off function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $0 < \eta < 1$ and $\eta \equiv 1$ on K. Multiplying (3.10) by η and integrating by parts, we obtain

$$kL_k r_k^{2+\alpha} \int_K \chi_{A_k}(x) \tilde{u}_k \tilde{v}_k \le \int_{\Omega_k} \frac{r_k^{2-\alpha}}{L_k} f\left(L_k r_k^\alpha \tilde{u}_k\right)\eta + \int_{\Omega_k} \tilde{u}_k \Delta \eta,$$

Since $\tilde{u}_k$ is uniformly Hölder continuous, it then form the boundedness of $\tilde{u}_k(0)$ that $\tilde{u}_k$ is uniformly bounded on compact set K. Therefore the right hand side of previous inequality is uniformly bounded. Because
$kL_k \left|x_k - y_k\right|^{2+\alpha} \to \infty$, we obtain

$$\lim_{k\to\infty} \int_K \chi_{A_k}(x) \tilde{u}_k \tilde{v}_k = 0,$$

which yields

$$\chi_T \tilde{u}_\infty \tilde{v}_\infty = 0 \text{ in } \mathbb{R}^n.$$

To sum up, we have

$$\begin{cases} -\Delta \tilde{u}_\infty \le 0, & \text{in } \mathbb{R}^n, \\ -\Delta \tilde{v}_\infty \le 0, & \text{in } \mathbb{R}^n, \\ \tilde{u}_\infty \tilde{v}_\infty = 0, & \text{in } T. \end{cases}$$

Thus we can infer from Corollary 2.6 that one of the limiting functions is identically zero and the other is a constant, which contradicts (3.9). The proof of Theorem 3.1 is complete. □

4. Conclusion and Further Works

The study of the asymptotic behavior of singular perturbed equations and system of elliptic or parabolic type is very broad and subject of research. In this paper, We study the large-interaction limit of solutions to a singularly perturbed elliptic system modeling the steady states of two species u and v which compete to some extent throughout a domain Ω but compete strongly on a subdomain $A \subset \Omega$. We improve the uniform convergence result of [28], proving bounds in Hölder norms whenever $A \subset \Omega$ is a smooth bounded domain.

Finally, we mention that there many interesting problems for further study. Note that we prove the uniform Hölder bounds to a singularly perturbed elliptic system, naturally to ask whether this result can be extended to the corresponding parabolic system? Up to our knowledge, the uniform Hölder bounds for parabolic setting is unknown, and both the asymptotics and the qualitative properties of the limit segregated profiles remain a challenge, this will be the object of a forthcoming paper.

Founding

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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