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About Upper and Lower Strong Fractional Weighted Mean Convergence by Moduli for Triple Sequences

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Abstract

In article, I present a study on upper and lower statistical convergence, and upper and lower strong fractional weighted mean convergence by moduli for triple sequences. One of the generalizations of the discrete operator Cesàro, was weighted mean operators, which are linear operators, too. Given a modulus function f, I established that a triple sequence that is f-upper or lower strong fractional weighted mean convergent, in some supplementary conditions, is also f-lower or upper statistically convergent. The results of this paper adapt the results obtained in [1] and [2] to upper and lower strong fractional weighted mean convergence and to triple sequence concept. Furthermore, new concepts can be applied to the approximation theory, topology, Fourier analysis, analysis interdisciplinary with applications electrical engineering, robotics and other domains.

Keywords

Upper and Lower Statistical Convergence, Triple Sequences, Unbounded Modulus Function, Numerical Analysis

1. Introduction and Motivation

Many areas of mathematics, such as mathematical optimizations, analysis, statistics, algebra, geometry use the opposing concepts of upper and lower. For examples, upper and lower solutions for first order problems with nonlinear boundary conditions [3], upper and lower probabilities generated by a random closed interval [4], upper and lower bounds for the Riemann-Stieltjes integral [5], upper and lower solutions method for fuzzy differential equations [6], upper limit superior and limit inferior of soft sequences [7] and many other very interesting and useful works.

In this paper, I discuss about upper and lower strong fractional weighted mean convergence by moduli for triple sequences, starting to the concept of the average operators.

Cesàro average operators have been used in many papers, in a discrete and integral form, and, also, generalizations of there, *i.e.* z-Cesàro operators, Rhaly operators, weighted mean operators, on different spaces of sequences of functions.

For a linear weighted mean operator, the weighted mean matrix A associated with this, is a lower triangular matrix with entries $a_{nk} = p_k / P_n$ where $p_1 > 0$, $p_k \geq 0$ $k \in \{2,3,\cdots\}$ and $P_n = \sum_{k=1}^{n} p_k$, where (p_k) is a bounded sequence of strictly positive numbers [8].

In present time, for these operators, some researched objectives are for statistical convergence and for generalizations, *i.e.* for λ-statistical convergence of order α [9], statistical convergence of order α in paranormed space [10]. These concepts have contributed to developing the fields of mathematical analysis, functional analysis, ergodic theory, fuzzy set theory, trigonometric series and approximation theory.

The concept of the weighted statistical convergence was given by Acar and Mohiuddine, in 2008 [11] and the concept was generalized by Aljimi and Sirimark in 2021 [12] [13]. In a modular space associated with a generalized double sequence of function, some work analyzed a particular concept, the deferred-weighted summability mean [14]. A paper published in 2022 studied double sequences of fuzzy numbers, and some cases of weighted ideal statistical convergence and strongly weighted ideal convergence [15]. The notions of ideal statistically convergence for sequence of fuzzy number were defined by the same author, in 2021, in the same time with the definition for the notions ideal statistically pre-Cauchy triple sequences [16].

In 2019 and 2022, the authors León-Saavedra, Listán-García, Perez Fernández and Romero de la Rosa, analized the statistical convergence and strong Cesàro convergence by moduli for double sequences [1] [17].

I extended the results from papers [1] [2], to triple sequences, because after researching the specialty literature, I realized that no other author has done this before. My motivation was generated by this. Moreover, I added new concepts, such as "f-upper strong fractional weighted mean convergent", "f-lower strong fractional weighted mean convergent" in order to bring new elements to the theory from mathematical analysis.

The concept of strong Cesàro convergence was given by Hardy-Littlewood [18] and Fekete [19].

In the recent specialty literature, in many papers the authors obtained result for different kinds of statistical convergence defined by moduli (*i.e.* in [1] [2] [17] [20]). In [21], Mursaleen and Edely obtained Connor's [22] result for double sequences. In [23], Şahiner *et al.* present results for triple sequences.

In this paper, I aim to obtain the results triple sequences and for different types of statistical convergence, which are defined by a density in N using a generalized compatible unbounded modulus function *f*.

Pringsheim introduced the definition of the convergence for double sequences in 1900. The concept of statistical convergence was first used by Fast in 1951. After, in year 2003, Tripathy and Mursaleen *et al.* used the statistical convergence for double sequences [24]. Extensions of the concept of statistical convergence were introduced by Kolk [25] which studied statistical convergence to normed spaces; by Maddox [26] which defined the locally convex Hausdorff topological linear spaces; Çakalli [27] which extended to topological Hausdorff groups; etc. Also, Fridy and Orhan presented in 1997, details about the statistical limit superior and limit inferior [28].

Kolk [29] extended the definition of statistical convergence with the help of nonnegative regular matrix $A = (a_{nk})$ calling it *A*-statistical convergence.

Also, in year 2009, the notion of the weighted statistical convergence was introduced and analyzed by Karakaya and Chishti [30]. After three years, in 2012, Mursaleen *et al.* [31] presented a modified concept. In 2013, Belen and Mohiuddine [32] created a generalization of this concept through de la Vallée-Poussin mean. After a year, in 2014, Esi [33] defined and analyzed studied the notion statistical summability through de la Vallée-Poussin mean in probabilistic normed spaces and Mohiuddine *et al.*, for a nonnegative regular matrix *A*, introduced the concept of weighted *A*-statistical convergence of a sequence and demonstrated the Korovkin approximation theorem by using this concept [34]. Recent, in year 2022, Özger *et al.* used the statistical approximation properties, the modulus of continuity and presented local approximation results [35].

The paper is organized as follows. In the "*Introduction and Motivation*" section, I wrote about literature review and motivation for the subject. The second section, "*Definition and Notations*", contains new definitions and new notations. In the third section, entitled "*Main results*", I suggest theorems with conditions in which a triple sequence is *f*-upper or lower strong fractional weighted mean convergent, or is *f*-lower or upper statistically convergent. The conclusions are stated in the "*Conclusions and Future Research Directions*" section.

2. Definitions and Notations

In this paper, $(X, \|\cdot\|)$ will denote a normed space, α is a proper fraction and (p_k) is a bounded sequence of strictly positive real numbers. $|B|$ denotes also the cardinality of the subset *B from the set of the natural numbers*.

Definition 2.1. [1] [17] *A sequence* $(x_n) \subset X$ *was said to be strong Cesàro convergent to L if* $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\|x_k - L\| = 0$.

In this paper, used the notions of lim sup, and lim inf from [22], I generalize this definition as follows:

Definition 2.2. *Let* (p_i) *a sequence of strictly positive real numbers and*

$P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$ *and* $\left[\frac{P_n}{p_n}\right] \ge n^{\alpha}$ *for all n. A sequence* $(x_n) \subset$ *X was said to be upper strong fractional weighted mean convergent to L if*

$$\limsup_{n\to\infty} \frac{1}{\left[\frac{P_n}{p_n}\right]^{-\alpha}} \sum_{k=1}^{n} \|x_k - L\| = 0 .$$

Definition 2.3. *Let* (p_i) *a sequence of strictly positive real numbers and* $P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$ *and* $\left[\frac{P_n}{p_n}\right] \ge n^{\alpha}$ *for all n. A sequence* $(x_n) \subset$ *X was said to be lower strong fractional weighted mean convergent to L if*

$$\liminf_{n\to\infty} \frac{1}{\left[\frac{P_n}{p_n}\right]^{-\alpha}} \sum_{k=1}^{n} \|x_k - L\| = 0 .$$

Let $\varepsilon > 0$ and $M_{\varepsilon} = \{n, \|x_k - L\| > \varepsilon\}$ a subset of the natural set of numbers.

Definition 2.4. [1] [17] *A sequence* $(x_n) \subset X$ *is called statistically convergent to L if for any* ε, M_{ε} *has zero density on the set of the natural numbers.*

Definition 2.5. [36] *A function* $f : R_+ \to R_+$ *is said to be a unbounded modulus function if it fulfills the following conditions:* $f(x) = 0$ *if and only* if $x = 0$; $f(x+y) \le f(x) + f(y)$ *for every* $x, y \in R_+$; *f is increasing, and continuous from the right at* 0; $\lim_{x\to\infty} f(x) = \infty$.

Throughout the paper, we denote by $|A|$ the cardinality of a finite set A. According to [20] [36] we have:

Definition 2.6. *Let A a subset of the set of natural numbers.*

1) *The lower fractional density of A is the limit*

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{\left| A \cap \left[1, \left[\frac{P_n}{p_n}\right]^{-\alpha}\right] \right|}{\left[\frac{P_n}{p_n}\right]^{-\alpha}} .$$

2) *For f, a unbounded modulus function, the f-lower fractional density of A is the limit* $\liminf_{n\to\infty} \dfrac{f\left(\left| A \cap \left[1, \left[\frac{P_n}{p_n}\right]^{-\alpha}\right] \right|\right)}{f\left(\left[\frac{P_n}{p_n}\right]^{-\alpha}\right)}$.

3) *The upper fractional density of A is the limit*

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{\left| A \cap \left[1, \left[\frac{P_n}{p_n}\right]^{-\alpha}\right] \right|}{\left[\frac{P_n}{p_n}\right]^{-\alpha}} .$$

4) *For f, a unbounded modulus function, the f-upper fractional density of A is*

the limit $\limsup_{n\to\infty} \dfrac{f\left(\left|A\cap\left[1,\left[\dfrac{P_n}{p_n}\right]^{-\alpha}\right]\right|\right)}{f\left(\left[\dfrac{P_n}{p_n}\right]^{-\alpha}\right)}$.

Every triple limit we use will be considered in Pringsheim's sense. Pringsheim [37], in 1900, defined the concept of convergence of real double sequences and of real triple sequences:

Definition 2.7. *A triple sequence* $X=(x_{mnl})$ $m,n,l\in\mathbb{N}$ *converges to* $a\in\mathbb{R}$, *if for every* $\varepsilon>0$ *there is* $n_0\in\mathbb{N}$ *such that* $|x_{mnl}-a|<\varepsilon$ *for all* $m,n,l>n_0$ (see [38], and also [17] [18]). *The limit a is called the Pringsheim limit of X.*

Because statistical convergence depends on the density of the subsets of N, then the concept of statistically convergent double sequences is a function of the density of subsets of $N\times N$, and the notion of statistically convergent triple sequences depends on the density of subsets of $N\times N\times N$.

The theory of triple sequences is a generalization of the single sequences and of the double sequences. A triple sequence of real numbers is a function $x:N\times N\times N\to R$. In the article, I will use the notation (x_{mnl}).

For example, for $x_{mnl}=\dfrac{1}{m+n+l}$, the limit a is obviously equal to 0.

According to [1] [17] we observed that the f-strong Cesàro convergence for triple sequences is a generalization of the f-strong Cesàro convergence for double sequences:

Definition 2.8. *Let f be the unbounded modulus function. A sequence* (x_{mnl}) *is said to be f upper strong Cesàro convergent to L if*

$$\limsup_{m,n,l\to\infty}\frac{f\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left\|x_{ijk}-L\right\|\right)}{f(mnl)}=0.$$

Definition 2.9. *Let f be the unbounded modulus function. A sequence* (x_{mnl}) *is said to be f-lower strong Cesàro convergent to L if*

$$\liminf_{m,n,l\to\infty}\frac{f\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left\|x_{ijk}-L\right\|\right)}{f(mnl)}=0.$$

Starting from f-strong Cesàro convergence for triple sequences, I define in this paper the notion of f-upper strong fractional weighted mean convergence for triple sequences f-lower strong fractional weighted mean convergence for triple sequences, as follows:

Definition 2.10. *Let* (p_i) *a sequence of strictly positive real numbers and* $P_n=\sum_{k=1}^{n}p_n$ *with* $\lim_{n\to\infty}P_n=\infty$, $\left[\dfrac{P_n}{p_n}\right]\ge n^{\alpha}$ *for all n. Let f be a unbounded modulus function. A sequence* (x_{mnl}) *is said to be f-upper strong fractional weighted mean convergent to L if*

$$\limsup_{m,n\to\infty}\frac{f\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left\|x_{ijk}-L\right\|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}=0.$$

Definition 2.11. *Let* (p_i) *a sequence of strictly positive real numbers, and* $P_n=\sum_{k=1}^{n}p_n$ *with* $\lim_{n\to\infty}P_n=\infty$, $\left[\frac{P_n}{p_n}\right]\geq n^{\alpha}$ *for all n. Let f be a unbounded modulus function. A sequence* (x_{mnl}) *is said to be f-lower strong fractional weighted mean convergent to L if*

$$\liminf_{m,n\to\infty}\frac{f\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left\|x_{ijk}-L\right\|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}=0.$$

In [38], by means of a new concept of density of a subset of *N*, was defined the following concept of convergence:

Definition 2.12. [38] *A sequence* (x_n) *is said to be f-statistically convergent to L if for every* $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{f\left(\left|k\leq n:\left\|x_k-L\right\|>\varepsilon\right|\right)}{f(mn)}=0.$$

Starting from [38] we can give the following definitions:

Definition 2.13. *Let* (p_i) *a sequence of positive real numbers and* $P_n=\sum_{k=1}^{n}p_n$ *with* $\lim_{n\to\infty}P_n=\infty$, $\left[\frac{P_n}{p_n}\right]\geq n^{\alpha}$ *for all n. Let f be a unbounded modulus function. A subset* $A\subseteq N\times N\times N$ *has f-upper fractional density if the following limit exists*

$$d_{3,f}(A)=\limsup_{m,n,l\to\infty}\frac{f\left(\left|(i,j,k)\in N\times N\times N:i\leq m,j\leq n,k\leq l\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}.$$

Definition 2.14. *Let* (p_i) *a sequence of positive real numbers and* $P_n=\sum_{k=1}^{n}p_n$ *with* $\lim_{n\to\infty}P_n=\infty$, $\left[\frac{P_n}{p_n}\right]\geq n^{\alpha}$ *for all n. Let f be a unbounded modulus function. A subset* $A\subseteq N\times N\times N$ *has f-lower fractional density if the following limit exists*

$$d_{3,f}(A)=\liminf_{m,n,l\to\infty}\frac{f\left(\left|(i,j,k)\in N\times N\times N:i\leq m,j\leq n,k\leq l\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}.$$

Also, starting from [10] we can give the following definitions:

Definition 2.15. *Let* (p_i) *a sequence of positive real number and* $P_n=\sum_{k=1}^{n}p_n$

with $\lim_{n\to\infty} P_n = \infty$, $\left[\frac{P_n}{p_n}\right] \geq n^{\alpha}$ *for all n. Let* (x_{ijk}) *be a triple sequence and* $L \in X$ *, for any* $\varepsilon > 0$, $p \leq m$, $q \leq n$, $r \leq l$ ($p,q,r,m,n,l \in N$).

Let the following subsets $M\varepsilon(p,q,r,m,n,l)$:

$$M\varepsilon(p,q,r,m,n,l)$$
$$=\left\{(i,j,k)\in N\times N\times N : p\leq i\leq m, q\leq j\leq n, r\leq k\leq l, \left\|x_{ijk}-L\right\|>\varepsilon\right\}.$$

Let f be an unbounded modulus function. Then (x_{ijk}) *is f-upper statistically convergent to L if*

$$\lim_{p,q,r\to\infty}\limsup_{m,n,l\to\infty}\frac{f\left(\left|M\varepsilon(p,q,r,m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}=0.$$

Definition 2.16. *Let* (p_i) *a sequence of positive real number and* $P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$, $\left[\frac{P_n}{p_n}\right] \geq n^{\alpha}$ *for all n. Let* (x_{ijk}) *be a triple sequence and* $L \in X$ *, for any* $\varepsilon > 0$, $p \leq m$, $q \leq n$, $r \leq l$ ($p,q,r,m,n,l \in N$).

Let the following subsets $M\varepsilon(p,q,r,m,n,l)$:

$$M\varepsilon(p,q,r,m,n,l)$$
$$=\left\{(i,j,k)\in N\times N\times N : p\leq i\leq m, q\leq j\leq n, r\leq k\leq l, \left\|x_{ijk}-L\right\|>\varepsilon\right\}.$$

Let f be an unbounded modulus function. Then (x_{ijk}) *is f-lower statistically convergent to L if*

$$\lim_{p,q,r\to\infty}\liminf_{m,n,l\to\infty}\frac{f\left(\left|M\varepsilon(p,q,r,m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}=0.$$

In the specialty literature, is says that, in many cases, the above limit may not exist (in Pringsheim's sense).

For double and analogous for triple sequences, in [17] it has been shown that Definition 2.14, 2.15 can be replaced to the definitions 2.17 and 2.18.

Definition 2.17. *Let* (p_i) *a sequence of positive real numbers and* $P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$, $\left[\frac{P_n}{p_n}\right] \geq n^{\alpha}$ *for all n. Let* (x_{ijk}) *be a triple sequence and* $L \in X$, $m,n,l \in N$ *. Let us define the subsets* $M\varepsilon(m,n,l)$:

$$M\varepsilon(m,n,l)=\left\{(i,j,k)\in N\times N\times N : i\leq m, j\leq n, k\leq l, \left\|x_{ijk}-L\right\|>\varepsilon\right\}.$$

Let f be a compatible unbounded modulus function. Then (x_{ijk}) *is f-upper statistically convergent to L if*

$$\limsup_{m,n,l\to\infty}\frac{f\left(\left|M\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}=0.$$

Definition 2.18. *Let* (p_i) *a sequence of positive real numbers and* $P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$, $\left[\frac{P_n}{p_n}\right] \ge n^{\alpha}$ *for all n. Let* (x_{ijk}) *be a triple sequence and* $L \in X$, $m,n,l \in N$. *Let us define the subsets* $M\varepsilon(m,n,l)$:

$$M\varepsilon(m,n,l) = \left\{(i,j,k) \in N \times N \times N : i \le m, j \le n, k \le l, \left\|x_{ijk} - L\right\| > \varepsilon\right\}.$$

Let f be a compatible unbounded modulus function. Then (x_{ijk}) *is f-lower statistically convergent to L if*

$$\liminf_{m,n,l\to\infty} \frac{f\left(\left|M\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha} l\right)} = 0.$$

Also, from [17] we known the following definition:

Definition 2.19. [17] *A modulus function f is said to be compatible if for any* $\varepsilon > 0$, *there exist* $\varepsilon' > 0$, $n_0(\varepsilon) \in N$ *such that the following inequality take place,* $\frac{f(n\varepsilon')}{f(n)} < \varepsilon$, *for all* $n \ge n_0$.

In this article, we define a generalized compatible modulus function, as follows:

Definition 2.20. *Let* (p_i) *a sequence of positive real numbers and* $P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$, $\left[\frac{P_n}{p_n}\right] \ge n^{\alpha}$ *for all n. Let f be an unbounded modulus function. The function f is said to be generalized compatible if for any* $\varepsilon > 0$, *there exist* $\varepsilon' > 0$ *and* $n_0(\varepsilon) \in N$ *such that* $\frac{f(n\varepsilon')}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha} l\right)} < \varepsilon$ *for all* $n \ge n_0$ and for all $m, l \in N$.

3. Main Results

Let $\varepsilon > 0$.

I denote by $M'\varepsilon(m,n,l) = M\varepsilon(m,n,l) - M\varepsilon(p,q,r,m,n,l)$.

Theorem 3.1. *Let* (p_i) *a sequence of positive real numbers,* $p_0 > 0$, *and* $P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$ *and* $\left[\frac{P_n}{p_n}\right] \ge n^{\alpha}$. *Let f be a generalized compatible unbounded modulus function. For which exist* $N \in \mathrm{N}$ *and for* $m \ge N > p$, $n \ge N > q$, $l \ge N > r$, *we have for all m, n,* $l \ge N$, *the inequality* $\frac{f\left(\left|M'\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha} l\right)} < \frac{\varepsilon}{2}$. *Then* (x_{ijk}) *is f lower statistically convergent to L if and only if for any* $\varepsilon > 0$,

$$\liminf_{m,n,l\to\infty}\frac{f\left(\left|M\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}=0.$$

Proof. The proof idea from Theorem 2.5 of [17] is applied.

We see that $M\varepsilon(p,q,r,m,n,l)\subseteq M\varepsilon(m,n,l)$; and, from the hypothesis that f is an increasing function, we have:

$$f\left(\left|M\varepsilon(p,q,r,m,n,l)\right|\right)\leq f\left(\left|M\varepsilon(m,n,l)\right|\right)$$

the implication follows dividing the above inequality by $f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)$ and taking limits as $(m, n, l) \to \infty$.

Let $\varepsilon > 0$. Since (x_{ijk}) is f-statistically convergent to L, there exist $(p,q,r)\in N\times N\times N$ and $n_0\in \mathrm{N}$ such that, if $m \geq n_0 > p$, $n \geq n_0 > q$, $l \geq n_0 > r$, we have the relations

$$\frac{f\left(\left|M\varepsilon(p,q,r,m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}<\frac{\varepsilon}{2}. \tag{1}$$

From hypothesis, for all $m, n, l \geq N$

$$\frac{f\left(\left|M'\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}<\frac{\varepsilon}{2}. \tag{2}$$

Finally, the result follows from (1) and (2). If $m,n,l\geq\max\{N,n_0\}$ I obtain the relations:

$$\frac{f\left(\left|M\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}\leq\frac{f\left(\left|M\varepsilon(p,q,r,m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}+\frac{f\left(\left|M'\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

which implies the equality from the theorem.

Theorem 3.2. *Let* (p_i) *a sequence of positive real numbers,* $p_0>0$, *and* $P_n=\sum_{k=1}^{n}p_n$ *with* $\lim_{n\to\infty}P_n=\infty$ *and* $\left[\frac{P_n}{p_n}\right]\geq n^{\alpha}$. *Let f be a generalized compatible unbounded modulus function. For which exist* $N\in\mathrm{N}$ *and for* $m \geq N >$ p, $n \geq N > q$, $l \geq N > r$, *we have for all* $m, n, l \geq N$, *the inequality* $\frac{f\left(\left|M'\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha}l\right)}<\frac{\varepsilon}{2}$. *Then* (x_{ijk}) *is f upper statistically convergent to L if and only if for any* $\varepsilon > 0$,

$$\limsup_{m,n,l\to\infty} \frac{f\left(\left|M\varepsilon(m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha} l\right)} = 0.$$

Proof. The proof is analogous which the proof for the Theorem 3.1.

Also, in [17] was proved that:

Theorem 3.3. [17] *Let f be a compatible unbounded modulus function. If* (x_{ij}) *is statistically convergent to L, then* (x_{ij}) *is f-statistically convergent to L.*

The theorem can be generalized and the demonstration is analogous, for the following:

Theorem 3.4. *Let f be a compatible unbounded modulus function. If* (x_{ijk}) *is statistically convergent to L, then* (x_{ijk}) *is f-lower statistically convergent to L.*

Theorem 3.5. *Let f be a compatible unbounded modulus function. If* (x_{ijk}) *is statistically convergent to L, then* (x_{ijk}) *is f-upper statistically convergent to L.*

Theorem 3.6. *Let* (p_i) *a sequence of positive real numbers,* $p_0 > 0$, *and* $P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$ *and* $\left[\frac{P_n}{p_n}\right] \ge n^{\alpha}$. *Let* ($x_{ijk}$) *be a triple sequence and let f be a generalized compatible unbounded modulus function. If* (x_{ijk}) *is f-upperr strong fractional weighted mean convergent to L, then* (x_{ijk}) *is f-upper statistically convergent to L.*

Proof. We show that (3) is true for all $t \in N$,

$$\lim_{p,q\to\infty} \limsup_{m,n,l\to\infty} \frac{f\left(\left|M_{\frac{1}{t}}(p,q,r,m,n,l)\right|\right)}{f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha} l\right)} = 0. \tag{3}$$

Let $\varepsilon > 0$ be small enough, then there exists $t \in \mathrm{N}$ such that $\frac{1}{t+1} \le \varepsilon < \frac{1}{t}$, which implies that for any $p,q,r,m,n,l \in N$

$$M_{\frac{1}{t}}(p,q,r,m,n,l) \subseteq M_{\varepsilon}(p,q,r,m,n,l) \subseteq M_{\frac{1}{t+1}}(p,q,r,m,n,l),$$

$$\left|M_{\frac{1}{t}}(p,q,r,m,n,l)\right| \le \left|M_{\varepsilon}(p,q,r,m,n,l)\right| \le \left|M_{\frac{1}{t+1}}(p,q,r,m,n,l)\right|.$$

Since f is increasing, dividing by $f\left(m\left[\frac{P_n}{p_n}\right]^{-\alpha} l\right)$, we get the result follows taking limits.

Thus, let $t \in \mathrm{N}$ be large enough, and we will show that (3) is satisfied. Let $p,q,r,m,n,l \in N$ with $p \le m$, $q \le n$, $r \le l$ then

$$f\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left\|x_{ijk}-L\right\|\right) \ge f\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\substack{k=1\\ \left\|x_{ijk}-L\right\|\ge\frac{1}{t}}}^{l}\left\|x_{ijk}-L\right\|\right)$$

$$\geq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ \|x_{ijk} - L\| \geq \frac{1}{t}}}^{l} \frac{1}{t} \right) \geq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ (i,j,k) \in M_{\frac{1}{t}}(m,n,l)}}^{l} 1 \right)$$
$$= f\left(\left| M_{\frac{1}{t}}(m,n,l) \right| \right) \geq f\left(\left| M_{\frac{1}{t}}(p,q,r,m,n,l) \right| \right). \tag{4}$$

Since (x_{ijk}) is strong weighted mean convergent to L, we have that

$$\liminf_{m,n,l \to \infty} \frac{f\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \|x_{ijk} - L\| \right)}{f\left(m \left[\frac{P_n}{p_n} \right]^{-\alpha} l \right)} = 0.$$

Therefore, dividing by $f\left(m \left[\frac{P_n}{p_n} \right] l \right)$ the inequalities (8), results

$$\liminf_{m,n,l \to \infty} \frac{f\left(\left| M_{\frac{1}{t}}(p,q,r,m,n,l) \right| \right)}{f\left(m \left[\frac{P_n}{p_n} \right]^{-\alpha} l \right)} = 0$$

for all $p,q,r \in N$, which gives that the sequence (x_{ijk}) is f-upper statistically convergent to L as in theorem.

Theorem 3.7. *Let* (p_i) *a sequence of positive real numbers,* $p_0 > 0$, *and* $P_n = \sum_{k=1}^{n} p_n$ *with* $\lim_{n\to\infty} P_n = \infty$ *and* $\left[\frac{P_n}{p_n} \right] \geq n^{\alpha}$. *Let* (x_{ijk}) *be a triple sequence and let f be a generalized compatible unbounded modulus function. If* (x_{ijk}) *is f-lower strong fractional weighted mean convergent to L, then* (x_{ijk}) *is f-lower statistically convergent to L.*

Proof. The proof is analogous which the proof for the Theorem 3.6.

4. Conclusions and Future Research Directions

A triple sequence that is f-upper and lower strong fractional weighted mean convergent, in some supplementary conditions, also is f-upper and lower statistically convergent.

The new concepts from this article can be applied to the approximation theory, topology, Fourier analysis, analysis interdisciplinary with applications electrical engineering, robotics and other domains.

Future research directions for our fractional models would be using the relation between f-upper and lower strong fractional weighted mean convergent and $M_{\lambda_{m,n,p}}$-statistical convergence for triple sequences (which was studied in [39]), and Wijsman lacunary statistical convergence, ϕ-convergence (which was studied in [40]), and statistical convergence of triple sequences in intuitionistic fuzzy normed spaces (which was studied in [41]).

Future research directions could be done by extending the results in the paper for triple sequences of fuzzy number and by analyzing their statistical convergence in software specific to solving mathematical problems, which apply procedures, codes, etc. Such software can be Maple, MATLAB, Python, C++, etc.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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