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Methods of Variations and Their Applications

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Abstract

In this review article, we begin with reviewing Calculus of variations giving few examples on its use to solve a large number of problems in geometry, physics, and other branches of knowledge. Afterwards, we direct our attention to different methods of variations which evolved during the last century and which include their use in eigenvalue problems and in finite difference methods and those adopted in classical and quantum mechanics. The methods used in evaluating products and quotients of functionals are also discussed along with variational iteration methods. Later on, a good number of applications in different areas are presented and discussed; then a concluding discussion is given.

Keywords

Variation, Iteration, Calculus, Method, Applications, Technique, Extreme

1. Introduction

There is no doubt that the importance of calculus of variations and the variational methods (in general) lies in their great use in solving so many problems in various areas and fields of learning. That usage is, immensely, manifested in the fields of geometry and physics; accordingly, this review article came into light.

In the next section, we present a quick review of calculus of variations giving some of its applications [1] [2]. In Section 3, we introduce the variational theory of eigenvalues [3] [4]. The Rayleigh-Ritz method is then introduced in Section 4 [3] [4]. In Section 5, we present a very important problem, namely, the variational problem of an elastic plate [3] [4] [5].

In Section 6, a very important physical problem is tackled which is the variational method in quantum mechanics [3] [6] [7]. Other selected different applications are given in Section 8 [8]. Finally, we conclude with a discussion in Section 9.

2. A Quick Review of Calculus of Variations

Calculus of variations (CV) deals with taking extreme values of some integrals, for instance when the curve connecting the two pints $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ is rotated about the x-axis so that the resulting surface, due to this roation, is minimum; then this problem concerns CV. In this geometric example and if we denote the surface as S, then S will be a function of the curve y, *i.e.* $S = S(y)$ and the problem is summarized in choosing y such that $S(y)$ is minimum. In other words the goal behind calculus of variations is to find the extreme values (also called stationary values) of an integral I of a certain function f. Here, the necessary conditions to attain such a result are discussed [1] [2].

2.1. One Independent and One Dependent Variables

If the considered integrand is $f(x, y, y')$, where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$; then what we are interested in is to find y such that $I(y)$ is stationary, where

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\mathrm{d}x \tag{1}$$

f is a given function which is differentiable with respect to x, y, and y'; y is the curve joining the two points (x_1, y_1) and (x_2, y_2). Now letting

$$y(x) \to y(x) + \alpha\eta(x), \quad \alpha \quad \text{is a very small parameter} \tag{2}$$

while $\eta(x)$ is an arbitrary function satisfying the condition $\eta(x_1) = \eta(x_2) = 0$. We see that Equation (2) represents an infinite number of curves in terms of the parameter η; we, now, compute δI to get

$$\delta I = \int_{x_1}^{x_2} \left\{ f(x, y + \alpha\eta, y' + \alpha\eta') - f(x, y, y') \right\} \mathrm{d}x \tag{3}$$

Note that the differentiation is with respect to x. From Equation (3), we see that δI can be written as

$$\delta I = \alpha I_1 + O(\alpha^2) \tag{4}$$

where I_1 is given by

$$I_1 = \int_{x_1}^{x_2} \left\{ \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right\} \mathrm{d}x \tag{5}$$

In order that I is an extreme value $I_1 = 0$, *i.e.*

$$\int_{x_1}^{x_2} \left\{ \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right\} \mathrm{d}x = 0 \tag{6}$$

with few mathematical manipulations, which include integration by parts and using the conditions at the end points, one gets

$$\int_{x_1}^{x_2} \eta \left\{ \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right) \right\} \mathrm{d}x = 0 \tag{7}$$

And since η is an arbitrary function, Equation (7) leads to

$$\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)=0 \tag{8}$$

This is the required and necessary condition for I to be an extreme value. Equation (8) is called Euler equation [1] [2].

There are few special cases:

1) f does not depend on y explicitly; this will lead to $\frac{\partial f}{\partial y'}=\text{constant}$.

2) f does not depend on $y' \rightarrow \frac{\partial f}{\partial y}=0$.

3) f does not depend on x explicitly; this implies that $f-y'\frac{\partial f}{\partial y}=\text{constant}$.

Example 1

To evaluate the least value of the surface of revolution (S) of the curve p_1p_2 around the x-axis; we see that $S=2\pi\int_{x_1}^{x_2} y\mathrm{d}s=2\pi\int_{x_1}^{x_2} y\sqrt{1+(y')^2}\mathrm{d}x$; f is then given by $f=y\sqrt{1+(y')^2}$, and using Euler equation we get the solution as $y=b\cosh\left(\frac{x}{b}+d\right)$. This is the well-known catenary curve. The constants b and d can be obtained using the two end points.

Example 2

Since $\mathrm{d}s=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}$ and looking for the shortest distance between two points, we see that $I=\int_1^2 \mathrm{d}s=\int_1^2 \mathrm{d}x\sqrt{1+(y')^2}$, and $f=\sqrt{1+(y')^2}$; using Euler equation, we get $y=ax+b$, a and b are constants; and hence our solution is a straight line as expected.

It is worthwhile to mention that the previous examples are just few examples showing the applications of the calculus of variations in geometry. Moreover, calculus of variations is also very beneficial in physics, especially in mechanics when we deal with several independent and dependent variables, as in the case of studying the motion of a system of particles where we get the famous Lagrange equations.

2.2. Hamilton Principle

As we mentioned before, one of the important applications of calculus of variations is to deal with a number of Euler equations describing a number of dependent variables and an independent one. In this case $f=L$, and L is the Lagrangian given by $L=T-V$, where T is the kinetic energy of the system and V is the potential energy. The independent variable in this case is the time (t), and the dependent variables are the coordinates of particles ($q_i(t), i=1,n$).

Now, mathematically, we state Hamilton principle, used in classical mechanics, as

$$\delta\int L\left(q_1,q_2,\cdots,q_n;q_1',\cdots,q_n',t\right)\mathrm{d}t=0 \tag{9}$$

And hence, applying the same technique, adopted in the subsection 2.1, we get

Euler or Lagrange equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q_i'}\right)-\frac{\partial L}{\partial q_i},\ i=1,2,\cdots,n \tag{10}$$

Note that we deal here with a conservative system (forces are derived from certain potentials) [1].

Example 3

If the Lagrangian is given by $L=m_0c^2\left(1-\sqrt{1-v^2/c^2}\right)-V\left(\vec{r}\right)$, then $f=L$ and $\frac{\partial f}{\partial x_i}=-\frac{\partial V}{\partial x_i}=F_i$; $\frac{\partial f}{\partial x_i'}=\frac{m_0x_i'}{\sqrt{1-v^2/c^2}}$. Applying Euler equations, we get $\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{m_0x_i'}{\sqrt{1-v^2/c^2}}\right\}=-\frac{\partial V}{\partial x_i}=F_i, i=1,2,3$. These equations are Lagrange equations for a relativistic particle moving in a potential V.

2.3. A Number of Independent Variables with One Dependent Variable

Assume, as a special case, that $u=u\left(x,y,z\right)$, and

$$I=\iiint f\left(u,u_x,u_y,u_z,x,y,z\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{11}$$

And where $u_x=\frac{\partial u}{\partial x},u_y=\frac{\partial u}{\partial y},u_z=\frac{\partial u}{\partial z}$; then, the problem of calculus of variations relies in fact on finding u which makes the integral I an extreme value *i.e.* we find the equation governing u, which means that

$$\delta I=\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0}=0 \tag{12}$$

In the same manner we put

$$u\left(x,y,z,\alpha\right)=u\left(x,y,z,0\right)+\alpha\eta\left(x,y,z\right) \tag{13}$$

with some mathematical manipulations we reach the result

$$\frac{\partial f}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u_y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial u_z}\right)=0 \tag{14}$$

Example 4 (Laplace equation)

In electrostatics, it is required that the integral $I=\iiint\nabla^2\varphi\left(x,y,z\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ has to be a minimum; this means that the energy of the system should be minimum. Accordingly we see that $f=\nabla^2\varphi\left(x,y,z\right)=\varphi_x^2+\varphi_y^2+\varphi_z^2$, and hence, applying Equation (14), we obtain $\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0$; which is Laplace equation [1] [2].

2.4. Several Independent and Dependent Variables

If

$$f=f\left(p,p_x,p_y,p_z,q,q_x,q_y,q_z,s,s_x,s_y,s_z,x,y,z\right) \tag{15}$$

where, p, q, and s are functions of the independent variables x, y and z. Following the same steps in the subsection 2.1, we put

$$\begin{aligned} p(\vec{r},\alpha) &= p(\vec{r},0)+\alpha\chi \\ q(\vec{r},\alpha) &= q(\vec{r},0)+\alpha\Psi \\ s(\vec{r},\alpha) &= s(\vec{r},0)+\alpha\varphi \end{aligned} \tag{16}$$

where χ, Ψ, and φ are arbitrary functions.

As in the procedure followed before we get

$$\begin{aligned} \frac{\partial f}{\partial p}-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial p_y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial p_z}\right)=0 \\ \frac{\partial f}{\partial q}-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial q_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q_y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial q_z}\right)=0 \\ \frac{\partial f}{\partial s}-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial s_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial s_y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial s_z}\right)=0 \end{aligned} \tag{17}$$

Equations (17) are Euler equations. In general, if the independent variables are $x_i\,(i=1,n)$ and the dependent ones are $y_j\,(j=1,m)$, then Euler equations are

$$\frac{\partial f}{\partial y_i}-\sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial y_{ij}}\right)=0,\ \text{where } y_{ij}=\frac{\partial y_i}{\partial x_j} \tag{18}$$

2.5. Lagrange Multipliers

Lagrange multipliers are involved when constraints are present in a physical problem; the constraint is usually a function relating the independent variables, for example the motion of the body is constrained to be on a plane (e.g. in the x-y plane where $z = 0$ in this case). In this situation Lagrange multipliers play a very vital role in solving the given problem in a simple way as will be shown in this section.

Let us consider the equation

$$\mathrm{d}f = f_x\mathrm{d}x + f_y\mathrm{d}y + f_z\mathrm{d}z = 0 \tag{19}$$

which is an equation leading to the extreme values of f; this requires that $f_x = f_y = f_z = 0$; we note that in most problems there exists one or more constraints of the type

$$\varphi(x,y,z)=0 \tag{20}$$

From which we see that

$$\mathrm{d}\varphi=\varphi_x\mathrm{d}x+\varphi_y\mathrm{d}y+\varphi_z\mathrm{d}z=0 \tag{21}$$

From Equation (19) and Equation (21), we get

$$\mathrm{d}f+\lambda\mathrm{d}\varphi=(f_x+\lambda\varphi_x)\mathrm{d}x+(f_y+\lambda\varphi_y)\mathrm{d}y+(f_z+\lambda\varphi_z)\mathrm{d}z=0 \tag{22}$$

Now we can choose x and y as independent variables so that

$$f_x+\lambda\varphi_x=0 \tag{23}$$

and

$$f_y+\lambda\varphi_y=0, \tag{24}$$

while we choose λ so that

$$f_z + \lambda \varphi_z = 0 \tag{25}$$

Accordingly, $\mathrm{d}f = 0$ implies that *f* has an extreme value. Note also that we have four variables $\{x, y, z, \lambda\}$ governed by four Equations (20, 23, 24, 25). In general, if we have *k* constraints, we will get Euler equations as [1]

$$\frac{\partial f}{\partial x_i} - \sum_k \lambda_k \frac{\partial \varphi_k}{\partial x_i} = 0; i = 1, 2, \cdots, n \tag{26}$$

Example 5 (Particle in a box)

Here, we present the benefit of using Lagrange multipliers in a quantum mechanical problem: A box, in the form of a rectangular parallelopiped shape whose sides are *a*, *b*, and *c*, contains a particle with mass *m* and energy $E(a,b,c) = \frac{h^2}{8m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$; we search for the shape of the box so as E is minimum with the constraint that the volume of the box is fixed (constant), *i.e.* $V(a,b,c) = abc = K$; *K* is a constant.

It is clear that $f = E$ and $\varphi = abc - K$; hence, adopting the Lagrange multipliers procedure, we get the equations $\frac{\partial E}{\partial a} + \lambda \frac{\partial V}{\partial a} = \frac{-h^2}{4ma^3} + \lambda bc = 0$, $\frac{\partial E}{\partial b} + \lambda \frac{\partial V}{\partial b} = \frac{-h^2}{4mb^3} + \lambda ac = 0$, and $\frac{\partial E}{\partial c} + \lambda \frac{\partial V}{\partial c} = \frac{-h^2}{4mc^3} + \lambda ab = 0$, from these equations, we get $a = b = c$, which means that the box, for the particle with minimum energy, is a cube [1].

Example 6 (The simple pendulum)

This is a well-known problem from classical mechanics and the Lagrangian, and hence *f*, here is given by $L = f = T - V = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$. While the constraint is $r = l = 0$. Note that we are using polar coordinates, and *m*, *g*, and *l* are the mass, the gravitational acceleration and the length of the string of the pendulum respectively.

Using the above Langrangian and Euler equations we get the two equations $\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta - \lambda = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) + mgr\sin\theta = 0$; λ gives the tension in the string (this is one of the beautiful outcomes of using calculus of variations). The second equation leads to the usual equation governing the motion of the simple pendulum. We note that if θ is small, then we get from the second equation: the equation $m\ddot{\theta} + \frac{g}{l}\sin\theta = 0$, which represents the equation of simple harmonic motion [1] [2].

Example 7 (Schrodinger wave equation)

This is a very interesting problem applied in quantum mechanics where we require that $\delta\iiint \Psi^*(x,y,z) H \Psi(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$; with the constraint $\iiint \Psi^*\Psi \mathrm{d}x\mathrm{d}y\mathrm{d}z = 1$. H is the Hamiltonian given by $H = -\frac{h^2}{8\pi m}\nabla^2 + V(x,y,z)$ which represents the total energy of the particle; while the constraint refers to the probability of finding the particle somewhere in space. Choosing a large surface and integrating we get $\delta\iiint\left[\frac{h^2}{8\pi m}\nabla\Psi^*\cdot\nabla\Psi+V\Psi^*\Psi\right]\text{d}x\text{d}y\text{d}z=0$, therefore, we obtain the function to deal with, in writing Euler equation, as $g=\frac{h^2}{8\pi m}\left[\nabla\Psi^*\cdot\nabla\Psi+V\Psi^*\Psi\right]-\lambda\Psi^*\Psi$. Using Ψ^* as the independent variable we get Schrodinger equation $-\frac{h^2}{8\pi m}\nabla^2\Psi+V(x,y,z)\Psi=\lambda\Psi$ [1] [2]. λ represents the energy eigenvalue.

3. Variational Theory of Eigenvalues

In this section, we study the relation of the variational methods with Sturm-Louiville problem; if we consider the following functional

$$I(y)=\int_a^b\left[P(x)y'^2+Q(x)y^2\right]\text{d}x \tag{27}$$

with the boundary conditions

$$y(a)=0=y(b) \tag{28}$$

and if $P(x)$ and $Q(x)$ are two continuous functions in the interval $a\le x\le b$ with $P(x)>0$ in the same interval, then the given functional satisfies Euler equation which is

$$\left[P(x)y'\right]'-Q(x)y=0 \tag{29}$$

Moreover, if we have the constraint on $y(x)$ as

$$\int_a^b w(x)y^2\text{d}x=1 \tag{30}$$

where $w(x)$ is a continuous in [*a*, *b*] and $w(x)>0$; then the functional $I(y)$ is bounded below and

$$I(y)\ge\int_a^b Q(x)y^2\text{d}x\ge\min_{[a,b]}\frac{Q(x)}{w(x)}\int_a^b w(x)y^2\text{d}x=\min_{[a,b]}\frac{Q(x)}{w(x)} \tag{31}$$

We put $y=y_1$, which satisfies Equation (28) and Equation (30) and which makes $I(y)$ to attain minimum values [3].

Using Lagrange multiplies technique, we can easily see that the function y_1 satisfies the equation

$$\left[P(x)y_1'\right]'-\left[Q(x)-\lambda_1 w(x)\right]y_1(x)=0 \tag{32}$$

Hence we rewrite Equation (32) in the shape

$$L[y_1]=\lambda_1 w(x)y_1 \tag{33}$$

and the linear operator L is given by

$$L(y)=-\left[P(x)y'(x)\right]'+Q(x)y \tag{34}$$

The last equation shows that $y_1(x)$ is an eigenfunction corresponding to the eigenvalue λ_1.

Now, if $U(x)$ and $V(x)$ are two functions satisfying the boundary conditions given by Equation (28), and using Equation (34) with integration by parts, we get

$$\int_a^b ULV\mathrm{d}x = \int_a^b VLU\mathrm{d}x \tag{35}$$

Note that the two functions $U(x)$ and $V(x)$ can be considered as elements from the Hilbert space $L_2[a,b]$. Hence L is a self-adjoint operator.

Once we compose the function y_1, we proceed to determine the function y_2 which makes the functional $I(y)$ a minimum value satisfying the boundary conditions (Equation (28)) and with the constraint given by Equation (30) and the constraint

$$\int_a^b w(x)\, yy_1 \mathrm{d}x = 1 \tag{36}$$

Again, we use Lagrange multipliers to get

$$L(y_2) = \mu_1 wy_1 + \lambda_2 wy_2 \tag{37}$$

with some mathematical manipulations and with reference to the involved equations, we reach the result that $\mu_1 = 0$. Hence, y_2 is an eigenfunction of L with eigenvalue λ_2.

We can proceed, in the same manner, to obtain other elements of the Hilbert space, $y_i, i = 3,4,\cdots$, which will form an infinite sequence of orthonormal set of functions [4].

Moreover we can easily prove, through integration by parts, that

$$I(y(x)) = \int_a^b L[y(x)]\, y(x)\mathrm{d}x + \left[P(x)\, yy'\right]_a^b \tag{38}$$

From Equation (28) and the equation $L(y_n) = \lambda_n y_n w$, we get

$$I[y_n] = \int_a^b w\lambda_n y_n^2 \mathrm{d}x = \lambda_n;\ n = 1,2,3,\cdots \tag{39}$$

and hence, we obtain

$$I[y_1] \le I[y_2] \le I[y_3] \le \cdots \tag{40}$$

Note that if one adds an additional constraint, then the set of functions under investigation will be reduced. For instance, if one takes into account the functional

$$I[y(x)] = \int_0^l y'^2 \mathrm{d}x \tag{41}$$

with the condition

$$y(0) = 0 = y(l) \tag{42}$$

Moreover if the weight function is of the form $w(x) = 1$; then, we get

$$y'' + \lambda y = 0 \tag{43}$$

In this case, it is clear that the solutions are given by

$$y_k = \sqrt{\frac{2}{l}} \sin\frac{k\pi x}{l};\ \lambda_k = \left(\frac{k\pi}{l}\right)^2,\ k = 1,2,3,\cdots \tag{44}$$

where the factor $\sqrt{\frac{2}{l}}$ is a normalization one [4].

We have to note that if $P(x)$ and $Q(x)$ or the two ends *a* and *b* were changed, then the eigenvalues also get changed. There are various cases which we summarize as follows:

If $P(x)$ and $Q(x)$ are replaced by $\bar{P}(x)$ and $\bar{Q}(x)$ respectively, satisfying the inequalities

$$\bar{P}(x) \geq P(x);\ \bar{Q}(x) \geq Q(x) \text{ with } a \leq x \leq b \tag{45}$$

Then every new eigenvalue $\bar{\lambda}_n$ is a neighbor of the corresponding one λ_n; e.g. $\bar{\lambda}_1 = I(\bar{y}_1) \geq I(y_1) = \lambda_1$.

To assert our conclusion, cited above, we introduce Courant's theorem which states that given an $n-1$ of piecewise continuous functions $\{p_1, p_2, \cdots, p_{n-1}\}$ in G, and if $d(p_1, p_2, \cdots, p_{n-1})$ is the greatest lower bound of the resulting eigenvalues for the functional $\bar{I}(\varphi)$, where φ is any continuous function with piecewise continuous derivatives with $H(\varphi) = 0$ and orthogonality conditions given as

$$H(\varphi, \rho_i) = 0, (i = 1, 2, \cdots, n-1) \tag{46}$$

Then, λ_n is the greatest eingenvalue which implies that the lower bound d is attained when the set of functions $\{p_1, p_2, \cdots, p_{n-1}\}$ ranges over all acceptable functions and the minimax is satisfied when

$$u = u_n, p_1 = u_1, p_2 = u_2, \cdots, p_{n-1} = u_{n-1} \tag{47}$$

Accordingly, λ_n is the greatest element among the eigenvalues, and this property is called the minimax property for the eigenvalues [4].

Moreover, if $Q(x) \to Q(x) + c$, then all the obtained values of the functional increase with the same amount *c*.

Now, if the eigenvalues are ordered in the manner

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots \tag{48}$$

And putting $z(t) = (wP)^{1/4} u$ and $t = \int_a^x \left(\frac{w}{P}\right)^{1/2} \mathrm{d}x$, and using the analysis performed before in the last subsection, we see that Sturm-Louiville problem will take the form

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} - f(t) z + \lambda z = 0 \tag{49}$$

where $f(t)$ is a continuous function. Equation (49) can be written as

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \lambda' z = 0 \tag{50}$$

On the interval $[0, \tau]$; and τ conforms with the eigenvalue *t*. The boundary conditions are, then, $z(0) = 0 = z(\tau)$; moreover, we see that

$$\lambda = \min \left\{ \int_0^\tau \left(\left[\frac{\mathrm{d}z}{\mathrm{d}t} \right]^2 + fz^2 \right) \mathrm{d}t \right\} \tag{51}$$

and

$$\lambda' = \min\left\{\int_0^\tau \left[\frac{\mathrm{d}z}{\mathrm{d}t}\right]^2 \mathrm{d}t\right\} \tag{52}$$

From which we get

$$\lambda \le \lambda' + \int_0^\tau f\left(z'\right)^2 \mathrm{d}t \tag{53}$$

Leading to the fact that λ differs from λ' by a finite quantity [3].

Note that, from Equation (50) along with boundary conditions, the eigenvalues are $\lambda_n' = \frac{n^2\pi^2}{\tau^2} = cn^2$, which implies that $\lambda_n \approx cn^2$ as $n \to \infty$. This reflects the asymptotic behavior of the eigenvalues [3] [4]. This is of course true when we deal with a finite interval; but when the range of t is not finite, the last result is not true. For instance, for Hermite equation with an infinite domain, the eigenvalues are proportional to n instead of n^2 [4].

4. Finite Difference Method (Euler's Equation)

In this case, the basic idea is that the values of the functional

$$I(y) = \int_a^b f\left(x, y, y'\right)\mathrm{d}x;\ \ y\left(x_1\right) = A, y\left(x_2\right) = B \tag{54}$$

are taken on multi-curves formed by a finite number of straight line segments defined by the coordinates $x_1(=a), x_1 + \Delta x, \cdots, x_1 + (n-1)\Delta x, x_2(=b)$; $\Delta x = \frac{x_2 - x_1}{n}$. Then the functional is reduced to the form $\varphi\left(y_1, y_2, \cdots, y_n\right)$; the extreme values of I are determined from the equations

$$\frac{\partial \varphi}{\partial y_i},\ \ i = 1, 2, , n \tag{55}$$

From which, one gets

$$\varphi_y\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\varphi_{y'}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \varphi_{y'}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right)}{\Delta x} = 0 \tag{56}$$

These equations give the approximate solution to the variational problem, while if we take the limits $\Delta x \to 0$ and $n \to \infty$, we obtain the usual Euler equation [3] [4]

$$\varphi_y - \frac{\mathrm{d}}{\mathrm{d}x}\left(\varphi_{y'}\right) = 0 \tag{57}$$

5. Rayleigh-Ritz Method

Here, the values of the functional are taken as a linear combination of the acceptable arbitrary functions, *i.e.* on y_n, where

$$y_n = \sum_{i=1}^n \alpha_i w_i(x) \tag{58}$$

α_i are constants and $\left\{w_1, w_2, w_3, \cdots, w_n\right\}$ is a suitable set of functions. These equations with the determination of the constraints will lead to a functional of the form $\varnothing(\alpha_1,\alpha_2,\cdots,\alpha_n)$ with the coefficients α_i's; so that $\varnothing$ attains an extreme value. Moreover, if the limit exists when $n\to\infty$, then we have a variational problem; otherwise, we get an approximate solution to the problem.

If the boundary conditions are of the homogeneous and linear type $\left[y(x_1)=0=y(x_2)\right]$, then the method of handling the variational problem will be easier in choosing the functions w_i; for example, we can choose the functions as

$$w_i=(x-x_1)(x-x_2)\varphi_i(x) \tag{59}$$

while if the boundary conditions are not homogeneous and of the form $y(x_1)=y_1, y(x_2)=y_2$; then we search for a solution of the type $y_n=\sum_{i=1}^{n}\alpha_i w_i(x)+w_0(x)$ with $w_0(x_1)=y_1$ and $w_0(x_2)=y_2$. This will produce the previous boundary conditions $y(x_1)=0=y(x_2)$.

It is easy to prove that the system of functions given as

$$w_i(x)=x^{i-1}(x-x_1)(x-x_2),\ i=1,2,3,\cdots \tag{60}$$

and

$$w_k=\sin\frac{k\pi(x-x_1)}{x_2-x_1},\ k=1,2,3,\cdots \tag{61}$$

Is linearly independent and form a complete set in $C^1[x_1,x_2]$ [3].

According to the previous piece of information, let us discuss getting the extreme value of the functional using the Rayleigh-Ritz method:

$$I(y)=\int_0^1\left(y'^2+y^2\right)\mathrm{d}x;\ y(0)=0, y(1)=1 \tag{62}$$

The exact solution of the Euler equation in this case is $y_{\text{exact}}=\frac{\sinh x}{\sinh 1}$ [4]. Now putting $w_0(x)=x$ and taking into account Equation (60), we get

For $n=1$

$$y_{1,\text{approximate}}=x+cx(1-x),\ c \text{ is a constant.} \tag{63}$$

Substituting Equation (63) in Equation (62), and putting $\frac{\mathrm{d}I}{\mathrm{d}c}=0$, we obtain the value of the constant as $c=-\frac{5}{22}$.

In the same manner, for $n=2$ we get $y_{2,\text{app.}}=x+c_1x(1-x)+c_2x^2(1-x)$, and following a similar procedure (by putting $\frac{\partial I}{\partial c_1}=0=\frac{\partial I}{\partial c_2}$),we find that the approximate solution is

$$y_{2,\text{app.}}=0.8541x-0.0169x^2+0.1628x^3. \tag{64}$$

Table 1 shows a comparison between the exact and approximate solutions [3] [4] [5].

6. Variational Problem of an Elastic Plate

If we consider the matter of evaluating the amplitude, of the very small deviations

Table 1. A comparison between the exact and approximate solutions for $n = 1$ and $n = 2$, using Ratleigh-Ritz Method.

x	$y(x)$	$y_1(x)$	$y_2(x)$	error1	error2
0	0	0	0	0	0
0.1	0.085234	0.079545	0.085404	0.005689	0.000170
0.2	0.171320	0.163636	0.171446	0.1549564	0.000126
0.3	0.259122	0.252273	0.259105	0.006849	0.000017
0.4	0.349517	0.345455	0.349355	0.004062	0.000162
0.5	0.443409	0.443182	0.443175	0.000227	0.000234
0.6	0.541740	0.545455	0.541541	0.003715	0.000199
0.7	0.645493	0.652273	0.645429	0.006780	0.000064
0.8	0.755705	0.763636	0.755818	0.007931	0.000113
0.9	0.873482	0.879545	0.873682	0.006063	0.000200
1	1	1	1	0	0

for an elastic plate with small thickness, g with $P(x,y)$ as the acting force; then the related differential equation is

$$D\nabla^4 g - \rho\omega^2 g - P = 0 \tag{65}$$

where D is the rigidity constant, ω is the angular velocity, and $\rho(x)$ is the linear mass density of the plate. Our plate is a rectangular one with dimensions $0 \le x \le a$ and $0 \le y \le b$. Multiplying Equation (65) by δg and integrating on the plate surface, we get

$$\begin{aligned}\nabla^4 g\delta g = {} & \delta\left(\frac{1}{2}g_{xx}^2\right) + \left(g_{xxx}\delta g - g_{xx}\delta g_x\right)_x + \delta\left[(1-\alpha)g_{xy}^2 + \alpha g_{xx}g_{yy}\right] \\ & + \left[(2-\alpha)g_{xyy}\delta g - \alpha g_{yy}\delta g_x\right]_x + \left[(2-\alpha)g_{xxy}\delta g - \alpha g_{xx}\delta g_y\right]_y \\ & - 2(1-\alpha)\left[g_{xy}\delta g\right]_{xy} + \delta\left(\frac{1}{2}g_{yy}^2\right) + \left[g_{yyy}\delta g - g_{yy}\delta g_y\right]_y\end{aligned} \tag{66}$$

α is an arbitrary parameter which takes the values between 0 and 1.5; it is called Poisson's ratio and depends on the matter of the plate [3]. Note that similar quantities are evaluated applying the given boundary conditions. Moreover, if the boundary conditions are such that the plate does not move or rotate (*i.e.* the plate is fixed), then $g = 0$ on the whole boundary which implies that $\frac{\partial g}{\partial x} = 0$ on the whole boundary ($\partial = c$) and $\frac{\partial g}{\partial y} = 0$ on $y = 0$ and $y = b$.

Having in mind that the variations go to zero according to the proposed conditions and performing integration by parts, the variational problem becomes

$$\delta\int_0^a\int_0^b\left\{\frac{1}{2}D\left[g_{xx}^2 + g_{yy}^2 + 2\alpha g_{xx}g_{yy} + 2(1-\alpha)g_{xy}^2\right] - Pg - \frac{1}{2}\rho\omega^2 g^2\right\}\mathrm{d}x\mathrm{d}x = 0 \tag{67}$$

D is the stress energy per unit area, while $-Pg$ is the additional stress energy due to loading, and the term $\frac{1}{2}\rho\omega^2 g^2$ represents the kinetic energy per unit area. All these quantities are estimated at maximum deflection; hence we obtain the natural boundary conditions for the problem via putting the boundary integrals of the integrand equal to zeros. For instance, on the boundaries $x = 0, x = a$, we have one of two alternatives:

$$D\frac{\partial \nabla^2 g}{\partial x} + (1-\alpha) D g_{xyy} = 0 \text{ ; values on } g \tag{68}$$

or

$$D\left(g_{xx} + \alpha g_{yy}\right) = 0 \text{ ; values on } g_x \tag{69}$$

Similarly, for the boundary conditions $y = 0, y = b$, we have the two choices

$$D\frac{\partial \nabla^2 g}{\partial y} + (1-\alpha) D g_{yxx} = 0 \text{ ; values on } g \tag{70}$$

or

$$D\left(g_{yy} + \alpha g_{xx}\right) = 0 \text{ ; values on } g_y \tag{71}$$

From the theory of elasticity, the amount $D\frac{\partial \nabla^2 g}{\partial x}$ represents the shear stress on the boundary $x = c$, c is a constant; and the amount $-(1-\alpha) D g_{xy}$ is the torsion, while the quantity $-D\left(g_{xx} + \alpha g_{yy}\right)$ is the torque (M_{xx}) [5]. Moreover, we see from Equation (67) that the effective force on the transverse edge along $x = c$ is δg and is equal to $R_x = D\frac{\partial \nabla^2 g}{\partial x} - \frac{\partial M_{xy}}{\partial y}$. This physical result led to the distinct advantage in the theory of small deflections for an elastic plate [5].

7. Variational Methods in Quantum Mechanics

It is a very difficult job to solve Schrodinger Equation (SE) for a multi-electrons atomic system, but using variational methods we can obtain approximate solutions for such a system. If we take into consideration the ground state of an arbitrary system, then the eigenfunction Ψ_0 corresponding to the energy E_0 obeys SE and

$$\mathbb{H}\Psi_0 = E_0 \Psi_0 \tag{72}$$

$\mathbb{H}$ is the Hamiltonian. From Equation (72), we can deduce that

$$E_0 = \frac{\int \Psi_0^* \mathbb{H} \Psi \, \mathrm{d}\tau}{\int \Psi_0^* \Psi \, \mathrm{d}\tau} \tag{73}$$

Now, replacing Ψ_0 by $\varnothing$, we get

$$E_\varnothing = \frac{\int \varnothing^* \mathbb{H} \varnothing \, \mathrm{d}\tau}{\int \varnothing^* \varnothing \, \mathrm{d}\tau} \tag{74}$$

Hence, from variational point of view, it is easy to prove that $E_\varnothing \geq E_0$ [3]. This can be seen as follows:

Since

$$\mathbb{H}\Psi_n = E_n\Psi_n \tag{75}$$

and $\varnothing$ is an approximate solution; hence it can be written as a linear combination in $\{\Psi_n\}$ as

$$\varnothing = \sum_n c_n \Psi_n \tag{76}$$

where

$$c_n = \int \Psi_n^* \varnothing \mathrm{d}\tau \tag{77}$$

It is simple to show, also, that for any eigenfunction Ψ_k, c_k is given by

$$c_k = \int \Psi_k^* \varnothing \mathrm{d}\tau \tag{78}$$

Note that we assume that the set $\{\Psi_n\}$ is a set of normalized eigenfunctions.

Substituting Equation (76) in the variational Equation (74), we get

$$E_\varnothing - E_0 = \frac{\sum_n c_n^* c_n (E_n - E_0)}{\sum_n c_n^* c_n} \tag{79}$$

From the last equation, it is clear that

$$E_\varnothing - E_0 \ge 0 \to E_\varnothing \ge E_0 \tag{80}$$

This is because $\sum_n c_n^* c_n > 0$ and $E_n \ge E_0$.

It is to be noted here that the reason behind presenting the above paragraph is to shed light on the process of finding an arbitrary wave function $\varnothing$ which gives the lowest possible value for the energy $E_\varnothing$ [3].

We choose $\varnothing$ which depends on certain parameters $\{\alpha, \beta, \gamma, \cdots\}$; these parameters are called the variational parameters and the energy is a function of them *i.e.*

$$E_\varnothing(\alpha, \beta, \gamma, \cdots) \ge E_0 \tag{81}$$

To proceed towards the evaluation of the lowest value of the energy $E_\varnothing$, which corresponds to that eigenstate nearest to Ψ_0, we give the following example:

The Helium Atom

The Hamiltonian for the He atom is given by

$$\mathbb{H} = -\frac{\hbar^2}{2m_e}\left(\nabla_1^2 + \nabla_1^2\right) - \frac{2e^2}{4\pi\varepsilon_0}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r_{12}},\ r_{12} = |r_1 - r_2| \tag{82}$$

Equation (82) can be written as

$$\mathbb{H} = \mathbb{H}_{H_1} + \mathbb{H}_{H_2} + \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r_{12}} \tag{83}$$

where

$$\mathbb{H}_{H_j} = -\frac{\hbar^2}{2m_e}\nabla_j^2 - \frac{2e^2}{4\pi\varepsilon_0}\frac{1}{r_j};\ j = 1, 2. \tag{84}$$

$\mathbb{H}_{H_j}$ represents the Hamiltonian for the two single electrons of the He atom and they satisfy the eigenvalue problem

$$\mathbb{H}_{H_j}\Psi_H\left(r_j,\theta_j,\varphi_j\right)=E_j\Psi_H\left(r_j,\theta_j,\varphi_j\right);\ j=1,2 \tag{85}$$

Ψ_H is the wave function similar to that of the Hydrogen atom [6], and the corresponding eigenenergy, E_j, is given by

$$E_j=-\frac{Z^2 m_e e^4}{32\pi^2\varepsilon_0^2\hbar^2 n_j^2};\ j=1,2 \tag{86}$$

If the repulsion between the two electrons is ignored, then the Hamiltonian is separable and the ground state wave function takes the form

$$\varnothing_0=\Psi_{1,s}\left(r_1\right)\Psi_{2,s}\left(r_2\right) \tag{87}$$

where

$$\Psi_{j,s}\left(r_j\right)=\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{1/2}\exp\left(-\frac{Zr_1}{a_0}\right);\ j=1,2 \tag{88}$$

a_0 is Bohr radius.

Now, we use $\varnothing_0$ as the arbitrary function and using Z as a variayional parameter, *i.e.* we write

$$E(Z)=\int\varnothing_0^*\left(r_1,r_2\right)\mathbb{H}\varnothing_0\left(r_1,r_2\right)\mathrm{d}r_1\mathrm{d}r_2 \tag{89}$$

Using the last equation one gets

$$E(Z)=\frac{m_e e^4}{16\pi^2\varepsilon_0^2\hbar^2}\left(Z^2-\frac{27}{8}Z\right) \tag{90}$$

Differentiating with respect to Z one gets the minimum value of the energy as $E_{\min}=-2.8477\left[\frac{m_e e^4}{16\pi^2\varepsilon_0^2\hbar^2}\right]=-2.8477\ \text{hartree}$. $E_{\min}$ is the ground state energy of the He atom.

The above calculations, related to the ground state energy for He atom, can also be obtained via perturbation techniques and the results are comparable [7].

Variational Method and the Characteristic Determinants

If there exists symmetry in the problem, the eigenfunction can be formulated accordingly. For instance, in the problem of a particle in a box, the ground state wave function is symmetrical about the center of the box [6].

As seen before the eigenfunction $\varnothing$ can be written as a linear combination of the functions $\{f_n\}$, *i.e.*

$$\varnothing=\sum_{n=1}^{n=N}c_n f_n \tag{91}$$

The c_n's are the variational parameters and $E_\varnothing$ is evaluated as

$$E_\varnothing=\frac{\int\varnothing^*\mathbb{H}\varnothing\mathrm{d}\tau}{\int\varnothing^*\varnothing\mathrm{d}\tau}=\frac{\sum_{n,m}c_n^*c_m E_m H_{nm}}{\sum_{n,m}c_n^*c_m S_{nm}} \tag{92}$$

where

$$H_{nm}=\int f_n^*\mathbb{H}f_n\mathrm{d}\tau\ \text{ and }\ S_{nm}=\int f_n^* f_n\mathrm{d}\tau \tag{93}$$

and since the Hamiltonian is Hermitian, $H_{nm}=H_{mn}$ and $S_{nm}=S_{mn}$. These quantities are called matrix elements. The energy is now expressed in terms of the coefficients $c_1, c_2, \cdots, c_N$ and

$$E_{\varnothing} = E_{\varnothing}\left(c_1, c_2, \cdots, c_N\right) \tag{94}$$

Differentiating with respect to the N coefficients, we get N equations in N in knowns.

Example 8 (Particle in a box)

If the boundaries of the box are at $x = 0$ and $x = 1$, and if we take the linear combination of the two functions $f_1 = x(1-x)$ and $f_2 = x^2(1-x)^2$; and since the Hamiltonian, in this case, is $\mathbb{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, we see that

$H_{12} = \int f_1 \mathbb{H} f_2 \mathrm{d}\tau = -\frac{\hbar^2}{2m}\int_0^1 x(1-x)\left[2 - 12x + 12x^2\right]\mathrm{d}x = \frac{\hbar^2}{30m}$; similarly we get the following results for the rest of the matrix elements

$H_{11} = \frac{\hbar^2}{6m}, H_{22} = \frac{\hbar^2}{105m}, S_{11} = \frac{1}{30}, S_{12} = \frac{1}{140}, S_{22} = \frac{1}{630}$.

Hence, the energy is given by $E(c_1, c_2) = \frac{c_1^2 H_{11} + 2c_1c_2H_{12} + c_2^2 H_{22}}{c_1^2 S_{11} + 2c_1c_2S_{12} + c_2^2 S_{22}}$; differentiating with respect to the two parameters, we get the two equations $c_1\left(H_{11} - ES_{11}\right) + c_2\left(H_{12} - ES_{12}\right) = 0$ and $c_1\left(H_{12} - ES_{12}\right) + c_2\left(H_{22} - ES_{22}\right) = 0$. Accordingly, we have a solution to these equations only if

$\begin{vmatrix} H_{11} - ES_{11} & H_{12} - ES_{12} \\ H_{12} - ES_{12} & H_{22} - ES_{22} \end{vmatrix} = 0$; from which we obtain the characteristic equation whose solution leads to $E_{\min} = 0.125002\frac{\hbar^2}{m}$ and this is exactly equal to the ground state energy (E_0) for the particle (m) in a box [3] [8].

8. More Applications

In this section we give few more applications on the use of variational methods in different areas.

8.1. Product and Quotient of Functionals

Calculus of variations, as mentioned before, deals with functionals of the form

$$I = \int_{x_0}^{x_1} f\left(x, y(x), y'(x)\right)\mathrm{d}x \tag{95}$$

where $\left(x, y(x), y'(x)\right)$ belongs to a suitable space. Note that there are functionals with their minimum values not necessarily equal to the extreme values of the functionals. However, we shall deal, here, with a functional which is given by

$$F = F\left[\int_{x_0}^{x_1} f\left(x, y(x), y'(x)\right)\mathrm{d}x\right] \tag{96}$$

with $f = f\left(f_1, f_2, \cdots, f_n\right)$ and F is a function of n independent variables.

In particular, if we consider ($n = 2$), then, we get a product and a quotient of the forms

$$P=\left(\int_{x_0}^{x_1} f_1\left(x,y(x),y'(x)\right)\mathrm{d}x\right)\left(\int_{x_0}^{x_1} f_2\left(x,y(x),y'(x)\right)\mathrm{d}x\right) \tag{97}$$

and

$$Q=\frac{\int_{x_0}^{x_1} f_1\left(x,y(x),y'(x)\right)\mathrm{d}x}{\int_{x_0}^{x_1} f_2\left(x,y(x),y'(x)\right)\mathrm{d}x} \tag{98}$$

Respectively. P can also be written in the form of double integral as

$$P=\int_{x_0}^{x_1}\int_{x_0}^{x_1} f_1\left(t,y(t),y'(t)\right) f_2\left(s,y(s),y'(s)\right)\mathrm{d}t\mathrm{d}s \tag{99}$$

These products and quotients will be of value in tackling some problems

8.2. Euler-Lagrange Equations

To get Euler-Lagrange equations related to the functional in Equation (95), one has to derive the ideal and necessary conditions involved in the theory of calculus of variations using Taylor series for the functional dependent on the first variational parameter. This is clarified by the following theorem.

Theorem 1

If the function in Equation (96) with the functional G_i is given by

$$G_i=\int_{x_0}^{x_1} f_i\left(x,y(x),y'(x)\right)\mathrm{d}x \tag{100}$$

and if it is expansion using Taylor series up to second order terms with respect to the variable i, then the Euler-Lagrange, related to F, is given by

$$\sum_{i=1}^{n} F_i'\left\{f_{iy}-\frac{\mathrm{d}}{\mathrm{d}x}\left(f_{iy'}\right)\right\}=0 \tag{101}$$

where F_i' is the partial derivative of F with respect to the i^{th} variable. Moreover the second variation is given by [8]

$$\begin{aligned}\delta^2F=&\frac{1}{2}\sum_{i=1}^{n}F_i'\left[f_{iyy}\delta y^2+2f_{iyy'}\delta y\delta y'+f_{iy'y'}\delta y'\delta y'\right]\mathrm{d}x\\&+\frac{1}{2}\sum_{i,j}^{n}F_{ij}''\int_{x_0}^{x_1}\int_{x_0}^{x_1}\left[f_{iy}-\frac{\mathrm{d}}{\mathrm{d}x}\left(f_{iy'}\right)\right]\left(r,y(r),y'(r)\right)\\&\times\left[f_{iy}-\frac{\mathrm{d}}{\mathrm{d}x}\left(f_{iy'}\right)\right]\left(s,y(s),y'(s)\right)\delta y(r)\delta y(s)\mathrm{d}r\mathrm{d}s\end{aligned} \tag{102}$$

Proof

Getting $G_i\left(x_0,x_1,y(x)\right)$ as a Taylor series up to second order implies the evaluation of δG_i as

$$\delta G_i=G_i\left(x_0,x_1,y(x)+\delta y(x)\right)-G_i\left(x_0,x_1,y(x)\right) \tag{102}$$

and hence, we get

$$\begin{aligned}\delta G_i=&\int_{x_0}^{x_1}\left\{f_{iy}-\frac{\mathrm{d}}{\mathrm{d}x}\left(f_{iy'}\right)\right\}\delta y\mathrm{d}x\\&+\frac{1}{2}\int_{x_0}^{x_1}\left[f_{iyy}\delta y^2+2f_{iyy'}\delta y\delta y'+f_{iy'y'}\delta y'y'\right]\mathrm{d}x\end{aligned} \tag{103}$$

Similarly

$$\begin{aligned}
\delta F &= F\left(G_1\left(x_0, x_1, y(x)+\delta y(x)\right), \cdots, G_n\left(x_0, x_1, y(x)+\delta y(x)\right)\right) \\
&\quad - F\left(G_1\left(x_0, x_1, y(x)\right), \cdots, G_n\left(x_0, x_1, y(x)\right)\right) \\
&\approx \sum_{i=1}^{n} \int_{x_0}^{x_1} F_1' \left[f_{iy} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f_{iy'}\right)\right] \delta y \mathrm{d}x \\
&\quad + \frac{1}{2} \sum_1^n \int_{x_0}^{x_1} F_1' \left[f_{iyy} \delta y^2 + 2 f_{iyy'\delta y \delta y'} + f_{iy'y'\delta y'^2} \right] \mathrm{d}x \\
&\quad + \frac{1}{2} \sum_1^n F_{ij}'' \left(\int_{x_0}^{x_1} \left[f_{iy} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f_{iy'}\right)\right] \delta y \mathrm{d}x \right) \left(\int_{x_0}^{x_1} \left[f_{iy} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f_{iy'}\right)\right] \delta y \mathrm{d}x \right)
\end{aligned} \tag{104}$$

Note that when $\delta y = 0$ in the first term of the last equations, we get Euler-Lagrange equation. Moreover, one can, also, derive the second order term of the expansion for the second variation [3] [8].

8.3. The Product Functional

For the product functional, we see that

$$\begin{aligned}
\delta P &= \int_{x_0+\alpha_0}^{x_1+\alpha_1} f_1\left(x, y(x)+w(x), y'(x)+w'(x)\right) \mathrm{d}x \\
&\quad \times \int_{x_0+\alpha_0}^{x_1+\alpha_1} f_2\left(x, y(x)+w(x), y'(x)+w'(x)\right) \mathrm{d}x \\
&\quad - \int_{x_0}^{x_1} f_1\left(x, y(x), y'(x)\right) \mathrm{d}x \int_{x_0}^{x_1} f_2\left(x, y(x), y'(x)\right) \mathrm{d}x
\end{aligned} \tag{105}$$

with $\left(\alpha_0, \alpha_1, w(x)\right)$ as the probable restricted variation.

Again, with the use of Taylor series, keeping the lowest order terms, and letting the variations in $y(x)$ to vanish, we get

$$\begin{aligned}
&\left[f_{1y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f_{1y'}\right)\right] \left(\int_{x_0}^{x_1} f_2\left(x, y(x), y'(x)\right) \mathrm{d}x \right) \\
&+ \left[f_{2y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f_{2y'}\right)\right] \left(\int_{x_0}^{x_1} f_1\left(x, y(x), y'(x)\right) \mathrm{d}x \right) = 0
\end{aligned} \tag{106}$$

leading to the related Euler-Lagrange equation [8].

Note that we can do the same thing with the quotient functional to obtain the related Euler-Lagrange equation [3].

8.4. Variational Iteration Method

In this subsection, we will use variational iteration methods to solve three kinds of non-linear partial differential equations; namely, the coupled Schrodinger-kdv equation, the generalized kdv equation and the shallow water equation.

To explain the variational iteration method, we consider the differential equation

$$Lu + Nu = g(x) \tag{107}$$

where L is a linear operator, N is a non-linear one and $g(x)$ is a function of *x*.

According to the method, we can build the functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left[Lu(\tau) + N\tilde{u}_n(\tau) - g(\tau) \right] \mathrm{d}\tau \tag{108}$$

λ is the Lagrange multiplier.

We assume that $\tilde{u}_n(\tau)$ is a constrained variation which implies that $\delta\tilde{u}_n(\tau)=0$.

8.4.1. Coupled Schrodinger-kdv Equations

If we consider the coupled Schrodinger-kdv equations

$$iu_1-(u_{xx}+uv)=0;\ v_1+6uv_x+v_{xxx}-\left(|u|^2\right)_x=0 \tag{109}$$

with the initial conditions

$$u(x,0)=6\sqrt{2}\mathrm{e}^{i\alpha x}k^2\operatorname{sech}^2(kx);\ v(x,0)=\frac{\alpha+16k^2}{3}-16k^2\tanh^2(kx) \tag{110}$$

where α and k are arbitrary constants.

To solve these equations for this special example using the variational method, we see that the functionals are given by $u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda_1\left(u_{nt}+i\left(u_{nxx}+u_n\tilde{v}_n\right)\right)\mathrm{d}\tau$ and $v_{n+1}(x,t)=v_n(x,t)+\int_0^t\lambda_2\left(v_{nt}+6u_n\tilde{v}_{nx}+v_{nxxx}-\tilde{u}^2\right)\mathrm{d}\tau$; where $\tilde{u}^2$ and $\delta u_n\tilde{v}_{nx}$ are constrained variations. Moreover, one can get the required conditions for stability as $\lambda_1'(\tau)=0$, $1+\lambda_1(t)=0$, $\lambda_2'(\tau)=0$, $1+\lambda_2(t)=0$. These equations are known to be Euler-Lagrange equations; from which one obtains that $\lambda_1=\lambda_2=-1$. Taking into account Equation (110) and the values of Lagrange multipliers, one gets the following results for u_1 and v_1

$$u_1(x,t)=6\sqrt{2}\mathrm{e}^{i\alpha x}k^2\operatorname{sech}^2(kx)+6\sqrt{2}k^2it\mathrm{e}^{i\alpha x}\operatorname{sech}^2(kx)\Big[\alpha^2+4i\alpha k\tanh(kx)+\frac{20}{3}k^2-10k^2\operatorname{sech}^2(kx)-\frac{1}{3}\Big]$$

and

$$\begin{aligned}v_1(x,t)=v_0(x,t)-tk^4\operatorname{sech}^2(kx)\Big[&256k\operatorname{sech}^2(kx)\tanh(kx)\\&-1152\sqrt{2}k\mathrm{e}^{i\alpha x}\operatorname{sech}^2(kx)\tanh(kx)+128k\tanh^3(kx)\\&+288k\mathrm{e}^{2i\alpha x}\operatorname{sech}^2(kx)\tanh(kx)-144i\alpha\mathrm{e}^{2i\alpha x}\operatorname{sech}^2(kx)\Big]\end{aligned}$$

Hence, we can continue to evaluate u_j and v_j on the basis of the obtained data using the variational iteration process [3]. The following figure (**Figure 1**) shows the numerical solution of $u(x,t)$; while in **Figure 2** we show that of $v(x,t)$ [3].

8.4.2. Shallow Water Equations

The shallow water system is governed by the two equations

$$u_t+vu_x+uv_x=0;\ v_t+u_x+vv_x=h'(x) \tag{111}$$

u and v represent the total depth of the canal and the water speed respectively; while $h(x)$ is the depth of the point x with respect to the water surface. The system is subject to the initial conditions

$$u(x,0)=h(x)+\frac{\operatorname{sech}x}{4};\ v(x,0)=0\text{ and }h(x)=\frac{\mathrm{e}^{-x^2}}{1+\mathrm{e}^{-x^2}} \tag{112}$$

Again, to solve the last equations using the variational iteration method, we write

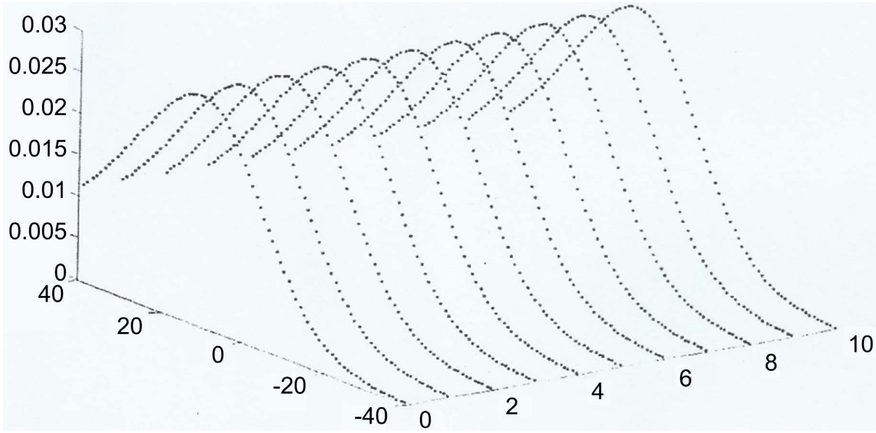

Figure 1. $u(x,t)$ in terms of the two parameters x and t.

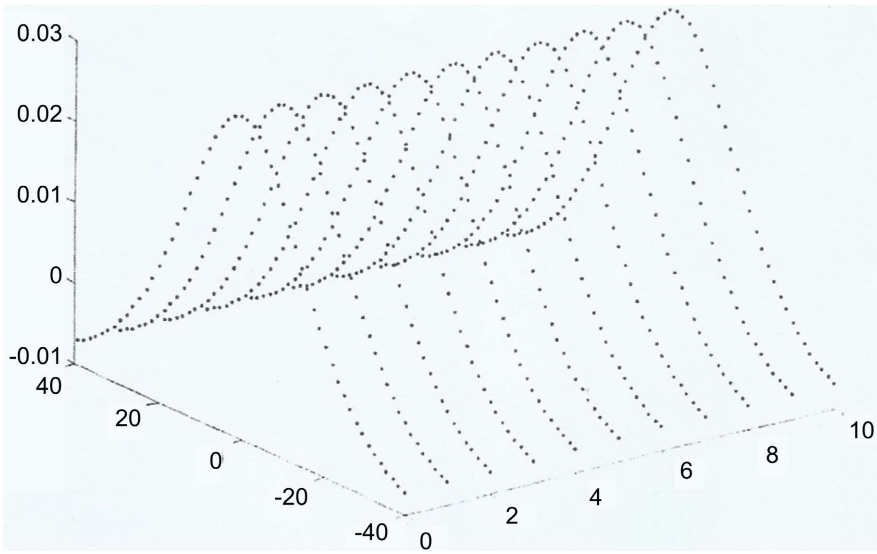

Figure 2. Graph of $v(x,t)$ numerically.

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1 \left(u_{nt} + \left(u_n \tilde{v}_{nx} + u_{nx} \tilde{v}_n\right)\right) \mathrm{d}\tau \tag{113}$$

and

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2 \left(v_{nt} + u_{nx} + u_n \tilde{v}_{nx} - h'(x)\right) \mathrm{d}\tau \tag{114}$$

$\delta u_n \tilde{v}_{nx}$ and $\delta v_n \tilde{u}_{nx}$ are constrained variations. Moreover, the required stability conditions are

$$\lambda_1'(\tau) = 0,\ 1 + \lambda_1(t) = 0,\ \lambda_2'(\tau) = 0, 1 + \lambda_2(t) = 0 \tag{115}$$

In the same manner, one gets $\lambda_1 = \lambda_2 = -1$.

Using the fore-mentioned recurrence relations with the conditions in Equation (112), we obtain

$$u_1(x,t) = \frac{1}{4}\left(\frac{4\exp\left(-x^2\right)\cosh x + 1 + \exp\left(-x^2\right)}{\cosh x\left(1 + \exp\left(-x^2\right)\right)}\right), \tag{116}$$

$$v_1(x,t)=\frac{1}{4}\frac{t\sinh x}{\cosh^2 x}, \tag{117}$$

$$\begin{aligned} u_2(x,t) &= u_1(x,t) \\ &+\frac{1}{32\cosh^4 x}\left[\left(\frac{6x\sinh x\cosh x-3\sinh^2 x+2x\exp\left(-x^2\right)\sinh^2 x}{1+2\exp\left(-x^2\right)\cosh x+\exp\left(-2x^2\right)}\right)\exp\left(-x^2\right)\right. \\ &\left.+\frac{\left(4\cosh x\exp\left(-x^2\right)+1+\exp\left(-x^2\right)\right)\left(\cosh^2 x-2\right)}{1+\exp\left(-x^2\right)}\right] \end{aligned} \tag{118}$$

and

$$\begin{aligned} v_2(x,t)=v_1(x,t)-&\left[\frac{t\sinh x}{4\cosh^2 x}-\frac{t^2\sinh x\left(\cosh^2 x-2\right)}{48\cosh^2 x}+\frac{2tx\exp\left(-x^2\right)}{\left(1+\exp\left(-x^2\right)\right)^2}\right. \\ &\left.-\frac{t\left[6x\cosh x-3\sinh x+2x\exp\left(-x^2\right)\sinh x\right]\exp\left(-x^2\right)}{4\left(1+2\exp\left(-x^2\right)\cosh x+\exp\left(-2x^2\right)\right)\cosh^2 x}\right] \end{aligned} \tag{119}$$

Of course, one can, now, make the desired calculations using certain values for the parameter t. **Figure 3** and **Figure 4** shows $u(x,t)$ and $v(x,t)$ for $t=0.8$ [3].

8.4.3. The Generalized-kdv Equation

In this model the generalized equation is of the form

$$u_t+u^p u_x+u_{xxx}=0 \tag{120}$$

with the initial condition

$$u(x,0)=\left[A\operatorname{sech}^2\left(kx-x_0\right)\right]^{1/p} \tag{121}$$

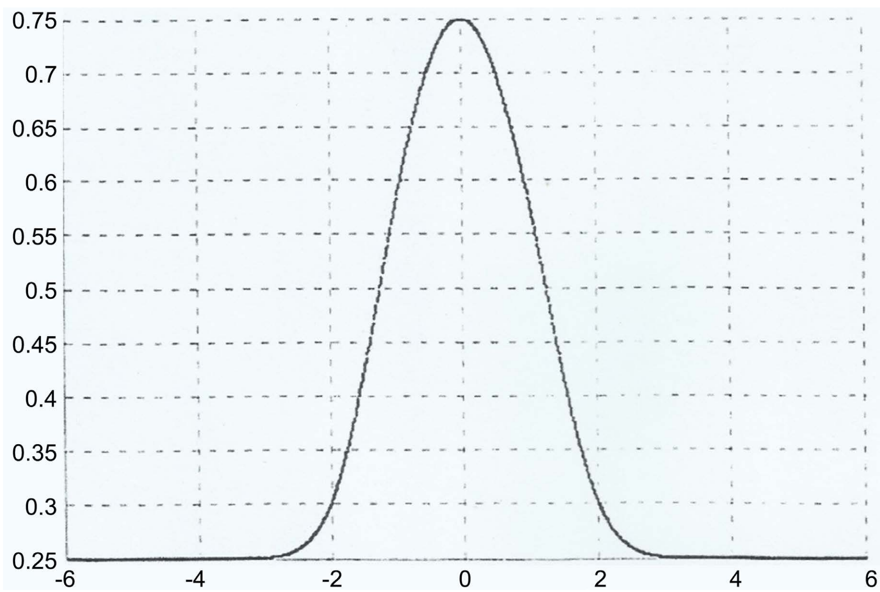

Figure 3. $u(x,t)$ for $t=0.8$ [3].

where $A = \dfrac{2(p+1)(p+2)}{m^2} k^2$ and k, m, x_0 are constants with $p \geq 2$.

Using the variational iteration method, we get

$$u_{j+1}(x,t) = u_j(x,t) + \int_0^t \lambda \left[u_{tn} + u_n^p \tilde{u}_{nx} + u_{nxxx} \right] \tag{122}$$

And $\delta u_n^p \tilde{u}_{nx} = 0$. Moreover, the stability condition yields $\lambda_1'(\tau) = 0$, $1 + \lambda_1(t) = 0$; which implies that $\lambda = -1$.

Putting $p = 4$, we obtain

$$\begin{aligned} u_1(x,t) = \frac{1}{8} \sqrt[4]{A} \sqrt{\operatorname{sech}(kx - x_0)} \big[& 8 + 4Akt \operatorname{sech}^2(kx - x_0) \tanh(kx - x_0) \\ & + k^3 t \tanh(kx - x_0) - 15tk^3 \sinh(kx - x_0) \operatorname{sech}^3(kx - x_0) \big] \end{aligned} \tag{123}$$

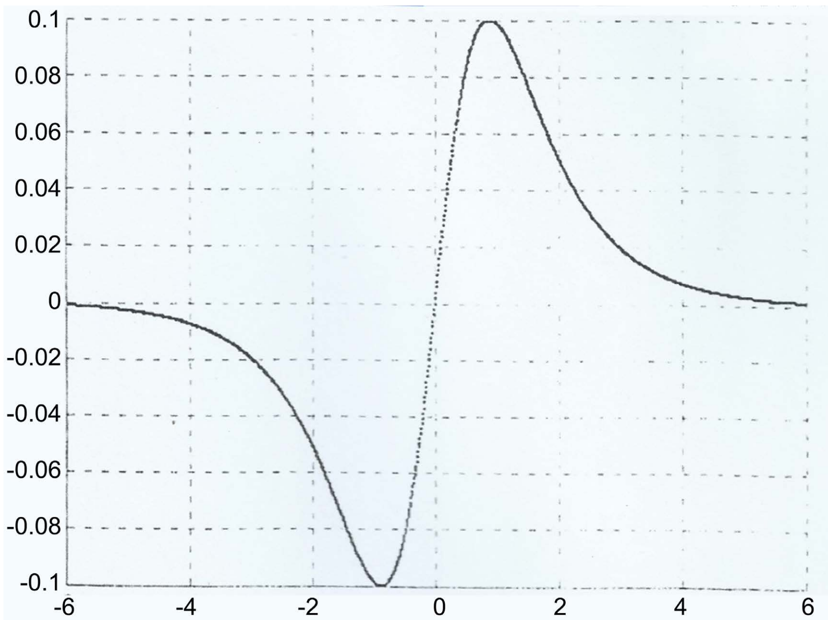

Figure 4. $v(x,t)$ for $t = 0.8$ [3].

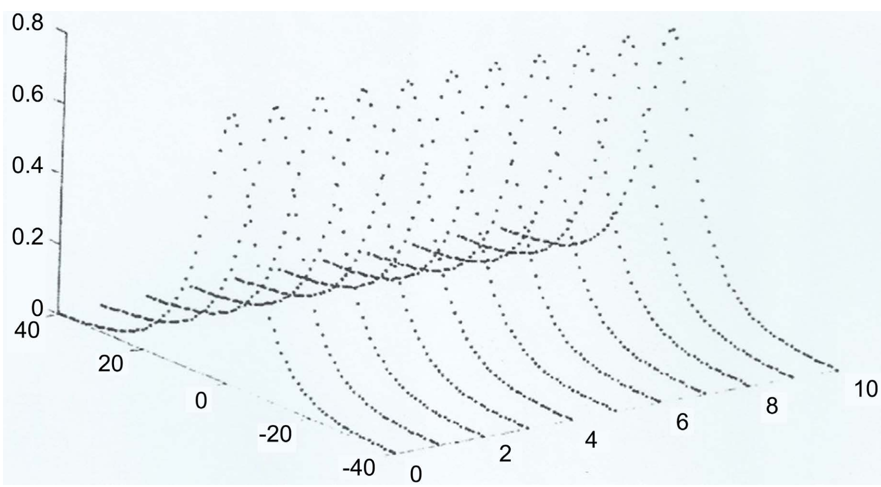

Figure 5. $u(x,t)$ for $A = 0.3375$, $k = 0.3$, and $c = 0.00675$ [3].

Hence

$$u(x,t)=\left[A\operatorname{sech}^2\left(kx-ct-x_0\right)\right]^{\frac{1}{p}},\ \text{where } c=\frac{4k^2}{m^2}. \tag{124}$$

Figure 5 shows $u(x,t)$ for $A=0.3375$, $k=0.3$, and $c=0.00675$ [3].

9. Concluding Discussion

The variational methods will remain to be the most powerful tool in the field of approximations in various directions and especially when dealing with Schrodinger eigenvalue equation when it cannot be solved exactly [9]. They will, always, be of importance in approximations for many subjects such as systems theory and control, optimization, analysis of complex systems, theoretical, mathematical and computational physics; and especially in the applications to non-linear partial differential equations and Hamiltonian systems [10].

Their applications are almost everywhere in the fields of learning: in fluid mechanics, in statistics, and in quantum neural networks, where variational methods are used in quantum machine learning based on a quantum mechanical network for the binary classification of points of a specific geometric pattern [11] [12] [13].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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