Journal of Applied Mathematics and Physics, 2023, 11, 1250-1262
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Global Optimization Method to Comprise Rotation-Minimizing Euler-Rodrigues Frames of Pythagorean-Hodograph Curve

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How to cite this paper: Peng, F.F., Pang, J.R. and Pan, Y.T. (2023) Global Optimization Method to Comprise Rotation-Minimizing Euler-Rodrigues Frames of Pythagorean-Hodograph Curve. *Journal of Applied Mathematics and Physics*, **11**, 1250-1262.
https://doi.org/10.4236/jamp.2023.115081

Received: February 14, 2023
Accepted: May 7, 2023
Published: May 10, 2023

Abstract

The rotation-minimizing frame is the main research object for a spatial curve. Although the mathematical description is not complicated, it is not easy to directly make an exact minimizing-rotation frame for the Euler-Rodrigues frame. The condition for the non-normalized Euler-Rodrigues frame of the Pythagorean-Hodograph curve to become the rotation-minimizing frame is given in this article, which is an ordinary differential equation with rational form, the analytical solution that does not always exist. To avoid calculating the solution of ordinary differential equations, a global optimization algorithm for the conditions is proposed, that has a weight function in the objective function. The quintic Pythagorean-Hodograph curve is analyzed concretely with the method, and its objective function and constraint conditions of optimization are clarified. The example is analyzed by using this method with different weight functions and contrasting that approach with its exact value.

Keywords

Rotation-Minimizing, Euler-Rodrigues Frames, Pythagorean-Hodograph Curves

1. Introduction

Pythagorean-Hodograph (PH) curve is a kind of polynomial parametric curve based on offset curve research. Its characteristic is that its rate function is also polynomial, and the arc length of curve can be calculated accurately. A polynomial parametric curve $\mathbf{P}(t)=[x(t),y(t),z(t)]$ is called spatial PH curve, if and only if the derivative $\mathbf{P}'(t)=\left[x'(t),y'(t),z'(t)\right]$ of the polynomial parametric curve satisfies the condition $\sigma(t)=\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}$ being a polynomial. The equivalent condition is that there are four polynomials $u(t),v(t),p(t),q(t)$ satisfying

$$\begin{cases} x'(t)=u^2(t)+v^2(t)-p^2(t)-q^2(t), \\ y'(t)=2u(t)q(t)+2v(t)p(t), \\ z'(t)=2v(t)q(t)-2u(t)p(t). \end{cases} \tag{1}$$

The arc length of the PH curve can be computed precisely by simply calculating a polynomial. The PH curve plays an important role in CAGD, which not only offers unique computational advantages over ordinary polynomial parametric curves in CAD/CAM applications, but also retains full compatibility with standard Bézier/B-spline representations [1].

In 1994, Farouki extended the planar PH curves to spatial PH curves and surfaces, and gave an explicit expression of developable surfaces with rational offsets [2]. At the same time, the further theoretical and applied research on PH curve was carried out in [3] [4]. Hermite interpolation and continuous analysis of PH curves have attracted much attention [5] [6] [7] [8] [9].

An orthogonal frame $\left(\boldsymbol{f}_1(t),\boldsymbol{f}_2(t),\boldsymbol{f}_3(t)\right)$ is on a given spatial curve $\mathbf{P}(t)$, if the $\boldsymbol{f}_1(t)$ is the unit tangent, and the others orthogonal unit vectors $\boldsymbol{f}_2(t),\boldsymbol{f}_3(t)$ span the normal plane. The derivative of the frame concerning to arc length s determines its angular velocity $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$ as follow

$$\frac{\mathrm{d}\boldsymbol{f}_1}{\mathrm{d}s}=\boldsymbol{\omega}\times\boldsymbol{f}_1,\frac{\mathrm{d}\boldsymbol{f}_2}{\mathrm{d}s}=\boldsymbol{\omega}\times\boldsymbol{f}_2,\frac{\mathrm{d}\boldsymbol{f}_3}{\mathrm{d}s}=\boldsymbol{\omega}\times\boldsymbol{f}_3. \tag{2}$$

where $\boldsymbol{\omega}\times\boldsymbol{f}_i$ represents the vector product of both vectors. There are many ways to comprise a frame [10]. A familiar case is the Frenet frame that is formed by the tangent $\boldsymbol{t}$, the principal normal $\boldsymbol{n}$, and the binormal $\boldsymbol{b}=\boldsymbol{t}\times\boldsymbol{n}$. If there are unit orthogonal vectors $\boldsymbol{u},\boldsymbol{v}$ in the normal plane, and the frame angular velocity satisfies $\boldsymbol{\omega}\cdot\boldsymbol{t}\equiv 0$, then the $(\boldsymbol{t},\boldsymbol{u},\boldsymbol{v})$ is named as a rotation-minimizing (RM) adapted frame [11] [12], which is also called as a Bishop frame. Another equivalent condition of RM frames is $\omega_1\equiv 0$. Its physical meaning is that the frames do not rotate around the tangent direction when it moves along the curve.

The RM frames of the spatial curve are of great value in research areas of computer graphics, computer animation, motion planning, and other research fields. It is specifically used in swept surface modeling, 3D roaming and motion interpolation, and has a wide range of application value. As there is hardly a way to find an exact formulation directly to calculate the RM frames, the computation of the RM frames to a spatial curve attracts a good deal of attention. Many effective geometric algorithms have been proposed; especially, using the Euler-Rodrigues (ER) frames to construct the RM frames with orthogonal transformation is a pretty good way [11] [13]. ER frame is a special rational adaptive frame base on spatial PH curve. It is always non-singular at the inflection point. It makes it different from the adaptive method, but this kind of approach does not offer a complete solution to solve the RM frame, and the authors have raised some open questions [14].

The perfect combination of PH curve and the RM frames can be achieved through the ER frames [15], and this frame is raised by the quaternion polynomial. But the frame is a rational form, which increases the difficulty to calculate the geometric attributes of a spatial curve [11] [13] [14]. In this paper, based on the ER frame of spatial PH curve, we propose an optimization method to approximate the RM frame, which avoids the existence to the ER frames with the transformation, and reduces the calculation of rational polynomials.

The remainder of this paper is organized as follows: In Section 2, the conditions for the non-normalized ER frames of the PH curve to become the RM frames are given, and the optimization algorithm for the conditions is proposed. The quintic PH curve is analyzed concretely, and the objective function and constraint conditions of optimization are clarified in Section 3. Some examples are analyzed by using this method in Section 4. The 5th section is the summary of the full text.

2. Preliminary Work

2.1. Quaternion

This section briefly introduces quaternion and the construction of PH curves with the quaternion.

A spatial Pythagorean-Hodograph (PH) curve can be represented in a compact form with the quaternion. Here is a brief introduction to the concept and basic operations of the quaternion. In the four-dimensional real vector space $\mathbf{R}^4$, the space formed by the quaternions represented by the standard base $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ is denoted by $\mathbf{H}$, which is defined as

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1.$$

and derive the relation

$$\mathbf{ij}=-\mathbf{ji}=\mathbf{k},\mathbf{jk}=-\mathbf{kj}=\mathbf{i},\mathbf{ki}=-\mathbf{ik}=\mathbf{j}.$$

A quaternion $\mathbf{A}$ in space $\mathbf{H}$ is written as $\mathbf{A}=\alpha_0+\alpha_1\mathbf{i}+\alpha_2\mathbf{j}+\alpha_3\mathbf{k}$, where $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ are the real numbers. For convenience, the quaternion is written in the form of $\mathbf{A}=(\alpha_0,\boldsymbol{\alpha})$, α_0 is called the scalar part of the quaternion, and $\boldsymbol{\alpha}=\alpha_1\mathbf{i}+\alpha_2\mathbf{j}+\alpha_3\mathbf{k}$ is the vector part, it is called the pure quaternion when $\alpha_0=0$, at this time it can be regarded as a vector in three-dimensional space, and the space it constitutes is isomorphic to $\mathbf{R}^3$. For any two quaternions $\mathbf{A}=(\alpha_0,\boldsymbol{\alpha})$, $\mathbf{B}=(\beta_0,\boldsymbol{\beta})$ the algorithm is defined as

$$\mathbf{A}\pm\mathbf{B}=(\alpha_0\pm\beta_0,\boldsymbol{\alpha}\pm\boldsymbol{\beta}),$$

$$\mathbf{AB}=(\alpha_0\beta_0-\boldsymbol{\alpha}\cdot\boldsymbol{\beta},\alpha_0\boldsymbol{\beta}+\beta_0\boldsymbol{\alpha}+\boldsymbol{\alpha}\times\boldsymbol{\beta}),$$

here $\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$ represents the scalar product of vectors. Therefore, the quaternion multiplication is non-commutative, that is $\mathbf{AB}\neq\mathbf{BA}$. The conjugate quaternion is defined as $\mathbf{A}^* = (\alpha_0, -\boldsymbol{\alpha})$, and its norm is $\|\mathbf{A}\| = \sqrt{\mathbf{A}\mathbf{A}^*} = \sqrt{\sum_{i=0}^{3} \alpha_i^2}$. $\mathbf{A}$ is called a unit quaternion when $\|\mathbf{A}\| = 1$, the pure unit quaternion can be regarded as a unit vector in $\mathbf{R}^3$.

Let $\mathbf{A}, \mathbf{B}$ be two real quaternions in $\mathbf{H}$, there define three commutative algebraic operations [16] as

$$\mathbf{A} \star \mathbf{B} = \frac{1}{2}\left(\mathbf{A}\mathbf{i}\mathbf{B}^* + \mathbf{B}\mathbf{i}\mathbf{A}^*\right), \tag{3}$$

$$\mathbf{A} \odot \mathbf{B} = \frac{1}{2}\left(\mathbf{A}\mathbf{j}\mathbf{B}^* + \mathbf{B}\mathbf{j}\mathbf{A}^*\right),$$

$$\mathbf{A} \otimes \mathbf{B} = \frac{1}{2}\left(\mathbf{A}\mathbf{k}\mathbf{B}^* + \mathbf{B}\mathbf{k}\mathbf{A}^*\right).$$

there are three squares denoted by $\mathbf{A}^{2\star} = \mathbf{A}\mathbf{i}\mathbf{A}^*, \mathbf{A}^{2\odot} = \mathbf{A}\mathbf{j}\mathbf{A}^*$ and $\mathbf{A}^{2\otimes} = \mathbf{A}\mathbf{k}\mathbf{A}^*$.

Now given a polynomial quaternion $\mathbf{A}(t) = u(t) + v(t)\mathbf{i} + p(t)\mathbf{j} + q(t)\mathbf{k}$, it follows from the formula (3) that as a pure quaternion $\mathbf{P}'(t) = \mathbf{A}^{2\star}$ meets the conditions Equation (1). The PH curve $\mathbf{P}(t)$ can be obtained through integral with its initial position. This PH curve can be presented by the Bézier method, and its control vertexes are directly expressed with the quaternion [17], the next review this method to produce the quintic PH curve.

2.2. Quintic PH Curve with Quaternion

For a quintic Bézier curve segment, it can be constructed as follows, given two points and its tangents $\{\boldsymbol{Q}_0, \boldsymbol{\varepsilon}_0\}$ and $\{\boldsymbol{Q}_5, \boldsymbol{\varepsilon}_1\}$.

Choose $\mathbf{A}_0$ and $\mathbf{A}_2$ as formula in [9] as

$$\mathbf{A}_i = \begin{cases} \sqrt{\|\boldsymbol{\varepsilon}_i\|}\dfrac{\dfrac{\boldsymbol{\varepsilon}_i}{\|\boldsymbol{\varepsilon}_i\|} + \mathbf{i}}{\left\|\dfrac{\boldsymbol{\varepsilon}_i}{\|\boldsymbol{\varepsilon}_i\|} + \mathbf{i}\right\|}, & \boldsymbol{\varepsilon}_i \neq -\mathbf{i}, \\ \sqrt{\|\boldsymbol{\varepsilon}_i\|}\mathbf{k}, & \text{else}, (i = 0, 2), \end{cases}$$

and $\mathbf{A}_1$ is free. Let

$$\Delta \boldsymbol{d}_0 = \boldsymbol{\varepsilon}_0 = \frac{1}{5}\mathbf{A}_0^{2\star},$$

$$\Delta \boldsymbol{d}_1 = \frac{1}{5}\left(\mathbf{A}_0 \star \mathbf{A}_1\right),$$

$$\Delta \boldsymbol{d}_2 = \frac{1}{15}\left(\mathbf{A}_0 \star \mathbf{A}_2 + \mathbf{A}_1^{2\star}\right),$$

$$\Delta \boldsymbol{d}_3 = \frac{1}{5}\left(\mathbf{A}_1 \star \mathbf{A}_2\right),$$

$$\Delta \boldsymbol{d}_4 = \boldsymbol{\varepsilon}_1 = \frac{1}{5}\mathbf{A}_2^{2\star}.$$

then the vertexes of Bézier curve are these

$$\boldsymbol{Q}_i = \boldsymbol{Q}_0 + \sum_{j=0}^{i-1} \Delta \boldsymbol{d}_j, i = 1, 2, 3, 4, 5,$$

the Bézier curve can be produced by the vertexes $\boldsymbol{Q}_i\left(i=0,1,\cdots,5\right)$, that is a spatial PH curve. The quaternions $\mathbf{A}_0$ and $\mathbf{A}_2$ can also be defined as others forms including the rotation angle parameters. This segment has good geometric properties, such as it having tangent $\boldsymbol{\varepsilon}_0$ at $\boldsymbol{Q}_0$ and $\boldsymbol{\varepsilon}_1$ at $\boldsymbol{Q}_5$. Because the quaternion $\mathbf{A}_1$ is free, there are some freedoms for adjusting and designing the spatial PH curve.

3. Rotation-Minimizing ER Frames

3.1. Condition of Rotation-Minimizing ER Frames

If the coefficient is quaternion $\mathbf{A}\left(t\right)$, construct velocity vector curve $\mathbf{P}'\left(t\right)=\mathbf{A}^{2\star}$, and then get a Bézier curve $\mathbf{P}\left(t\right)$ through integration. It has an ER orthogonal frame $\left(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\right)=\frac{1}{\sigma}\left(\mathbf{A}^{2\star},\mathbf{A}^{2\odot},\mathbf{A}^{2\otimes}\right)$, in which $\sigma\left(t\right)=\left\|\mathbf{AiA}^*\right\|=\left\|\mathbf{AjA}^*\right\|=\left\|\mathbf{AkA}^*\right\|$. To comprise the RM frames for the ER frames, set the rotation angle as $\theta\left(t\right)$, and construct the orthogonal transformation

$$\begin{bmatrix}\overline{\boldsymbol{e}}_2\left(t\right)\\ \overline{\boldsymbol{e}}_3\left(t\right)\end{bmatrix}=\begin{bmatrix}\cos2\theta & -\sin2\theta\\ \pm\sin2\theta & \pm\cos2\theta\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_2\left(t\right)\\ \boldsymbol{e}_3\left(t\right)\end{bmatrix}. \tag{4}$$

In Equations (4), there is a rotation transformation when it takes the positive signs, otherwise, it is the specular transformation. To make $\left(\boldsymbol{e}_1,\overline{\boldsymbol{e}}_2,\overline{\boldsymbol{e}}_3\right)$ become a RM frame, it must satisfy $\overline{\boldsymbol{e}}_2'\cdot\overline{\boldsymbol{e}}_3=0$, for details, refer to the literature [9], and deduce the relationship (All of the derivatives are with respect to parameter t in this text unless noted otherwise)

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{2}\left(\boldsymbol{e}_2'\cdot\boldsymbol{e}_3\right), \tag{5}$$

For the convenience of calculation, when rotating the above ER frame, we do not consider the normalization of the coordinates. It is still an orthogonal frame, and directly rotate the isometric frame $\left(\mathbf{AiA}^*,\mathbf{AjA}^*,\mathbf{AkA}^*\right)$, which are not rational polynomials. At the same time, according to the above marks, it can also be recorded as $\left(\mathbf{A}^{2\star},\mathbf{A}^{2\odot},\mathbf{A}^{2\otimes}\right)$, and a similar relation can be obtained as

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{2\sigma^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{2\odot}\cdot\mathbf{A}^{2\otimes}\right)\triangleq g\left(t\right), \tag{6}$$

A brief proof process is followed, according to equations (4), we can obtain

$$\begin{bmatrix}\boldsymbol{u}\left(t\right)\\ \boldsymbol{v}\left(t\right)\end{bmatrix}=\begin{bmatrix}\cos2\theta & -\sin2\theta\\ \sin2\theta & \cos2\theta\end{bmatrix}\begin{bmatrix}\mathbf{A}^{2\odot}\\ \mathbf{A}^{2\otimes}\end{bmatrix}, \tag{7}$$

thus we have

$$\boldsymbol{u}'\left(t\right)=-2\left(\sin2\theta\mathbf{A}^{2\odot}+\cos2\theta\mathbf{A}^{2\otimes}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}+\left[\cos2\theta\left(A^{2\odot}\right)'-\sin2\theta\left(A^{2\otimes}\right)'\right]$$

and

$$\boldsymbol{v}\left(t\right)=\sin2\theta\mathbf{A}^{2\odot}+\cos2\theta\mathbf{A}^{2\otimes},$$

let's take the scaler product of them

$$\boldsymbol{u}'(t)\cdot\boldsymbol{v}(t)=2\sigma^2\frac{\mathrm{d}\theta}{\mathrm{d}t}+\sin 2\theta\cos 2\theta\left[\left(A^{2\odot}\right)'\cdot\mathbf{A}^{2\odot}-\left(A^{2\otimes}\right)'\cdot\mathbf{A}^{2\otimes}\right]$$
$$+\cos^2 2\theta\left(A^{2\odot}\right)'\cdot\mathbf{A}^{2\otimes}-\sin^2 2\theta\left(A^{2\otimes}\right)'\cdot\mathbf{A}^{2\odot}. \tag{8}$$

Here exist $\mathbf{A}^{2\odot}\cdot\mathbf{A}^{2\odot}=\mathbf{A}^{2\otimes}\cdot\mathbf{A}^{2\otimes}=\sigma^2$. This thereby, $\left(\mathbf{A}^{2\odot}\right)'\cdot\mathbf{A}^{2\odot}=\left(\mathbf{A}^{2\otimes}\right)'\cdot\mathbf{A}^{2\otimes}=\sigma'\sigma$, and $\mathbf{A}^{2\odot}\cdot\mathbf{A}^{2\otimes}=0$, then $\left(\mathbf{A}^{2\odot}\right)'\cdot\mathbf{A}^{2\otimes}+\mathbf{A}^{2\odot}\cdot\left(\mathbf{A}^{2\otimes}\right)'=0$. These submit to the Equation (8) make it to zero, so the Equation (6) is hold.

Theoretically, setting the initial value of θ_0 (set $\theta_0=0$ in subsequent examples), we can calculate θ by integrating equation (6) as follows,

$$\theta=\theta_0+\int_0^t g(t)\mathrm{d}t.$$

But in general, the expression of θ is difficult to obtain. It is an ordinary differential problem with initial values, and we can obtain the value through discrete numerical calculation. Using the Trapezoidal integral formula, we can get the solution of discrete point sequence $t_0,t_1,\cdots,t_m$ with $\theta(t_0)=\theta_0$,

$$\begin{gathered}\theta(t_k)=\theta(t_{k-1})+\frac{\Delta t}{2}\left[g(t_{k-1})+g(t_k)\right]+O(\Delta t)^3,\\ \Delta t=t_k-t_{k-1},k=1,2,\cdots,m.\end{gathered} \tag{9}$$

Although this method can achieve approximate minimum rotation, there is a challenge for each θ to calculate the integral of a rational expression in practical application. Because of the delay in the calculation, it is very detrimental to a predictable adaptive method for the entire curve.

To build an adapted rotation-minimizing frame, the numerical calculation method cannot meet practical requirements, so we select a continuous angular function

$$\theta=\arctan\phi(t)$$

to approximate the exact solution. The corresponding rotation matrix is

$$\begin{bmatrix}\boldsymbol{u}(t)\\ \boldsymbol{v}(t)\end{bmatrix}=\frac{1}{1+\phi^2(t)}\begin{bmatrix}1-\phi^2(t) & -2\phi(t)\\ \pm 2\phi(t) & \pm\left[1-\phi^2(t)\right]\end{bmatrix}\begin{bmatrix}\mathbf{A}^{2\odot}\\ \mathbf{A}^{2\otimes}\end{bmatrix}, \tag{10}$$

The angle function is often chosen as

$$\phi(t)=\frac{a(t)}{b(t)},$$

where $a(t)$ and $b(t)$ are the reduced polynomials, *i.e.* $gcd\left(a(t),b(t)\right)=1$.

The orthogonal transformation

$$\begin{bmatrix}\boldsymbol{u}(t)\\ \boldsymbol{v}(t)\end{bmatrix}=\frac{\begin{bmatrix}b^2(t)-a^2(t) & -2a(t)b(t)\\ \pm 2a(t)b(t) & \pm\left[b^2(t)-a^2(t)\right]\end{bmatrix}}{a^2(t)+b^2(t)}\begin{bmatrix}\mathbf{A}^{2\odot}\\ \mathbf{A}^{2\otimes}\end{bmatrix} \tag{11}$$

Equations (11) are minor changes of the transformation matrix that compares with the literature [11], which is just for the convenience of expression. The final comprised frame $(\boldsymbol{e}_1, \boldsymbol{e}_u, \boldsymbol{e}_v)$ is a RM frames, here $\boldsymbol{e}_u = \frac{\boldsymbol{u}(t)}{\|\boldsymbol{u}(t)\|}$, $\boldsymbol{e}_v = \frac{\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|}$. Polynomials $a(t), b(t)$ need to meet the relation (we only consider the rotation transformation here)

$$\frac{1}{2\sigma^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{2\odot}\cdot\mathbf{A}^{2\otimes}\right)=\frac{ab'-a'b}{a^2+b^2}, \tag{12}$$

In fact, for one general spatial curve, the polynomials $a(t)$ and $b(t)$ does not always exist, so we adopt the optimization calculation method to find the approximate RM frames.

3.2. Optimization Method of ER Frames

The optimization method is as follows,

1) Find that the polynomial $a(t)$ and $b(t)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{2\odot}\cdot\mathbf{A}^{2\otimes}=ab'-a'b$, the sufficient condition for the equation to hold is that the degree of $a(t)$ and $b(t)$ must be same. The degree of $a(t), b(t)$ selected is the same as that of the polynomial $\mathbf{A}^{2\odot}(t)$, so there are still 2 degrees of freedom left. Otherwise, the degree of freedom will be reduced and not even.

2) Use the above 2 freedoms to approximate and optimize the above Equation (12) as

$$1=\frac{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{2\odot}\cdot\mathbf{A}^{2\otimes}}{ab'-a'b}\approx\frac{2\sigma^2}{a^2+b^2},$$

and construct the following objective function

$$\arg\min_{a,b}\int_0^1\rho(t)\left[a^2(t)+b^2(t)-2\sigma^2\right]^2\mathrm{d}t. \tag{13}$$

where $\rho(t)$ is the weight function. Generally, we can select $\rho(t)=1$ or $\rho(t)=\sigma(t)$ and so on, which can realize the optimization effect of arc length.

Although it is not an exact RM frames, it can realize the minimum rotation problem on the whole curve at one time.

4. RM Frames of Quintic PH Curve

Let $\mathbf{A}(t)=(1-t)^2\mathbf{A}_0+2(1-t)t\mathbf{A}_1+t^2\mathbf{A}_2$, where $\mathbf{A}_i\ (i=0,1,2)$ are three real quaternion polynomials, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}\mathbf{j}\mathbf{A}^*\right)=\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{2\odot}=\mathbf{A}\mathbf{j}\mathbf{A}^{*\prime}+\mathbf{A}'\mathbf{j}\mathbf{A}^*=4\sum_{i=0}^{3}B_{3,i}(t)\mathbf{P}_i, \tag{14}$$

where

$$\boldsymbol{p}_0=\mathbf{A}_0\odot(\mathbf{A}_1-\mathbf{A}_0),$$
$$\boldsymbol{p}_1=\frac{1}{3}\left[\mathbf{A}_0\odot(\mathbf{A}_2-\mathbf{A}_1)\right]+\frac{2}{3}\left[\mathbf{A}_1\odot(\mathbf{A}_1-\mathbf{A}_0)\right],$$

$$\begin{aligned}\boldsymbol{p}_2 &= \frac{1}{3}\left[\mathbf{A}_2 \odot \left(\mathbf{A}_1 - \mathbf{A}_0\right)\right] + \frac{2}{3}\left[\mathbf{A}_1 \odot \left(\mathbf{A}_2 - \mathbf{A}_1\right)\right],\\ \boldsymbol{p}_3 &= \mathbf{A}_2 \odot \left(\mathbf{A}_2 - \mathbf{A}_1\right).\end{aligned} \tag{15}$$

All of these $\boldsymbol{p}_i\left(i = 0,1,2,3\right)$ are the pure quaternions, and $B_{n,i}\left(t\right) = C_n^i\left(1-t\right)^{n-i} t^i$ is the Bernstein basis function, then the Equation (14) also can be presented by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{AjA}^*\right) = \sum_{i=0}^{3} B'_{4,i}\left(t\right)\mathbf{r}_i, \tag{16}$$

where

$$\begin{aligned}\boldsymbol{r}_0 &= -\left(\mathbf{p}_0 + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3\right),\\ \boldsymbol{r}_1 &= -\left(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3\right),\\ \boldsymbol{r}_2 &= -\left(\mathbf{p}_2 + \mathbf{p}_3\right),\\ \boldsymbol{r}_3 &= -\mathbf{p}_3.\end{aligned} \tag{17}$$

The following expression is parallelled deduced as above,

$$\mathbf{AkA}^* = \mathbf{A}^{2\otimes} = \sum_{i=0}^{4} B_{4,i}\left(t\right)\mathbf{q}_i, \tag{18}$$

where

$$\begin{aligned}\boldsymbol{q}_0 &= \mathbf{A}_0^{2\otimes},\\ \boldsymbol{q}_1 &= \mathbf{A}_0 \otimes \mathbf{A}_1,\\ \boldsymbol{q}_2 &= \frac{1}{3}\left[\mathbf{A}_0 \otimes \mathbf{A}_2\right] + \frac{2}{3}\left[\mathbf{A}_1^{2\otimes}\right],\\ \boldsymbol{q}_3 &= \mathbf{A}_1 \otimes \mathbf{A}_2,\\ \boldsymbol{q}_4 &= \mathbf{A}_2^{2\otimes}.\end{aligned} \tag{19}$$

After calculating the scaler product of (16) and (18), we obtain follow equation

$$\mathbf{A}^{2\odot\prime} \cdot \mathbf{A}^{2\otimes} = \sum_{i=0}^{3}\sum_{j=0}^{4} B'_{4,i}\left(t\right)B_{4,j}\left(t\right)\left(\boldsymbol{r}_i \cdot \boldsymbol{q}_j\right). \tag{20}$$

Set two polynomials to be $a\left(t\right) = \sum_{i=0}^{4} a_i B_{4,i}\left(t\right)$ and $b\left(t\right) = \sum_{i=0}^{4} b_i B_{4,i}\left(t\right)$. we can deduce follow equations,

$$a\left(t\right)b'\left(t\right) = \sum_{i=0}^{3}\sum_{j=0}^{4} B'_{4,i}\left(t\right)B_{4,j}\left(t\right)\left(b_i - b_4\right)a_j,$$

and

$$a'\left(t\right)b\left(t\right) = \sum_{i=0}^{3}\sum_{j=0}^{4} B'_{4,i}\left(t\right)B_{4,j}\left(t\right)\left(a_i - a_4\right)b_j,$$

then

$$a\left(t\right)b'\left(t\right) - a'\left(t\right)b\left(t\right) = \sum_{i=0}^{3}\sum_{j=0}^{4} B'_{4,i}\left(t\right)B_{4,j}\left(t\right)\left[\left(b_i - b_4\right)a_j - \left(a_i - a_4\right)b_j\right]. \tag{21}$$

The result of (21) is equal to the above Equations (20), there is an equation on two septic polynomials being the same, it is

$$\sum_{i=0}^{3}\sum_{j=0}^{4} B'_{4,i}\left(t\right)B_{4,j}\left(t\right)M_{ij} = 0. \tag{22}$$

where $M_{ij} = \left[(b_i - b_4)a_j - (a_i - a_4)b_j\right] - \boldsymbol{r}_i \cdot \boldsymbol{q}_j = \det\begin{pmatrix} a_4 - a_i & a_j \\ b_4 - b_i & b_j \end{pmatrix} - \boldsymbol{r}_i \cdot \boldsymbol{q}_j$. Here we define that $B_{n,-1}(t) = 0$ and $C_n^{-1} = 0$, use the formula $B'_{4,i}(t) = -4\left[B_{3,i}(t) - B_{3,i-1}(t)\right]$, the equations can be rewritten as

$$\sum_{i=0}^{3}\sum_{j=0}^{4}\left[B_{3,i}(t) - B_{3,i-1}(t)\right]B_{4,j}(t)M_{ij} = 0. \tag{23}$$

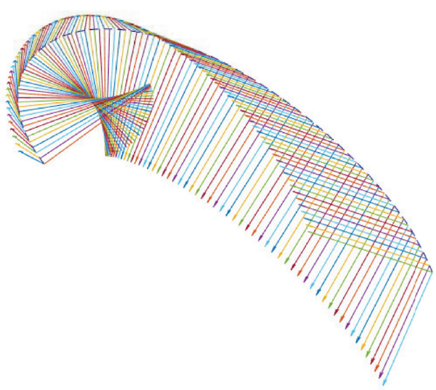
Due to the coefficient of the septic polynomial being zero, we can obtain a system with 8 equations and 10 variables $a_i, b_i\ (i = 0,2,3,4)$, there are

$$\begin{cases} M_{00} - M_{10} = 0, \\ 3M_{10} + 4M_{01} - 3M_{20} - 4M_{11} = 0, \\ M_{20} + 4M_{11} + 2M_{02} - M_{30} - 4M_{21} - 2M_{12} = 0, \\ M_{30} + 12M_{21} + 18M_{12} + 4M_{03} - 12M_{31} - 18M_{22} - 4M_{13} = 0, \\ 4M_{31} + 18M_{22} + 12M_{13} + M_{04} - 18M_{32} - 12M_{23} - M_{14} = 0, \\ 2M_{32} + 4M_{23} + M_{14} - 4M_{33} - M_{24} = 0, \\ 4M_{33} + 3M_{24} - 3M_{34} = 0, \\ M_{34} = 0. \end{cases} \tag{24}$$

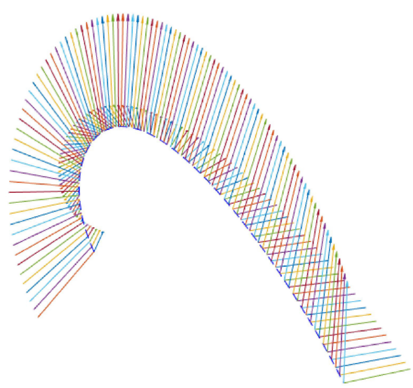
The constraint Equations (24) is a system of quadratic equations with polynomial coefficients $a_i, b_i\ (i = 0,1,2,3,4)$, which are solved by optimization method, so obtain two polynomials $a(t), b(t)$. The ER frames are further rotated and transformed by the formula (10), and the next step normalizes these two components, then obtain the global rotation-minimizing (GRM) frames. Of course, this is an approximately GRM frames by optimization method, because the exact polynomials are always non-existent.

5. Examples

We use the three quaternions $\mathbf{A}_0 = (0,1,0,2)^{\mathrm{T}}$, $\mathbf{A}_1 = (0,2,1,-1)^{\mathrm{T}}$, $\mathbf{A}_2 = (0,3,-2,-1)^{\mathrm{T}}$ in follow examples. Figure 1(a) is the original ER frame, Figure 1(b) is the GRM frames through optimization with $\rho = 1$. After calculating, the two polynomials are



(a) ER frames



(b) GRM frames with $\rho = 1$

Figure 1. Comparison of two different frames.

$$a(t) = 6.5789B_{4,0}(t) + 3.1495B_{4,1}(t) - 2.1811B_{4,2}(t)$$
$$+15.3863B_{4,3}(t) + 17.1115B_{4,4}(t),$$
$$b(t) = -0.8943B_{4,0}(t) + 1.0685B_{4,1}(t) - 0.6119B_{4,2}(t)$$
$$+1.1118B_{4,3}(t) + 3.9292B_{4,4}(t).$$

Figure 2 describes the swept surface with ER frames and RM frames with $\rho = 1$, Figure 2(a) is corresponding to the ER frames, and Figure 2(b) is to the GRM frames.

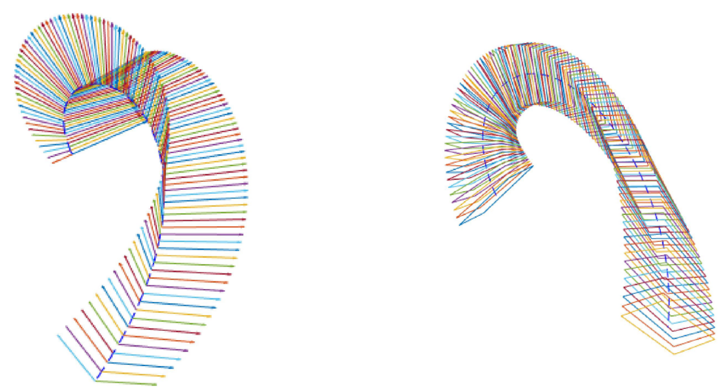
Figure 3 shows the GRM frame through optimization with $\rho = |\tau(t)\kappa(t)|$, in this $\tau(t)$ is the torsion of the curve, and $\kappa(t)$ is the curvature. Using this optimizing method, the polynomials are

$$a(t) = 9.8125B_{4,0}(t) + 1.1353B_{4,1}(t) - 1.1893B_{4,2}(t)$$
$$+15.1672B_{4,3}(t) + 9.1231B_{4,4}(t),$$
$$b(t) = -1.4645B_{4,0}(t) + 0.8832B_{4,1}(t) - 0.4792B_{4,2}(t)$$
$$-0.0104B_{4,3}(t) + 2.7759B_{4,4}(t).$$

Figure 4 describes the exact RM frames through the ordinary differential equation numerical calculation method, Figure 4(a) is the frames, and Figure 4(b) is its swept surface.

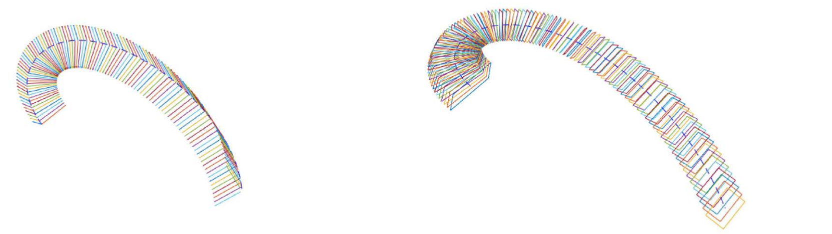
(a) ER frames swept surface (b) GRM frames swept surface with $\rho = 1$

Figure 2. Comparison of two swept surfaces with different frames.



(a) GRM frames with $\rho = |\tau(t)\kappa(t)|$ (b) GRM frames swept surface with $\rho = |\tau(t)\kappa(t)|$

Figure 3. GRM frames with $\rho = |\tau(t)\kappa(t)|$.



(a) The RM frames is rotated with ER frames, which uses the exact $\theta(t)$ by formula (9)

(b) The swept surface of the RM frames through the ordinary differential equation method

Figure 4. The RM frames raised by the ordinary differential equation method.

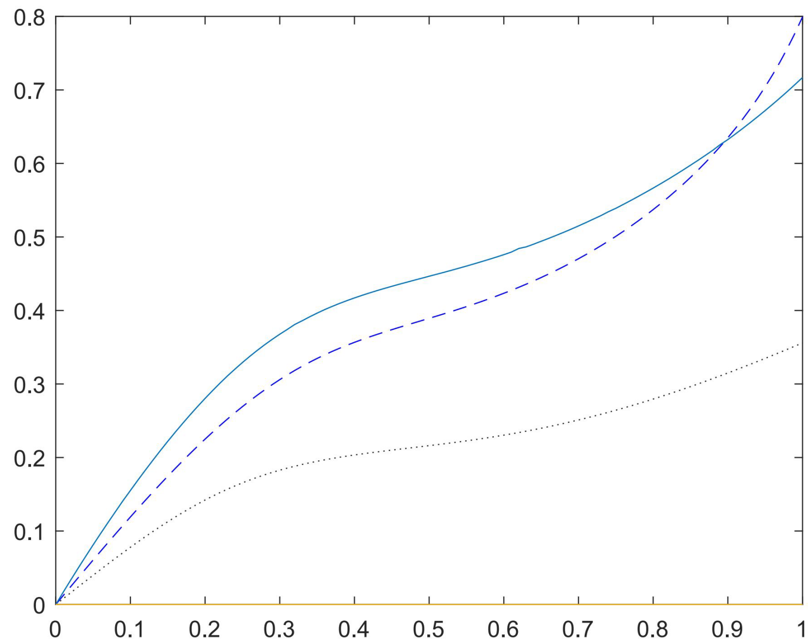


Figure 5. Angle of rotation $\theta(t)$ change curve, solid curve is to $\rho = 1$, dash curve is to $\rho = |\tau(t)\kappa(t)|$, and the dotted curve is the exact value of the RM frames which is calculated by calculating integral, the arc length of the curve is $\frac{17}{3}$.

Figure 5 describes the rotation angle changing along parameter t with different weight function. The exact value is the lower limit for each method of rotation minimizing. The rotation amount of the optimization method is greater than the exact method, but the curve arc length is $\frac{17}{3}$, as an approximation method, the effect can be acceptable. The choice of weight function may reduce the rotation amount, we usually define it as $\rho = 1$.

6. Conclusions

To a spatial PH curve built by the quaternion, ER frames are natural orthogonal frames. An ordinary differential equation of the rotation angle function is deduced by differentiation. The RM frames of those ER frames can be obtained by calculating its integral. The optimization method to construct an approximate rotation-minimizing frames is proposed on it ER frames, that avoid the existence to the ER frames with the transformation of rational form, and the weight function of the optimization objective function can optimize its effect. Moreover, we try to avoid the derivation and integration of rational polynomials in the process of its implementation, to reduce the amount of calculation. In terms of experimental results, it has a good practical effect.

The effect is good when the arc length of the curve is not large, but when the arc length is large, the effect of the objective function close to zero is not very ideal. When the torsion transformation of the curve itself is very large, its effect also has a certain impact.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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