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Riemann Boundary Value Problems on the Curve of Parabola

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Abstract

In this paper, we study Riemann boundary value problems on the Curve of Parabola. We characterized the functions which are intergrable on the Curve of Parabola. We also study the asymptotic behaviors of Cauchy-type integral and Cauchy principal value integral on the Curve of Parabola at infinity. At the end, we discuss the Riemann boundary value problems for sectionally holomorphic functions with the Curve of Parabola as their jump curve and obtain the explicit form.

Keywords

Riemann-Hilbert Problem, Plemelj Formula, The Cauchy-Type Integral

1. Introduction

As we all know, monographs [1] [2] [3] systematically study the boundary value theory of analytic functions. The monograph [4] systematically studies the Riemann boundary value problem on the positive real axis and introduces the concept of the principal part and order of a holomorphic function on the complex plane sectioned along the positive real axis at the origin and infinity. Some scholars have also discussed the boundary value problem of analytic functions on some curves wider than smooth curves [5]-[11]. On all sides among the value problems, the Riemann boundary value problem is the basic boundary value problem, and many boundary value problems can be transformed into Riemann boundary value problems to solve [12]-[17]. References [1] [2] [3] have discussed the boundary value problem on finite curves in detail. Although the Riemann boundary value problem on infinitely long curves is very important, so far there is not much research in this area, and it is not perfect. Reference [4] discusses the Riemann boundary value problem on the positive real axis and the concepts of principal parts and orders are extended. In this paper, some Riemann boundary value problems on the Curve of Parabola ($L = x + ix^2$) are presented, and the definitions of principal parts and orders of infinity point in reference [4] are used. L that appears alone in this paper defaults to the Curve of Parabola and $\widehat{L_a L_b}$ defaults to the closed arc segment from L_a to L_b on L without special case description, and L_a defaults to the point $x_a + ix_a^2$.

2. Preliminary

We denote by $l_a = x_a + ix_a^2$ for $x_a \in \mathbb{R}$. As same as the case of the real axis, we may consider its image, **Figure 1**, by mapping $\rho(z) = \frac{1}{z+i}$, then get a new shape after the transformation, **Figure 2**.

Definition 2.1. Let f be defined on the sub-arc L' (closed or open, finite or infinite) of L. If

$$\left| f(t') - f(t'') \right| \le M \left| t' - t'' \right|^{\mu}, 0 < \mu \le 1, \tag{2.1}$$

for arbitrary points t', t'' on L', where M, μ are constants, then f is said to satisfy Hölder condition of order μ on L, denoted by $f \in H^{\mu}(L')$,where μ is called the Hölder index, If the index μ is not emphasized, it may be denoted briefly by $f \in H(L')$.

Definition 2.2. Let f be defined on the sub-arc L' (The arc formed by all points on the Curve of Parabola whose modulus length is greater than or equal to Δ, where $\Delta > 0$). If

$$\left| f(t') - f(t'') \right| \le M \left| \frac{1}{t'} - \frac{1}{t''} \right|^{\mu}, 0 < \mu \le 1, \tag{2.2}$$

for arbitrary points t', t'' on L', where M, μ are definite constants, then f is said to satisfy condition of μ order in the neighborhood of ∞, denoted by

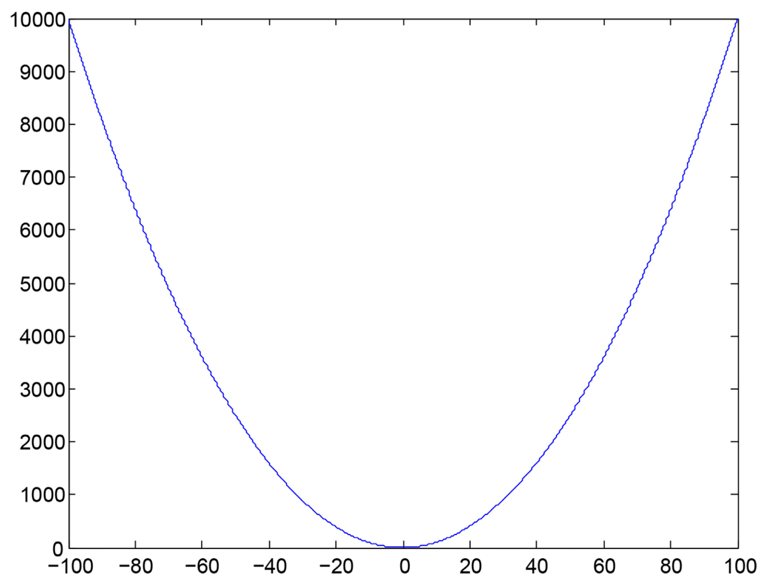

Figure 1. Σ_1.

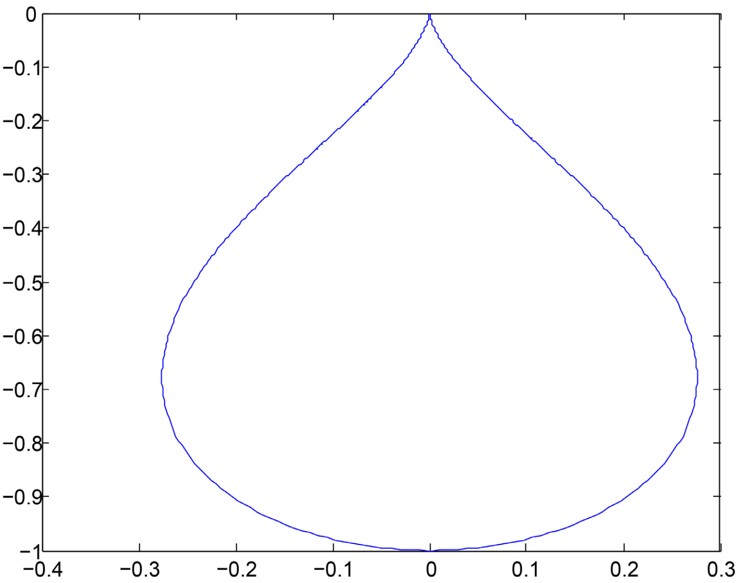

Figure 2. Σ_1.

$f \in \hat{H}^{\mu}(\infty)$, or denoted briefly by $f \in \hat{H}(\infty)$. If $f \in \hat{H}^{\mu}(\infty)$ and $f(\infty)=0$, then denoted by $f \in \hat{H}_0^{\mu}(\infty)$ or denoted briefly by $f \in \hat{H}_0(\infty)$.

Definition 2.3. For any closed segment of a parabola $L'\left(L'=\widehat{L_aL_b}\right)$, L_a and L_b are any two points on L', thus $f \in H^{\mu}(L')$, it may be denoted by $f \in H_c^{\mu}(L')$ or denoted briefly by $f \in H_c(L')$. If exists rbitrary point L_δ, such that $f \in H^{\mu}(L)$, it may be denoted by $f \in H^{\mu}(0)$ or denoted briefly by $f \in H(0)$.

Let *f* be a function defined on *L*. There exists $\Delta > 0$ such that

$$f(t)=\frac{f^*(t)}{t^{\nu}}, |t| \geq \Delta, \tag{2.3}$$

where ν is a real number and f^* is a bounded function, or equivalently,

$$f(t)=O\left(t^{-\nu}\right), t \to \infty, \tag{2.4}$$

then denoted by $f \in O^{\nu}(\infty)$.

Let $f_m(\tau)=\tau^m f(\tau)$. If $f_m \in \hat{H}^{\mu}(\infty)$ and $f_m \in \hat{H}_{\nu}^{\mu}(\infty)$, denoted by $f \in \hat{H}_m^{\mu}(\infty)$ and $f \in \hat{H}_{\nu,m}^{\mu}(\infty)$, respectively.

Sometimes, we also need to consider the following types of functions:

$$f(t)=\frac{f^*(t)}{t^{\lambda}}, \text{ where } f^* \in H^{\mu}(0), \tag{2.5}$$

and

$$\lambda=\alpha+i\beta, 0 \leq \alpha < 1. \tag{2.6}$$

3. Sectional Holomorphic Functions Jumping on the Curve of Parabola

To properly formulate the Riemann boundary value problem on the Curve of parabola, we must introduce the partitioned holomorphic function with the Curve of parabola as the jumping curve and its generalized principal part at the end ∞ of the Curve of parabola.

If F is holomorphic in the complex plane cut by the Curve of parabola, then denoted by $F \in A(\mathbb{C} \setminus L)$. To better solve the boundary value problem, we need to introduce the Cauchy-type integral on the Curve of parabola.

Definition 3.1. Let f be defined on L and locally integrable. Then

$$C(f)(z) = \frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - z}\mathrm{d}\tau,\ z \notin L \tag{3.1}$$

Denote by

$$C(f)(z) = \lim_{r\to 0}\frac{1}{2\pi i}\int_{|s-m|>r}\frac{f\left(s+is^2\right)|1+2is|}{(s-m)+i\left(s^2-m^2\right)}\mathrm{d}s,\ z = m + im^2 \notin L \tag{3.2}$$

is called the Cauchy-type integral with kernel density f along L.

Remark. Obviously, the above integral can be regarded as the sum of the following two integrals

$$C_{\widehat{L_{-\infty}L_b}}(f)(z) = \frac{1}{2\pi i}\int_{L_{-\infty}}^{L_b}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau,\ z \notin \widehat{L_{-\infty}L_b} \tag{3.3}$$

and

$$C_{\widehat{L_bL_{+\infty}}}(f)(z) = \frac{1}{2\pi i}\int_{L_b}^{L_{+\infty}}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau,\ z \notin \widehat{L_bL_{+\infty}}. \tag{3.4}$$

That is

$$C(f)(z) = C_{\widehat{L_{-\infty}L_b}}(f)(z) + C_{\widehat{L_bL_{+\infty}}}(f)(z),\ z \notin L \tag{3.5}$$

Assume $f \in O^{\nu}(\infty)(\nu > 0)$. Let $z \notin L$. The closest point z_L to z can be found on L. Take Δ large enough that $|\tau - z| \ge |\tau - z_L|$ and $|\tau - z_L| \ge \left|x^2 - Im\, z_L\right| \ge \left|x^2\right|$ and $2\sqrt{2}x^2 \ge |1+2ix| \ge 1$ for $x > \Delta$.

$$\left|\int_{L_\Delta}^{L_{+\infty}}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau\right| = \left|\int_{L_\Delta}^{L_{+\infty}}\frac{f^*(\tau)}{\tau^{\nu}(\tau - z)}\mathrm{d}\tau\right| \tag{3.6}$$

$$\le \int_{L_\Delta}^{L_{+\infty}}\frac{M\left|d\left(x+ix^2\right)\right|}{\left|x+ix^2\right|^{\nu}\left|x+ix^2-z\right|} \tag{3.7}$$

$$\le \int_{L_\Delta}^{L_{+\infty}}\frac{M|1+i2x|}{|x|^{\nu}\left|x+ix^2-z\right|}\mathrm{d}x \tag{3.8}$$

$$\le \int_{L_\Delta}^{L_{+\infty}}\frac{4M|x|}{|x|^{\nu}\left|x^2\right|}\mathrm{d}x \tag{3.9}$$

$$\le \int_{L_\Delta}^{L_{+\infty}}\frac{4M}{x^{\nu+1}}\mathrm{d}x. \tag{3.10}$$

Thus (3.1) must exist.

Remark. Since L is not the Lyapunov curve, the operator C is neither L^2 bonded nor H^{μ} bounded on L. So we cannot deal with both Cauchy-type and Cauchy principal value integral as same as the references [10] [11] [12]. We have to reprove some results in this paper.

But we still can get some elementary results by transferring it to the integral on the finite curve $\rho(L)$. It is easy to check that (2.1) holds if and only if

$$\left| f\left(\frac{1}{w_1}\right) - f\left(\frac{1}{w_2}\right) \right| \le M \left| w_1 - w_2 \right|^{\mu} \tag{3.11}$$

near $w = 0$, [10]. Thus we have, see [11], Map the plane 1 to the plane 2 with the following transformation.

$$w = \frac{1}{\tau + i} \tag{3.12}$$

$$\frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - z}\mathrm{d}\tau \overset{\tau=\frac{1}{w}-i}{=} \frac{1}{2i\pi}\int_{\rho(L)} \frac{f\left(\frac{1}{w}-i\right)}{\frac{1}{w}-i-z}\mathrm{d}\left(\frac{1}{w}-i\right) \tag{3.13}$$

$$= \frac{1}{z+i}\frac{1}{2i\pi}\int_{\rho(L)} \frac{f\left(\frac{1}{w}-i\right)}{w}\frac{1}{w-\frac{1}{z+i}}\mathrm{d}w,\ z \notin L. \tag{3.14}$$

where the integral is the generalized integral with the weakly singular point $-i$. So it is obvious that $C[f] \in A(\mathbb{C} \setminus L)$.

Lemma 3.2. (*Analytical Properties of Cauchy-Type Integrals*) *Let* $f \in O^{\nu}(\infty)(\nu > 0)$ *and locally integrable on* $\widehat{L_a L_b} \subset L$. *Then*

$$C(f) \in A(\mathbb{C} \setminus L). \tag{3.15}$$

Proof. For $w \notin L$, the closest point w_L to w can be found on *L*. Let

$$d(w) = \left| w - w_L \right|. \tag{3.16}$$

Obviously, $C(f)(z)$ converges uniformly on $U = \{ z \mid |z - w| \le 0.5 d(w) \}$, so

$$\lim_{z \to w} \frac{C(f)(w) - C(f)(z)}{w - z} = \frac{1}{2\pi i}\int_L \lim_{z \to w} \frac{f(\tau)\left((\tau - w)^{-1} - (\tau - z)^{-1}\right)}{w - z}\mathrm{d}\tau, \tag{3.17}$$

thus

$$\left(C(f)\right)'(w) = \frac{1}{2\pi i}\int_L \frac{f(\tau)}{(\tau - w)^2}\mathrm{d}\tau,\ w \notin L. \tag{3.18}$$

which shows. □

Lemma 3.3. *Let* $f \in O^{\nu}(\infty)(\nu > 0)$ *and locally integrable on L. Then*

$$C_{\widehat{L_{-\infty} L_a}}(f) \in A\left(\mathbb{C} \setminus \widehat{L_b L_{+\infty}}\right). \tag{3.19}$$

$$C_{\widehat{L_b L_{+\infty}}}(f) \in A\left(\mathbb{C} \setminus \widehat{L_b L_{+\infty}}\right). \tag{3.20}$$

Proof. As same as 3.2, it is Obvious that the lemma holds since $\dfrac{f\left(\frac{1}{w}-i\right)}{w}$ is just weak singular at origin. □

Referring to [4], we introduce the concept of generalized principal part and order below.

Definition 3.4. Let $F \in A(\mathbb{C} \setminus L)$ If there exists an integer function $E(z)$ such that

$$\lim_{z \to \infty} \left(F(z) - E(z) \right) = 0, \tag{3.21}$$

then $E(z)$ is called the generalized principal part of $F(z)$ at ∞, denoted by $G.P[F,\infty](z)$.

Remark. If F has an isolated singularity ∞, then it has the Laurent series

$$F(z) = \sum_{k=0}^{+\infty} a_k z^k + \sum_{k=1}^{+\infty} a_{-k} z^{-k} \tag{3.22}$$

near ∞, the principal part in the classical sense (with constant a_0) is

$$P.P(F,\infty)(z) = \sum_{k=0}^{+\infty} a_k z^k,\ z \in \mathbb{C}, \tag{3.23}$$

then we can prove

$$G.P(F,\infty) = P.P(F,\infty). \tag{3.24}$$

Easy to prove

$$\lim_{z \in \mathbb{C} \setminus L, z \to \infty} \left(G.P(F,\infty)(z) - P.P(F,\infty)(z) \right) = 0. \tag{3.25}$$

Noting that both $G.P(F,\infty)$ and $P.P(F,\infty)$ are integral functions, we know that is equivalent to

$$\lim_{z \in \mathbb{C}, z \to \infty} \left(G.P(F,\infty)(z) - P.P(F,\infty)(z) \right) = 0. \tag{3.26}$$

so, we can get

$$G.P(F,\infty) = P.P(F,\infty) \tag{3.27}$$

Remark. In general, may not be an isolated singularity of, so he has no principal part in the classical sense, for example,

$$F(z) = \frac{\ln(-z)}{z^m},\ m = 1, 2, \cdots, \tag{3.28}$$

where the logarithmic function $\ln w$ selects the main branch on the complex plane cut by $\Gamma = x + ix^2, x \in \mathbb{R}$, but

$$G.P(F,\infty)(z) = 0 \tag{3.29}$$

It can be seen from this that the general concept of main part is more extensive than the concept of main part in the classical sense, and it is a kind of extension.

Remark. The generalized principal part is unique. For proof, referring to [4].

Definition 3.5. Let $F \in A(\mathbb{C} \setminus L)$. If

$$0 < \beta_m = \limsup_{z \to \infty} \left| z^{-m} F(z) \right| < +\infty, \tag{3.30}$$

then F is said to be of order m at ∞, denoted by $\operatorname{Ord}(F,\infty) = m$.

Remark. Obviously, if $G.P(z^{-m}F,\infty) = \beta_m \neq 0$, then $\operatorname{Ord}(F,\infty) = m$. When $G.P(F,\infty)$ is a polynomial of degree m, we have $\operatorname{Ord}(F,\infty) = m$.

Remark. If $\operatorname{Ord}(F,\infty)$, then $G.P(z^{-m-1}F,\infty) = 0$.

Theorem 3.6. (*Generalized principal part of a Cauchy-type integral at infinity*)

Let $f \in \hat{H}_{\nu}^{\mu}(\infty) \cap O^{\nu}(\infty)(\nu > 0)$ *and locally integrable on L, then*

$$G.P\left[C(f), \infty\right] = 0. \tag{3.31}$$

Proof. Pick a fixed point L_{Δ} arbitrarily on *L*, take $\Delta > 1$ and large enough such that

$$\left|f(t_1) - f(t_2)\right| \le M\left|\frac{1}{t_1} - \frac{1}{t_2}\right|^{\mu}, \forall t_1, t_2 \in \widehat{L_{-\infty} L_{-\Delta}} \cap \widehat{L_{+\Delta} L_{+\infty}} \tag{3.32}$$

Given $z = x + iy \notin L$ and take $z_L \in L$. Such that $\left|z - z_L\right| = dist\{z, L\}$. For any $\tau \in L$, there's always

$$\left|\tau - z_L\right| \le \left|\tau - z\right| + \left|z - z_L\right| \le 2\left|\tau - z\right| \tag{3.33}$$

And when $z \to \infty$, $z_L \to \infty$ accordingly. There is

$$C(f)(z) = \frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - z}\mathrm{d}\tau \tag{3.34}$$

$$= \frac{1}{2\pi i}\int_{L_{-\infty}}^{L_{-1}} \frac{f(\tau)}{\tau - z}\mathrm{d}\tau + \frac{1}{2\pi i}\int_{L_{-1}}^{L_{1}} \frac{f(\tau)}{\tau - z}\mathrm{d}\tau + \frac{1}{2\pi i}\int_{L_{1}}^{L_{+\infty}} \frac{f(\tau)}{\tau - z}\mathrm{d}\tau \tag{3.35}$$

$$= \frac{1}{2\pi i}\left(I_1 + I_2 + I_3\right). \tag{3.36}$$

For I_1,

$$I_1 = \int_{L_{-\infty}}^{L_{-1}} \frac{f(\tau)}{\tau - z}\mathrm{d}\tau \tag{3.37}$$

$$= \int_{L_{-\infty}}^{L_{-\Delta}} \frac{f(\tau)}{\tau - z}\mathrm{d}\tau + \int_{L_{-\Delta}}^{L_{-1}} \frac{f(\tau)}{\tau - z}\mathrm{d}\tau \tag{3.38}$$

$$= I_{11} + I_{12} \tag{3.39}$$

For I_{11},

$$I_{11} = \int_{L_{-\infty}}^{L_{-\Delta}} \frac{f(\tau)}{\tau - z}\mathrm{d}\tau = \int_{L_{-\infty}}^{L_{-\Delta}} \frac{f^*(\tau)}{\tau^{\nu}(\tau - z)}\mathrm{d}\tau\left(\tau = x + ix^2\right) \tag{3.40}$$

$$\left|I_{11}\right| = \left|\int_{L_{-\infty}}^{L_{-\Delta}} \frac{f^*(\tau)}{\tau - z}\mathrm{d}\tau\right| \tag{3.41}$$

$$\le \int_{L_{-\infty}}^{L_{-\Delta}} \frac{\left|f^*(\tau)\right|}{\left|\tau^{\nu}\right|\left|\tau - z\right|}\mathrm{d}\tau \tag{3.42}$$

$$\le \int_{L_{-\infty}}^{L_{-\Delta}} \frac{M\left|\mathrm{d}\left(x + ix^2\right)\right|}{\left|x + ix^2\right|^{\nu}\left|x + ix^2 - z\right|} \tag{3.43}$$

We know that $|x| > 1$. We have

$$1 < \left|1 + 2ix\right| \le 4x, \left|x\right|^{\nu} \le \left|x + ix^2\right|^{\nu} \le \left|\sqrt{2}x^2\right|^{\nu}, \tag{3.44}$$

$$\left|\tau - z_L\right| \le \left|\tau - z\right| + \left|z - z_L\right| \le 2\left|\tau - z\right|, \left|\tau - z_L\right| \ge \frac{1}{2}\left|\tau - z\right| \tag{3.45}$$

thus

$$|I_{11}| \le \int_{L_{-\infty}}^{L_{-\Delta}} \frac{M\left|\mathrm{d}\left(x+ix^2\right)\right|}{\left|x+ix^2\right|^v \left|x+ix^2-z\right|} \tag{3.46}$$

$$\le \int_{L_{-\infty}}^{L_{-\Delta}} \frac{M|1+2ix|}{\left|x+ix^2\right|^v \left|x+ix^2-z\right|}\mathrm{d}x \tag{3.47}$$

$$\le \int_{L_{-\infty}}^{L_{-\Delta}} \frac{8M|x|}{|x|^v |\tau - z_L|}\mathrm{d}x \tag{3.48}$$

$$\le \int_{L_{-\infty}}^{L_{-\Delta}} \frac{8M|x|}{|x|^v |x - x_L|}\mathrm{d}x\left(x_L \text{ is abscissa of } z_L\right) \tag{3.49}$$

$$= 8M\int_{L_{\Delta}}^{L_{+\infty}} \frac{1}{t^v\left(t+x_L\right)}\mathrm{d}t\left(t=-x\right) \tag{3.50}$$

By Hölder inequality, we obtain $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, Call p,q a pair of conjugate numbers.

$$\int_{\Delta}^{+\infty} \frac{1}{t^v\left(t+x_L\right)}\mathrm{d}t \le \left(\int_{\Delta}^{+\infty}\left(\frac{1}{t^v}\right)^p \mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\Delta}^{+\infty}\left(\frac{1}{t+x_L}\right)^q \mathrm{d}t\right)^{\frac{1}{q}} \tag{3.51}$$

while $0<v<1$.

$$\int_{\Delta}^{+\infty}\left(\frac{1}{t^v}\right)^p \mathrm{d}t = \int_{\Delta}^{+\infty}\frac{1}{t^{vp}}\mathrm{d}t = \frac{1}{1-vp}x^{1-vp}\Big|_{\Delta}^{+\infty} \tag{3.52}$$

while $v \ge 1$.

$$\int_{\Delta}^{+\infty} \frac{1}{t^v\left(t+x_L\right)}\mathrm{d}t \le \int_{\Delta}^{+\infty} \frac{1}{t^{\frac{1}{2}}\left(t+x_L\right)}\mathrm{d}t \tag{3.53}$$

By Hölder inequality,

$$\int_{\Delta}^{+\infty} \frac{1}{t^v\left(t+x_L\right)}\mathrm{d}t = 0\left(\text{when } t \to +\infty\right) \tag{3.54}$$

thus $I_{11}=0$. For I_{12},

$$|I_{12}| = \left|\int_{L-\Delta}^{L-1}\frac{f(\tau)}{\tau - z}\right| \le \int_{L-\Delta}^{L-1}\frac{|f(\tau)|}{|\tau - z|}|\mathrm{d}\tau| \le \int_{L-\Delta}^{L-1}\frac{M}{|\tau - z|}|\mathrm{d}\tau| \tag{3.55}$$

Because this is a closed interval, so $I_{12} \to 0$ (when $z \to \infty$). So $I_1 = 0$.

For I_2, $|x|<1$.

$$|I_2| = \left|\int_{L_{-1}}^{L_1}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau\right| \le \int_{L_{-1}}^{L_1}\frac{M}{|\tau - z|}|\mathrm{d}\tau| \tag{3.56}$$

Because this is a closed interval, so $I_2 \to 0$ (when $z \to \infty$). For I_3,

$$I_3 = \int_{L_1}^{L_{+\infty}}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau \tag{3.57}$$

$$= \int_{L_1}^{L_{\Delta}}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau + \int_{L_{\Delta}}^{L_{+\infty}}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau \tag{3.58}$$

$$= I_{31} + I_{32} \tag{3.59}$$

Because $[1,\Delta]$ with $[-\Delta,-1]$ are the interval of symmetry. $I_{12}, I_{31} \to 0$ similarly (when $z \to \infty$).

Because $[-\infty,-\Delta]$ with $[\Delta,\infty]$ are the interval of symmetry (when $z \to \infty$). $I_{11}, I_{32} \to 0$ similarly (when $z \to \infty$). Thus $I_3 = 0$.

To sum up, when $z \to \infty$,

$$\frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - z}\mathrm{d}\tau \to 0 \tag{3.60}$$

$$G.P\left[C(f),\infty\right] = 0 \tag{3.61}$$

is evidenced. □

Corollary 3.7. (*Finite Expansion of Generalized Principal Parts of Cauchy-Type Integrals at Infinity*) *Let* $f \in \hat{H}^{\mu}_{\lambda,0}(\infty) \cap O^{\nu}(\infty)(\nu > 0)$, *where* λ *is a positive integer, and locally integrable, then*

$$G.P\left(z^{\lambda}C(f),\infty\right)(z) = \sum_{k=0}^{\lambda-1}\frac{z^{\lambda-1-k}}{2\pi i}\int_L f(\tau)\tau^k \mathrm{d}\tau.$$

Proof. Let $f_k(\tau) = \tau^k f(\tau)(k = 0,1,\cdots,\lambda)$, since $f \in O^{\nu}(\infty)$, $f_k \in O^{\nu+\lambda-k}(\infty)$, from which it follows at once $\frac{1}{2\pi i}\int_L \frac{f(\tau)\tau^{\lambda}}{\tau - z}\mathrm{d}\tau$ is well defined. □

$$z^{\lambda}\left(C(f)\right)(z) = \frac{1}{2\pi i}\int_L \frac{f(\tau)\left(z^{\lambda} - \tau^{\lambda}\right)}{\tau - z}\mathrm{d}\tau + \frac{1}{2\pi i}\int_L \frac{f(\tau)\tau^{\lambda}}{\tau - z}\mathrm{d}\tau \tag{3.62}$$

$$= -\sum_{k=0}^{\lambda-1}\frac{z^k}{2\pi i}\int_L f(\tau)\tau^{\lambda-1-k}\mathrm{d}\tau + \frac{1}{2\pi i}\int_L \frac{f(\tau)\tau^{\lambda}}{\tau - z}\mathrm{d}\tau, \tag{3.63}$$

by Theorem 3.1 and $f \in O^{\nu}(\infty)$,

$$\lim_{z\in\mathbb{C}\setminus L, z\to\infty}\frac{1}{2\pi i}\int_L \frac{f(\tau)\tau^{\lambda}}{\tau - z}\mathrm{d}\tau = 0,$$

then is proved.

Let $F \in A(\mathbb{C}\setminus L)$. For $t \in L$, if F is continuous up to both sides of L, denote by $F^{\pm}(t)$ the boundary values respectively. To analyze the boundary value of the Cauchy-type integral, we introduce the Cauchy principal value integral. Let $t = x + ix^2$. Since f is Hölder continuous,

$$\lim_{\delta\to 0^+}\frac{1}{2\pi i}\left(\int_{L_{-\infty}}^{L_{\alpha-\delta}}\frac{f(\tau)}{\tau - t}\mathrm{d}\tau + \int_{L_{\alpha+\delta}}^{L_{\infty}}\frac{f(\tau)}{\tau - t}\mathrm{d}\tau\right)$$

the Cauchy-type integral exists, denoted by

$$C[f](t) = \frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - t}\mathrm{d}\tau, t \in L \tag{3.64}$$

the Cauchy principal value integral with kernel density f along L. Similarly, we define the Cauchy principal value integral with kernel density f along L

Remark. The Cauchy principal value integral can also be divided into two parts, that is

$$C(f)(t)=\left(C_{L_\infty L_b}(f)\right)(t)+\left(C_{L_b L_\infty}(f)\right)(t), t\in L.$$

Lemma 3.8. (*Boundary Values of Cauchy-Type Integrals*) *Let* $f\in H_c^\mu(L)\cap O^\nu(\infty)(\nu>0)$. *Then the boundary values of the Cauchy-type integral exist and the following Plemelj formula holds*

$$\begin{cases} C(f)^+(z)=\frac{1}{2}f(z)+\frac{1}{2\pi i}\int_L\frac{f(\tau)}{\tau-z}\mathrm{d}\tau, \\ C(f)^-(z)=-\frac{1}{2}f(z)+\frac{1}{2\pi i}\int_L\frac{f(\tau)}{\tau-z}\mathrm{d}\tau \end{cases} \tag{3.65}$$

for $z\in L$.

Proof. Let $z\in L$.Rewrite $C(f)(z)$ as

$$C(f)^\pm(z)=C_{L_{-\infty}L_c}[f]^\pm(z)+C_{L_cL_\infty}[f]^\pm(z) \tag{3.66}$$

$$=C_{L_{-\infty}L_c}[f]^\pm(z)+C_{L_{-c}L_c}[f]^\pm(z)+C_{L_cL_\infty}[f]^\pm(z) \tag{3.67}$$

$$=\pm\frac{1}{2}f(z)+\frac{1}{2\pi i}\int_{L_{-\infty}}^{L_c}\frac{f(\tau)}{\tau-z}\mathrm{d}\tau+\frac{1}{z\pi i}\int_{L_{-c}}^{L_c}\frac{f(\tau)}{\tau-z}\mathrm{d}\tau+\frac{1}{z\pi i}\int_{L_c}^{L_+\infty}\frac{f(\tau)}{\tau-z}\mathrm{d}\tau \tag{3.68}$$

$$=\pm\frac{1}{2}f(z)+\frac{1}{z\pi i}\int_{L_{+\infty}}^{L_{-\infty}}\frac{f(\tau)}{\tau-z}\mathrm{d}\tau \tag{3.69}$$

□

Corollary 3.9. *If* $f\in O^\nu(\infty)(\nu>0)$, *and for any finite closed arc* $\widehat{L_aL_b}\subset L$, *have* $f\in H^a\left(\widehat{L_aL_b}\right)$,*thus their Cauchy-Type integrals* $C_{L_{-\infty}L_b}$ *and* $C_{L_aL_{+\infty}}$ *have the positive and negative boundary values, and satisfy the following Plemelj formula*:

$$\begin{cases} C_{L_{-\infty}L_b}(f)^+(z)=\frac{1}{2}f(z)+\frac{1}{2\pi i}\int_{L_{-\infty}}^{L_b}\frac{f(\tau)}{\tau-z}\mathrm{d}\tau, \\ C_{L_{-\infty}L_b}(f)^-(z)=-\frac{1}{2}f(z)+\frac{1}{2\pi i}\int_{L_{-\infty}}^{L_b}\frac{f(\tau)}{\tau-z}\mathrm{d}\tau \end{cases}$$
$$\begin{cases} C_{L_aL_{+\infty}}(f)^+(z)=\frac{1}{2}f(z)+\frac{1}{2\pi i}\int_{L_a}^{L_{+\infty}}\frac{f(\tau)}{\tau-z}\mathrm{d}\tau, \\ C_{L_aL_{+\infty}}(f)^-(z)=-\frac{1}{2}f(z)+\frac{1}{2\pi i}\int_{L_a}^{L_{+\infty}}\frac{f(\tau)}{\tau-z}\mathrm{d}\tau \end{cases} \tag{3.70}$$

Theorem 3.10. *If* $f\in H_\alpha^\mu(0)\cap H_c^\mu(L)\cap O^\nu(\infty)(\alpha<1,\nu>0)$, *then the Cauchy-type integral given by is a sectionally holomorphic function with L as its jump curve.*

Proof. By Lemma3.1, Lemma 3.2, and Lemma 3.3, then Theorem3.3 is proved. □

4. Continuity and Singularity at the Origin of the Cauchy Principal Value Integral

This section discusses the continuity of the Cauchy principal value integral, as well as singularity and hölder properties at infinity. First, we have the following Privalov theorem.

Theorem 4.1. (*Hölder Continuity of Cauchy Principal Value Integrals*)

Let $f \in H_\nu^\mu(L)(\nu > 0, 0 < \mu < 1)$ *, then, for the Cauchy principal value integral* $C[f]$ *and the boundary value* $(C[f])^{\pm}$ *, have*

$$C[f], C[f]^{\pm} \in H_\nu^\mu(L) \cap \hat{H}_0^{\frac{1}{2}\mu}(\infty) \tag{4.1}$$

Proof. The first part of the theorem, $C[f](t), C[f]^{\pm}(t) \in H_\nu^\mu(L)$, follows from the Privalov theorem on the finite curve which argument is similar the proof or Theorem 3.2. The proof of the second part had been shown in the proof of Lemma3.8.

5. Boundary Value Problem

Consider the Riemann boundary value problem on *L*: Find a sectional holomorphic function $\Phi(z)$ with the jumping curve *L*, satisfying the boundary value condition and the growth condition at infinity

$$\begin{cases} \Phi^+(t) = G(t)\Phi^-(t) + g(t), t \in L \\ G.P\left(z^{-(m+1)}\Phi, \infty\right) = 0, \end{cases} \tag{5.1}$$

where m is an integer, G and g are given functions on *L*. The Riemann boundary value problem (5.1) is denoted by R_m. To solve the conditions that G and g need to satisfy in this problem, we will discuss them one by one in the following.

The simplest R_m problem is the Liouville problem. Let's discuss the Liouville problem first.

Problem. (The Liouville Problem) Find a sectionally holomorphic function $\Phi(z)$ satisfying

$$\begin{cases} \Phi^+(t) = \Phi^-(t), t \in L, \\ G.P\left(z^{-(m+1)}\Phi, \infty\right) = 0, \end{cases} \tag{5.2}$$

Lemma 5.1. *When* $m \ge 0$ *, the solution of Liouville problem is an arbitrary polynomial of degree not exceeding m. When* $m < 0$ *, there exist only trival solution* $\Phi(z) = 0$ *.*

Problem. (The Jump Problem R_m) Find a sectional holomorphic function $\Phi(z)$ satisfying

$$\begin{cases} \Phi^+(t) = \Phi^-(t) + g(t), t \in L \\ G.P\left(z^{-(m+1)}\Phi, \infty\right) = 0, \end{cases} \tag{5.3}$$

where

$$g \in H_c(L) \cap O^\nu(\infty) \cap \hat{H}_{\mu_0,0}(\infty)(\nu > 0), m_0 = \max\{0, -(m+1)\} \tag{5.4}$$

when $m = -1$, we get the R_{-1} problem.

Problem. (The Jump Problem R_{-1}) Find a sectional holomorphic function $\Phi(z)$ satisfying

$$\begin{cases} \Phi^+(t) = \Phi^-(t) + g(t), t \in L \\ G.P(\Phi, \infty) = 0, \end{cases} \tag{5.5}$$

where g is the same as the above problem.

$$g \in H_c(L) \cap O^{\nu}(\infty) \cap \hat{H}_0(\infty)(\nu > 0) \tag{5.6}$$

Lemma 5.2. *The unique solution of* R_{-1} *is*

$$\Phi(z) = (C(g))(z) = \frac{1}{2\pi i}\int_L \frac{g(\tau)}{\tau - z}\mathrm{d}\tau,\ z \in \mathbb{C} \setminus L. \tag{5.7}$$

Theorem 5.3. *When* $m \geq 0$, *the solution of* R_m *is*

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{g(\tau)}{\tau - z}\mathrm{d}\tau + P_m(z),\ z \in \mathbb{C} \setminus L, \tag{5.8}$$

where $P_m(z)$ *is arbitrary polynomial of degree not greater than m. When* $m = -1$, *the unique solution of problem* R_m *is the solution of problem* R_{-1}. *When* $m < -1$, *it is (uniquely) solvable with the solution if and only if the following conditions are fulfilled*

$$\frac{1}{2\pi i}\int_L g(\tau)\tau^k \mathrm{d}\tau = 0,\ k = 0,1,2,\cdots,-m-2. \tag{5.9}$$

Proof. When $m \geq 0$, from Lemma 5.2, $C(g)$ is the solution of the problem . Therefore, Φ is a solution of the problem if and only if $\Delta = \Phi - C(g)$ is a solution of the following Liouville problem:

$$\begin{cases} \Delta^+(t) = \Delta^-(t),\ t \in L, \\ G.P\left(z^{-(m+1)}\Delta, \infty\right) = 0, \end{cases} \tag{5.10}$$

Thus, by Lemma 5.2, we prove the theorem.

When $m < 0$, it is obvious that the solution of problem is the solution of problem. Thus, by Lemma 5.2, the unique solution of problem is $C(g)$ if and only if the following conditions are fulfilled

$$G.P\left(z^{-(m+1)}C(g), \infty\right) = 0. \tag{5.11}$$

By

$$g \in H_c(L) \cap O^{\nu}(\infty) \cap \hat{H}_{\mu_0,0}(\infty)(\nu > 0), m_0 = \max\{0, -(m+1)\} \tag{5.12}$$

and Inference 3.2, we obtain. □

Remark. When $m < 0$, by Lemma 5.2 and(5.11), the solution of (5.5) can be rewritten as

$$\Phi(z) = \frac{z^{m+1}}{2\pi i}\int_L \frac{\tau^{-(m+1)}g(\tau)}{\tau - z}\mathrm{d}\tau,\ z \in \mathbb{C} \setminus L.$$

$$\Phi^{\pm}(z) \in \hat{H}_{-(m+1),0}(\infty). \tag{5.13}$$

Problem. (The Jump Problem O_m) Find a sectional holomorphic function $\Phi(z)$ with L as its jump curve such that

$$\begin{cases} \Phi^+(t) = \Phi^-(t) + g(t),\ t \in L, \\ G.P\left(z^{-m}\Phi, \infty\right) = 1, \end{cases} \tag{5.14}$$

where $g \in H_c(L)$. When $m \geq 0$, $g \in \hat{H}_0(\infty)$. When $m < 0$, $g \in \hat{H}_{-m,0}(\infty)$.

Remark. By Note 3.6, the order of the partitioned holomorphic function Φ at infinity is a fixed m order, this problem is called the fixed-order jump problem. When $g=0$, the problem is called the Liouville problem of fixed order.

Lemma 5.4. *When* $m \geq 0$, *the solution of is*

$$\Phi(z)=\frac{1}{2\pi i}\int_L \frac{g(\tau)}{\tau-z}\mathrm{d}\tau+P_m(z), z\in\mathbb{C}\setminus L, \tag{5.15}$$

where P_m *is an arbitrary polynomial of degree m and the leading coefficient is* 1. *When* $m<0$, *the solution of is*

$$\Phi(z)=\frac{1}{2\pi i}\int_L \frac{g(\tau)}{\tau-z}\mathrm{d}\tau, z\in\mathbb{C}\setminus L \tag{5.16}$$

if and only if the following conditions are fulfilled

$$\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_L g(\tau)\tau^k\mathrm{d}\tau=0, k=0,1,2,\cdots,-m-2,\\ \dfrac{1}{2\pi i}\displaystyle\int_L g(\tau)\tau^{-m-1}\mathrm{d}\tau=-1.\end{cases} \tag{5.17}$$

Proof. By Theorem 5.1 and Inference 3.2, Lemma 5.4 can be proved. □

Problem. (Homogeneous Boundary Value Problems) Find a sectional holomorphic function $\Phi(z)$ satisfying

$$\begin{cases}\Phi^+(t)=G(t)\Phi^-(t), t\in L,\\ G.P\left(z^{-m-1}\Phi,\infty\right)=0,\end{cases} \tag{5.18}$$

where $G\in H(L)$ and $G(t)\neq 0, t\in L$. In addition, G satisfies the infinity growth condition

$$G(\infty)=1, \log G\in \hat{H}_0(\infty), \tag{5.19}$$

where $\log G$ is the single-valued continuous branch such that $\log G(\infty)=0$.

Remark. Because of the regularity condition, we can choose a single-valued continuous branch of $\log G$ on *L*. In addition, because of $G(\infty)=1$, there is $\log G(\infty)=0$ on the branch we choose.

Assume

$$\frac{\log G(0)}{2\pi i}=\alpha+i\beta, \tag{5.20}$$

then,

$$\kappa=-[\alpha] \tag{5.21}$$

is called the index of homogeneous boundary value problem, where $[\alpha]$ is the largest integer not exceeding α.

Problem. (Canonical Problems) Find a sectional holomorphic function with *L* as its jump curve such that

$$\begin{cases}\Phi^+(t)=G(t)\Phi^-(t), t\in L\\ G.P\left(z^{\kappa}\Phi,\infty\right)=1.\end{cases} \tag{5.22}$$

Denote by

$$\Gamma(z)=\frac{1}{2\pi i}\int_L \frac{\log G(\tau)}{\tau-z}\mathrm{d}\tau,\ z\in\mathbb{C}\setminus L. \tag{5.23}$$

By (5.19), we gain

$$\log G\in H(L)\cap\hat{H}_0(\infty) \tag{5.24}$$

Thus, by Theorem 3.1 and Lemma3.2, we obtain

$$\Gamma(\infty)=0,\Gamma(z)=-(\alpha+i\beta)\log(-z)+\Delta(z), \tag{5.25}$$

where Δ is holomorphic in the neighborhood cut by L near 0, and $\lim_{z\to 0}\Delta(z)$ exists.

Let

$$X(z)=z^{-\kappa}\mathrm{e}^{\Gamma(z)},\ z\in\mathbb{C}\setminus L, \tag{5.26}$$

by (5.24), we have

$$G.P(zX,0)=0,G.P(z^{\kappa}X,\infty)=1, \tag{5.27}$$

then, we have

$$X^-(t)=G(t)X^+(t),\ t\in L. \tag{5.28}$$

so, by(5.27) and (5.28), X is the solution to the canonical problem.

Remark. Reference to [2] [3], we classify node 0 in detail. By, and, when α is an integer in, X^{-1} is bounded near 0, at this time, the 0 is called a special node; when α is not an integer in, the 0 is called an ordinary node, at this time, since $X^{-1}(z)$ contains factor $(-z)^{\alpha-[\alpha]}$, so $X^{-1}(0)=0$.

The uniqueness of the solution is shown below. If Φ is the solution of the canonical problem, let

$$Q(z)=\frac{\Phi(z)}{X(z)},\ z\in\mathbb{C}\setminus L, \tag{5.29}$$

Q given in is a sectional holomorphic function, so, by (5.22) and (5.27),

$$\begin{cases}Q^+(t)=Q^-(t),\ t\in L,\\ G.P(Q,\infty)=1.\end{cases} \tag{5.30}$$

so, by Lemma 5.4, $Q=1$ that is, $Q=X$.

Summarizing the above discussion, we have the following theorem.

Lemma 5.5. *Under conditions, the canonical problem has a unique solution X, which is given by* (5.23).

Remark. X in is called the canonical solution of the homogeneous problem.

If Φ is the solution of the homogeneous problem, then Q in is the solution of the following Liouville problem

$$\begin{cases}Q^+(t)=Q^-(t),\ t\in L,\\ G.P\left(z^{-(m+1+\kappa)}Q,\infty\right)=0.\end{cases} \tag{5.31}$$

By Lemma 5.2, one gets

$$\Phi(z)=X(z)P_{\kappa+m}(z),\ z\in\mathbb{C}\setminus L, \tag{5.32}$$

where κ is the index given in, where $P_{\kappa+m}(z)$ is arbitrary polynomial of degree not greater than $m+\kappa$, if $\kappa+m<0$, $P_{\kappa+m}=0$. In addition, it is easy to prove that Φ in is the solution of the homogeneous problem.

Summarizing the above discussion, we have the following theorem.

Lemma 5.6. *When* $\kappa\geq -m$, Φ *in is the solution of the homogeneous problem*; *When* $\kappa<-m$, *the unique solution of problem is* $\Phi=0$.

Next, solve the problem . Using X in, let

$$F(z)=\frac{\Phi(z)}{X(z)}, z\in\mathbb{C}\setminus L, \tag{5.33}$$

F is a sectional holomorphic function, problem can be transfer into the following jump problem:

$$\begin{cases} F^+(t)=F^-(t)+\dfrac{g(t)}{X^+(t)}, t\in L, \\ G.P\left(z^{-(m+1+\kappa)}F,\infty\right)=0. \end{cases} \tag{5.34}$$

Under the conditions, and the additional condition $\nu-1>2\mu$, it can be proved that the following formula holds

$$\frac{g}{X^+}\in H_c(L)\cap O^{\nu}(\infty)\cap\hat{H}_{m_k,0}(\infty)(\nu>0), \tag{5.35}$$

where $m_\kappa=\max\{0,-(m+1+\kappa)\}$.

By Theorem 4.1 and

$$\Gamma^+\in H_c(L)\cap\hat{H}_0(\infty), \tag{5.36}$$

So

$$X^+\in H_c(L), \mathrm{e}^{\Gamma^+(t)}\in\hat{H}(\infty). \tag{5.37}$$

Let $\mu'=\min\{\mu,\mu_1\}$, $\nu'=\nu-2\mu-\min\{\mu,\mu_1\}$. By $\nu-2\mu>1$, we have $\mu'\in(0,1),\nu'>1$. Then

$$\frac{g}{X^+}\in\hat{H}^{\mu'}_{\nu',m_\kappa}(\infty). \tag{5.38}$$

By Lemma 5.1, when $m+\kappa\geq 0$, the solution to the jumping problem $\mathrm{R}_{m+\kappa}$ is

$$F(z)=\frac{1}{2\pi i}\int_L\frac{g(\tau)}{X^+(\tau)(\tau-z)}f\tau+P_{m+\kappa}(z), z\in\mathbb{C}\setminus L, \tag{5.39}$$

where $P_{m+\kappa}$ is an arbitrary polynomial of degree not greater than $m+\kappa$. When $m+\kappa<0$, the unique solution of problem $\mathrm{R}_{m+\kappa}$ is $C\left(g/X^+\right)$, or $p_{m+\kappa}=0$ in, if and only if the following conditions are fulfilled:

$$\frac{1}{2\pi i}\int_L\frac{g(\tau)}{X^+(\tau)}\tau^k\mathrm{d}\tau=0, k=0,1,\cdots,-m-2. \tag{5.40}$$

when $m+\kappa=-1$, the condition does not appear.

If Φ is the solution to problem, then

$$\Phi(z)=\frac{X(z)}{2\pi i}\int_L \frac{g(\tau)}{X^+(\tau)(\tau-z)}\mathrm{d}\tau + X(z)P_{m+\kappa}(z),\ z\in\mathbb{C}\setminus L. \tag{5.41}$$

Summarizing the above discussion, we have the following theorem.

Theorem 5.7. *Under the conditions, and the additional condition* $\nu-1>2\mu$, *the solution of* R_m *problem is* (5.41), *where* $P_{m+\kappa}(z)$ *is arbitrary polynomial of degree not greater than* $m+\kappa$. *When* $m+\kappa=-1$, *the solution of* R_m *problem is* (5.41) *and* $p_{m+\kappa}=0$. *When* $m+\kappa<-1$, *the unique solution of* R_m *problem is* (5.41) *and* $p_{m+\kappa}=0$ *if and only if the solvability condition* (5.40) *is satisfied.*

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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