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Semigroup of Weakly Continuous Operators Associated to a Generalized Schrödinger Equation

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Abstract

In this work, we prove the existence and uniqueness of the solution of the generalized Schrödinger equation in the periodic distributional space P'. Furthermore, we prove that the solution depends continuously respect to the initial data in P'. Introducing a family of weakly continuous operators, we prove that this family is a semigroup of operators in P'. Then, with this family of operators, we get a fine version of the existence and dependency continuous theorem obtained. Finally, we provide some consequences of this study.

Keywords

Semigroups Theory, Weakly Continuous Operators, Existence of Solution, Generalized Schrödinger Equation, Distributional Problem, Periodic Distributional Space

1. Introduction

We know from [1], that the generalized Schrödinger equation

$$u_t - i\mu\partial_x^m u = 0 \in P' \tag{1}$$

with initial data in the periodic distributional space: P', has a solution in $C(I\!R, P')$. This also happens for a generalized Schrödinger type homogeneous model given in [2].

Now, if we add a dissipative term to the problem (1), it is natural to set up the model:

$$u_t - i\mu\partial_x^m u = \beta\partial_x^q u \in P' \tag{2}$$

with initial data in P', which we will solve following the ideas of [1] and [3].

That is, we will prove that (2) has a solution and that it is unique. Furthermore, we will demonstrate that the solution depends continuously with respect to the initial data in P', considering the weak convergence in P'. And we will prove that the introduced family of operators forms a semigroup of weakly continuous operators. Thus, with this family we will rewrite our result in an elegant version.

In [2], the conservative nature of the problem allowed to obtain a group of weakly continuous operators, and here from ($P_{m,q}$) we will generate a semigroup of weakly continuous operators.

We also want to highlight the wealth of information from Terence [4], Kato [5], Liu-Zheng [6], Muñoz [7] and references [8] and [9].

Our article is organized as follows. In Section 2, we indicate the methodology used and cite the references used. In Section 3, we put the results obtained from our study. This section is divided into three subsections. Thus, in 3.1 we prove that the problem ($P_{m,q}$) has a unique solution and also demonstrate that the solution depends continuously with respect to the initial data. In Subsection 3.2, we introduce families of weakly continuous linear operators in P' that manage to form a semigroup. In Subsection 3.3 we improve Theorem 3.1.

Finally, in Section 4 we give the conclusions of this study.

2. Methodology

As theoretical framework in this article we use the references [1] [3] [10] [11] and [12] for Fourier Theory in periodic distributional space, periodic Sobolev spaces, topological vector spaces, weakly continuous operators, semigroup of operators and existence of solution of a distributional differential equation.

Thus, for a quick review of some definitions necessary for the development of this work, we cite [2].

We will use this theory in the analysis of the existence and continuous dependence of the solution of ($P_{m,q}$), carrying out a series of calculations and approximations in the process.

3. Main Results

The presentation of the results obtained has been organized in subsections and is as follows.

3.1. Solution of the Generalized Schrödinger Equation ($P_{m,q}$)

In this subsection we will study the existence of a solution to the problem ($P_{m,q}$) and the continuous dependence of the solution with respect to the initial data in P'.

Theorem 3.1 *Let* $\mu > 0$, $\beta > 0$, *m and q are even number not a multiple of four, and the distributional problem*

$$\left(P_{m,q}\right)\quad \left|\begin{array}{l} u \in C\left(\left[0,+\infty\right), P'\right) \\ \partial_t u - i\mu\partial_x^m u = \beta\partial_x^q u \in P' \\ u(0) = f \in P'. \end{array}\right.$$

then ($P_{m,q}$) has a unique solution $u \in C^1\left(\left(0,+\infty\right),P'\right)$. Furthermore, the solution depends continuously on the initial data. That is, given $f_n, f \in P'$ such that $f_n \xrightarrow{P'} f$ implies $u_n(t) \xrightarrow{P'} u(t)$, $\forall t \in \left[0,+\infty\right)$, where u_n is solution of ($P_{m,q}$) with initial data f_n and u is solution of ($P_{m,q}$) with initial data *f*.

Proof.- We have organized the proof as follows.

1) Suppose there exists $u \in C\left(\left[0,+\infty\right),P'\right)$ satisfying ($P_{m,q}$); this will allow us to obtain the explicit form of *u*. Then taking the Fourier transform to the equation

$$\partial_t u - i\mu\partial_x^m u = \beta\partial_x^q u,$$

we get

$$-\beta k^q \hat{u} = \beta\left(ik\right)^q \hat{u} = \partial_t \hat{u} - i\mu\left(ik\right)^m \hat{u} = \partial_t \hat{u} + i\mu k^m \hat{u},$$

which for each $k \in \mathbb{Z}$ is an ODE with initial data $\hat{u}(k,0) = \hat{f}(k)$.

Thus, we propose an uncoupled system of homogeneous first-order ordinary differential equations

$$\left(\Omega_k\right) \left| \begin{array}{l} \hat{u} \in C\left(\left(0,+\infty\right),S'(\mathbb{Z})\right) \\ \partial_t \hat{u}(k,t) + i\mu k^m \hat{u}(k,t) = -\beta k^q \hat{u}(k,t) \\ \hat{u}(k,0) = \hat{f}(k) \text{ with } \hat{f} \in S'(\mathbb{Z}), \end{array} \right.$$

$\forall k \in \mathbb{Z}$ and we get

$$\hat{u}(k,t) = \mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\hat{f}(k),$$

from where we obtain the explicit expression of *u*, candidate for solution:

$$u(t) = \sum_{k=-\infty}^{+\infty} \hat{u}(k,t)\phi_k = \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\hat{f}(k)\phi_k \tag{1}$$

$$= \left[\left(\hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\right)_{k\in Z}\right]^{\vee}. \tag{2}$$

Since $f \in P'$ then $\hat{f} \in S'(Z)$. Thus, we affirm that

$$\left(\hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\right)_{k\in Z} \in S'(Z), \quad \forall t \geq 0. \tag{3}$$

Indeed, let $t \geq 0$, since $\hat{f} \in S'(Z)$ then satisfies: $\exists C > 0$, $\exists N \in I\!N$ such that $\left|\hat{f}(k)\right| \leq C\left|k\right|^N$, $\forall k \in Z - \{0\}$, using this we get

$$\left|\hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\right| = \left|\hat{f}(k)\mathrm{e}^{-i\mu k^m t}\right|\underbrace{\mathrm{e}^{-\beta k^q t}}_{\leq 1} \leq \left|\hat{f}(k)\right|\underbrace{\left|\mathrm{e}^{-i\mu k^m t}\right|}_{=1} = \left|\hat{f}(k)\right| \leq C\left|k\right|^N.$$

Then,

$$\left(\hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\right)_{k\in Z} \in S'(Z).$$

If we define

$$u(t) := \left[\left(\hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\right)_{k\in Z}\right]^{\vee}, \text{ for all } t \geq 0, \tag{4}$$

we have that $u(t) \in P'$, $\forall t \geq 0$, since we apply the inverse Fourier transform to

$\left(\hat{f}(k)e^{-i\mu k^m t}e^{-\beta k^q t}\right)_{k\in Z} \in S'(Z)$.

2) We will prove that u defined in (4) is solution of ($P_{m,q}$) and $u \in C^1((0,\infty), P')$.

Evaluating (2) at $t=0$, we obtain

$$u(0) = \left[\left(\hat{f}(k)\right)_{k\in Z}\right]^{\vee} = \left[\hat{f}\right]^{\vee} = f.$$

Also, the following statements are verified.

a) $\partial_t u(t) = i\mu \partial_x^m u(t) + \beta \partial_x^q u(t)$ in P', $\forall t>0$. That is, we will prove that it is satisfied:

$$\underbrace{\lim_{h\to 0}\left\langle \frac{u(t+h)-u(t)}{h}, \varphi \right\rangle}_{\langle \partial_t u(t),\varphi\rangle :=} = i\mu\left\langle \partial_x^m u(t), \varphi\right\rangle + \beta\left\langle \partial_x^q u(t),\varphi\right\rangle, \forall \varphi \in P$$

and for all $t>0$.

Indeed, let $t>0$, $\varphi \in P$ and $0<|h|<t$, we denote

$$I_{h,t} := \left\langle \frac{u(t+h)-u(t)}{h}, \varphi \right\rangle.$$

Thus, we get

$$\begin{aligned}
I_{h,t} &= \frac{1}{h}\left\{\left\langle u(t+h),\varphi\right\rangle - \left\langle u(t),\varphi\right\rangle\right\} \\
&= \frac{1}{h}\left\{\lim_{n\to+\infty}\left\langle \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m(t+h)} e^{-\beta k^q (t+h)}\phi_k, \varphi\right\rangle \right. \\
&\quad \left. - \lim_{n\to+\infty}\left\langle \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\phi_k, \varphi\right\rangle\right\} \\
&= \frac{1}{h}\left\{\lim_{n\to+\infty}\left\langle \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\left(e^{-i\mu k^m h}e^{-\beta k^q h}-1\right)\phi_k, \varphi\right\rangle\right\} \\
&= \lim_{n\to+\infty}\left\langle \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\left(\frac{e^{-i\mu k^m h}e^{-\beta k^q h}-1}{h}\right)\phi_k, \varphi\right\rangle \\
&= \lim_{n\to+\infty}\left\{ \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\left(\frac{e^{-i\mu k^m h}e^{-\beta k^q h}-1}{h}\right)\underbrace{\left\langle\phi_k, \varphi\right\rangle}_{=2\pi\hat{\varphi}(-k)}\right\} \\
&= \lim_{n\to+\infty} 2\pi\left\{ \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\left(\frac{e^{-i\mu k^m h}e^{-\beta k^q h}-1}{h}\right)\hat{\varphi}(-k)\right\} \\
&= 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\left(\frac{e^{-i\mu k^m h}e^{-\beta k^q h}-1}{h}\right)\hat{\varphi}(-k).
\end{aligned} \tag{5}$$

Let $h>0$, we have

$$\begin{aligned}
e^{-i\mu k^m h}e^{-\beta k^q h} - 1 &= \int_0^h \left[e^{-i\mu k^m s}e^{-\beta k^q s}\right]' \mathrm{d}s \\
&= \int_0^h \left(-i\mu k^m - \beta k^q\right) e^{-i\mu k^m s}e^{-\beta k^q s}\mathrm{d}s.
\end{aligned} \tag{6}$$

Taking norm to equality (6) we obtain

$$\begin{aligned}\left|\mathrm{e}^{-i\mu k^m h}\mathrm{e}^{-\beta k^q h}-1\right| &\le \int_0^h \left\{\mu|k|^m+\beta|k|^q\right\}\underbrace{\left|\mathrm{e}^{-i\mu k^m s}\right|}_{=1}\underbrace{\mathrm{e}^{-\beta k^q s}}_{\le 1}\mathrm{d}s \\ &= \left\{\mu|k|^m+\beta|k|^q\right\}\underbrace{\int_0^h \mathrm{d}s}_{=h} \\ &= \left\{\mu|k|^m+\beta|k|^q\right\}h.\end{aligned} \tag{7}$$

That is, from (7) we get

$$\left|\frac{\mathrm{e}^{-i\mu k^m h}\mathrm{e}^{-\beta k^q h}-1}{h}\right| \le \mu|k|^m+\beta|k|^q. \tag{8}$$

Using the inequality (8) and that $\hat{f}\in S'(Z)$ we obtain

$$\begin{aligned}&\sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\underbrace{\left|\mathrm{e}^{-i\mu k^m t}\right|}_{=1}\underbrace{\mathrm{e}^{-\beta k^q t}}_{\le 1}\left|\hat{\varphi}(-k)\right|\left|\frac{\mathrm{e}^{-i\mu k^m h}\mathrm{e}^{-\beta k^q h}-1}{h}\right| \\ &\le \sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right|\left\{\mu|k|^m+\beta|k|^q\right\} \\ &= \mu\sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right||k|^m+\beta\sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right||k|^q \\ &\le C\left\{\mu\sum_{k=-\infty}^{+\infty}|k|^{N+m}\left|\hat{\varphi}(\underbrace{-k}_{=J})\right|+\beta\sum_{k=-\infty}^{+\infty}|k|^{N+q}\left|\hat{\varphi}(\underbrace{-k}_{=J})\right|\right\} \\ &= C\left\{\mu\sum_{J=-\infty}^{+\infty}|J|^{N+m}\left|\hat{\varphi}(J)\right|+\beta\sum_{J=-\infty}^{+\infty}|J|^{N+q}\left|\hat{\varphi}(J)\right|\right\}<\infty\end{aligned}$$

since $\hat{\varphi}\in S(Z)$.

If $h<0$ we have

$$\left|\frac{\mathrm{e}^{-i\mu k^m h}\mathrm{e}^{-\beta k^q h}-1}{h}\right| \le \left\{\mu|k|^m+\beta|k|^q\right\}\mathrm{e}^{-\beta k^q h}. \tag{9}$$

Using the inequality (9), $0<|h|<t$ and that $\hat{f}\in S'(Z)$ we obtain

$$\begin{aligned}&\sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\underbrace{\left|\mathrm{e}^{-i\mu k^m t}\right|}_{=1}\mathrm{e}^{-\beta k^q t}\left|\hat{\varphi}(-k)\right|\left|\frac{\mathrm{e}^{-i\mu k^m h}\mathrm{e}^{-\beta k^q h}-1}{h}\right| \\ &\le \sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right|\underbrace{\mathrm{e}^{-\beta k^q (t+h)}}_{\le 1}\left\{\mu|k|^m+\beta|k|^q\right\} \\ &= \mu\sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right||k|^m+\beta\sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right||k|^q \\ &\le C\left\{\mu\sum_{k=-\infty}^{+\infty}|k|^{N+m}\left|\hat{\varphi}(\underbrace{-k}_{=J})\right|+\beta\sum_{k=-\infty}^{+\infty}|k|^{N+q}\left|\hat{\varphi}(\underbrace{-k}_{=J})\right|\right\} \\ &= C\left\{\mu\sum_{J=-\infty}^{+\infty}|J|^{N+m}\left|\hat{\varphi}(J)\right|+\beta\sum_{J=-\infty}^{+\infty}|J|^{N+q}\left|\hat{\varphi}(J)\right|\right\}<\infty\end{aligned}$$

since $\hat{\varphi}\in S(Z)$.

Using the Weierstrass M-Test, the series $I_{h,t}$ is absolute and uniformly convergent. Then we can take limit and get

$$\lim_{h\to 0} I_{h,t} = 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \hat{\varphi}(-k) \underbrace{\lim_{h\to 0}\left\{\frac{e^{-i\mu k^m h} e^{-\beta k^q h} - 1}{h}\right\}}_{=-i\mu k^m - \beta k^q}$$

$$= (-i\mu) 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \hat{\varphi}(-k) k^m - \beta 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \hat{\varphi}(-k) k^q. \tag{10}$$

Using (10) and that $\langle T^{(r)}, \varphi \rangle = (-1)^r \langle T, \varphi^{(r)} \rangle = \langle T, \varphi^{(r)} \rangle$ for $\varphi \in P$, $T \in P'$ with r a even number, we have

$$\lim_{h\to 0} I_{h,t} = (-i\mu) 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \underbrace{\hat{\varphi}(-k)}_{=\frac{1}{2\pi}\langle \varphi, \phi_k \rangle} k^m - \beta 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \underbrace{\hat{\varphi}(-k)}_{=\frac{1}{2\pi}\langle \varphi, \phi_k \rangle} k^q$$

$$= i\mu \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \left\langle \varphi, \underbrace{-k^m \phi_k}_{=(ik)^m \phi_k} \right\rangle + \beta \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \left\langle \varphi, \underbrace{-k^q \phi_k}_{=(ik)^q \phi_k} \right\rangle$$

$$= i\mu \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \underbrace{\left\langle \varphi, \phi_k^{(m)} \right\rangle}_{=\langle \varphi^{(m)}, \phi_k \rangle} + \beta \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \underbrace{\left\langle \varphi, \phi_k^{(q)} \right\rangle}_{=\langle \varphi^{(q)}, \phi_k \rangle}$$

$$= i\mu \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \left\langle \phi_k, \varphi^{(m)} \right\rangle + \beta \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \left\langle \phi_k, \varphi^{(q)} \right\rangle$$

$$= i\mu \lim_{n\to+\infty} \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \left\langle \phi_k, \varphi^{(m)} \right\rangle + \beta \lim_{n\to+\infty} \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \left\langle \phi_k, \varphi^{(q)} \right\rangle$$

$$= i\mu \lim_{n\to+\infty} \left\langle \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \phi_k, \varphi^{(m)} \right\rangle + \beta \lim_{n\to+\infty} \left\langle \sum_{k=-n}^{n} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \phi_k, \varphi^{(q)} \right\rangle \tag{11}$$

$$= i\mu \left\langle u(t), \varphi^{(m)} \right\rangle + \beta \left\langle u(t), \varphi^{(q)} \right\rangle$$

$$= i\mu \left\langle \partial_x^m u(t), \varphi \right\rangle + \beta \left\langle \partial_x^q u(t), \varphi \right\rangle.$$

Therefore,

$$\langle \partial_t u(t), \varphi \rangle = i\mu \langle \partial_x^m u(t), \varphi \rangle + \beta \langle \partial_x^q u(t), \varphi \rangle, \quad \forall \varphi \in P, \quad \forall t > 0.$$

That is,

$$\partial_t u(t) = i\mu \partial_x^m u(t) + \beta \partial_x^q u(t) \quad \text{in } P', \quad \forall t > 0.$$

b) $u \in C([0,+\infty), P')$. That is, we will prove that

$$u(t+h) \xrightarrow{P'} u(t) \text{ when } h \to 0, \forall t \geq 0.$$

In effect, let $t > 0$, $\varphi \in P$, we will prove that

$$H_{t,h} := \langle u(t+h) - u(t), \varphi \rangle \to 0, \quad \text{when } h \to 0.$$

We know that if $\varphi \in P$ then $\hat{\varphi} \in S(Z)$. Using (5) we have

$$H_{t,h} = 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \left(e^{-i\mu k^m h} e^{-\beta k^q h} - 1 \right) \hat{\varphi}(-k).$$

Let $0 < h < 1$, from (8) we get

$$\left| e^{-i\mu k^m h} e^{-\beta k^q h} - 1 \right| \leq \mu |k|^m |h| + \beta k^q |h| < \mu |k|^m + \beta |k|^q. \tag{12}$$

Using (12) and that $\hat{f} \in S'(Z)$ we obtain

$$\sum_{k=-\infty}^{+\infty} \left|\hat{f}(k)\right| \underbrace{\left|e^{-i\mu k^m t}\right|}_{=1} \underbrace{e^{-\beta k^q t}}_{\leq 1} \left|e^{-i\mu k^m h} e^{-\beta k^q h} - 1\right| \left|\hat{\varphi}(-k)\right|$$

$$\leq C\mu \sum_{k=-\infty}^{+\infty} |k|^{N+m} \left|\hat{\varphi}(\underbrace{-k}_{=J})\right| + C\beta \sum_{k=-\infty}^{+\infty} |k|^{N+q} \left|\hat{\varphi}(\underbrace{-k}_{=J})\right|$$

$$= C\mu \sum_{J=-\infty}^{+\infty} |J|^{N+m} \left|\hat{\varphi}(J)\right| + C\beta \sum_{J=-\infty}^{+\infty} |J|^{N+q} \left|\hat{\varphi}(J)\right| < \infty$$

since $\hat{\varphi} \in S(Z)$.

Let $h < 0$, $|h| < 1$ from (9) we get

$$\left|e^{-i\mu k^m h} e^{-\beta k^q h} - 1\right| \leq e^{-\beta k^q h} \left\{\mu |k|^m + \beta |k|^q\right\}. \tag{13}$$

Using (13), $0 < |h| < t$ and that $\hat{f} \in S'(Z)$ we obtain

$$\sum_{k=-\infty}^{+\infty} \left|\hat{f}(k)\right| \underbrace{\left|e^{-i\mu k^m t}\right|}_{=1} e^{-\beta k^q t} \left|e^{-i\mu k^m h} e^{-\beta k^q h} - 1\right| \left|\hat{\varphi}(-k)\right|$$

$$\leq C\mu \sum_{k=-\infty}^{+\infty} |k|^{N+m} \left|\hat{\varphi}(\underbrace{-k}_{=J})\right| + C\beta \sum_{k=-\infty}^{+\infty} |k|^{N+q} \left|\hat{\varphi}(\underbrace{-k}_{=J})\right|$$

$$= C\mu \sum_{J=-\infty}^{+\infty} |J|^{N+m} \left|\hat{\varphi}(J)\right| + C\beta \sum_{J=-\infty}^{+\infty} |J|^{N+q} \left|\hat{\varphi}(J)\right| < \infty$$

since $\hat{\varphi} \in S(Z)$.

Using the Weierstrass M-Test we conclude that the series $H_{t,h}$ converges absolute and uniformly. Then it is possible to take limit and obtain

$$\lim_{h\to 0} H_{t,h} = 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \hat{\varphi}(-k) \underbrace{\lim_{h\to 0} \left\{e^{-i\mu k^m h} e^{-\beta k^q h} - 1\right\}}_{=0} = 0. \tag{14}$$

Doing $t = 0$ in $H_{t,h}$ with $h > 0$, we have

$$H_{0,h} = 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) \hat{\varphi}(-k) \left\{e^{-i\mu k^m h} e^{-\beta k^q h} - 1\right\}.$$

Using (12) and that $\hat{f} \in S'(Z)$ we obtain

$$\sum_{k=-\infty}^{+\infty} \left|\hat{f}(k)\right| \left|\hat{\varphi}(-k)\right| \left|e^{-i\mu k^m h} e^{-\beta k^q h} - 1\right|$$

$$\leq C\mu \sum_{k=-\infty}^{+\infty} |J|^{N+m} \left|\hat{\varphi}(J)\right| + C\beta \sum_{k=-\infty}^{+\infty} |J|^{N+q} \left|\hat{\varphi}(J)\right| < \infty$$

since $\hat{\varphi} \in S(Z)$.

Using the Weierstrass M-Test we conclude that the series $H_{0,h}$ converges absolute and uniformly. Then is possible to take limit and obtain

$$\lim_{h\to 0^+} H_{0,h} = 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) \hat{\varphi}(-k) \underbrace{\lim_{h\to 0^+} \left\{e^{-i\mu k^m h} e^{-\beta k^q h} - 1\right\}}_{=0} = 0. \tag{15}$$

From (14) and (15) we can conclude that

$$u \in C\left([0,\infty), P'\right).$$

c) $\partial_t u \in C\left(I\!R^+, P'\right)$. That is, we will prove that

$$\partial_t u(t+h) \xrightarrow{P'} \partial_t u(t) \text{ when } h \to 0, \forall t \in I\!R^+.$$

In effect, let $t \in I\!R^+$ and $\varphi \in P$, using item a) we have

$$\begin{aligned} &\left\langle \partial_t u(t+h), \varphi \right\rangle - \left\langle \partial_t u(t), \varphi \right\rangle \\ &= i\mu\left\{\left\langle \partial_x^m u(t+h), \varphi \right\rangle - \left\langle \partial_x^m u(t), \varphi \right\rangle\right\} + \beta\left\{\left\langle \partial_x^q u(t+h), \varphi \right\rangle - \left\langle \partial_x^q u(t), \varphi \right\rangle\right\} \\ &= i\mu \underbrace{\left\{\left\langle u(t+h), \varphi^{(m)} \right\rangle - \left\langle u(t), \varphi^{(m)} \right\rangle\right\}}_{\to 0} + \beta \underbrace{\left\{\left\langle u(t+h), \varphi^{(q)} \right\rangle - \left\langle u(t), \varphi^{(q)} \right\rangle\right\}}_{\to 0} \to 0 \end{aligned} \tag{16}$$

when $h \to 0$, since item b) is valid with $\varphi^{(r)} \in P$ for $r \in \{m, q\}$.

From b) and c) we have that $u \in C^1\left(I\!R^+, P'\right)$.

3) Now, we will prove that the solution depends continuously respect to initial data. That is, if $f_n \xrightarrow{P'} f$ we will prove that:

$$u_n(t) \xrightarrow{P'} u(t), \quad \forall t \in I\!R^+.$$

We know that if $f_n \xrightarrow{P'} f$ then $\hat{f}_n \xrightarrow{S'(Z)} \hat{f}$, that is

$$\left\langle \hat{f}_n - \hat{f}, \xi \right\rangle \to 0 \quad \text{when } n \to +\infty, \quad \forall \xi \in S(Z). \tag{17}$$

For $t \in I\!R^+$ fixed and arbitrary, we want to prove that

$$\left\langle u_n(t), \psi \right\rangle \to \left\langle u(t), \psi \right\rangle \text{ when } n \to +\infty, \quad \forall \psi \in P.$$

Thus, let $t \in I\!R^+$ be fixed and $\psi \in P$, using the generalized Parseval identity, we obtain the following equalities:

$$\left\langle u_n(t), \psi \right\rangle = 2\pi \left\langle \left(\hat{f}_n(k) e^{-i\mu k^m t} e^{-\beta k^q t} \right)_{k \in Z}, \tilde{\hat{\psi}} \right\rangle \tag{18}$$

$$\left\langle u(t), \psi \right\rangle = 2\pi \left\langle \left(\hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \right)_{k \in Z}, \tilde{\hat{\psi}} \right\rangle. \tag{19}$$

From (18) and (19) we obtain:

$$\left\langle u_n(t), \psi \right\rangle - \left\langle u(t), \psi \right\rangle = 2\pi \sum_{k=-\infty}^{+\infty} \left\{ \hat{f}_n(k) - \hat{f}(k) \right\} \underbrace{e^{-i\mu k^m t} e^{-\beta k^q t} \tilde{\hat{\psi}}(k)}_{\xi_k :=} \to 0$$

when $n \to +\infty$, since $\xi := (\xi_k)_{k \in Z} \in S(Z)$ and (17) holds.

□

Corollary 3.1 *Let* $\mu > 0$, $\beta > 0$, *m and q are even number not a multiple of four, then the unique solution of*($P_{m,q}$) *is*

$$u(t) = \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \phi_k = \left[\left(\hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \right)_{k \in Z} \right]^{\vee},$$

where $\phi_k(x) = e^{ikx}$, $x \in I\!R$.

3.2. Semigroup of Operators in *P'*

Let's remember that P' is the topological dual of *P*, where *P* is a complete metric space.

In this subsection, we will introduce families of operators $\left\{T_{\mu,\beta}(t)\right\}_{t \ge 0}$ in P', with $\mu > 0$ and $\beta > 0$; and we will prove that these operators are continuous in the weak sense. That is, $T_{\mu,\beta}(t)$ is continuous from P' to P' with the weak topology of P', which we will call the weakly continuous operator.

Furthermore, we will prove that $T_{\mu,\beta}(t)$ satisfies the semigroup properties.

For simplicity, we will denote this family of operators by $\{T(t)\}_{t\ge 0}$.

Theorem 3.2 *Let* $t \ge 0$, $\mu > 0$ *and* $\beta > 0$, *we define*:

$$T(t): P' \to P'$$
$$f \to T(t) f := \left[\left(\hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\right)_{k\in Z}\right]^{\vee} \in P',$$

then the following statements are satisfied:

1) $T(0) = I$.

2) $T(t)$ is $\mathbb{C}$ - linear and weakly continuous $\forall t \ge 0$. That is, for every $t \ge 0$, if $f_n \xrightarrow{P'} f$ then $T(t) f_n \xrightarrow{P'} T(t) f$.

3) $T(t+r) = T(t) \circ T(r)$, $\forall t, r \ge 0$.

4) $T(t) f \xrightarrow{P'} f$ when $t \to 0^+$, $\forall f \in P'$.

That is, for each $f \in P'$ fixed, the following is satisfied

$$\langle T(t) f, \psi \rangle \to \langle f, \psi \rangle, \text{ when } t \to 0^+, \forall \psi \in P.$$

Proof. - Let $f \in P'$ then $\hat{f} \in S'(Z)$. Then, from (3) we have

$$\left(\hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\right)_{k\in Z} \in S'(Z);$$

taking the inverse Fourier transform, we obtain

$$\underbrace{\left[\left(\hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t}\right)_{k\in Z}\right]^{\vee}}_{=T(t)f} \in P', \quad \forall t \ge 0.$$

That is, $T(t)$ is well defined for all $t \ge 0$.

1) We easily obtain:

$$T(0) f = \left[\left(\hat{f}(k) e^{-i\mu k^m 0} e^{-\beta k^q 0}\right)_{k\in Z}\right]^{\vee} = \left[\left(\hat{f}(k)\right)_{k\in Z}\right]^{\vee} = \left[\hat{f}\right]^{\vee} = f, \quad \forall f \in P'.$$

2) Let $t \in \mathbb{R}^+$, we will prove that $T(t): P' \to P'$ is $\mathbb{C}$-linear. In effect, let $a \in \mathbb{C}$ and $(\phi, \psi) \in P' \times P'$, we have

$$\begin{aligned}
T(t)(a\phi + \psi) &= \left[\left(e^{-i\mu k^m t} e^{-\beta k^q t} \left[a\phi + \psi\right]^{\wedge}(k)\right)_{k\in Z}\right]^{\vee} \\
&= \left[\left(e^{-i\mu k^m t} e^{-\beta k^q t} \left[a\hat{\phi}(k) + \hat{\psi}(k)\right]\right)_{k\in Z}\right]^{\vee} \\
&= \left[a\left(e^{-i\mu k^m t} e^{-\beta k^q t} \hat{\phi}(k)\right)_{k\in Z} + \left(e^{-i\mu k^m t} e^{-\beta k^q t} \hat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\
&= a\left[\left(e^{-i\mu k^m t} e^{-\beta k^q t} \hat{\phi}(k)\right)_{k\in Z}\right]^{\vee} + \left[\left(e^{-i\mu k^m t} e^{-\beta k^q t} \hat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\
&= aT(t)\phi + T(t)\psi.
\end{aligned}$$

Now, for $t \in \mathbb{R}^+$ we will prove that $T(t): P' \to P'$ is weakly continuous. That is, if $f_n \xrightarrow{P'} f$ we will prove that $T(t) f_n \xrightarrow{P'} T(t) f$.

We know that if $f_n \xrightarrow{P'} f$ then $\hat{f}_n \xrightarrow{S'} \hat{f}$, that is,

$$\left\langle \hat{f}_n, \xi \right\rangle \to \left\langle \hat{f}, \xi \right\rangle, \quad \text{when } n \to +\infty, \quad \forall \xi \in S(Z).$$

That is,

$$\left\langle \hat{f}_n - \hat{f}, \xi \right\rangle \to 0, \quad \text{when } n \to +\infty, \quad \forall \xi \in S(Z). \tag{20}$$

We want to prove that:

$$\left\langle T(t) f_n, \psi \right\rangle \to \left\langle T(t) f, \psi \right\rangle \text{ when } n \to +\infty, \quad \forall \psi \in P.$$

Thus, let $t \in I\!R^+$ fixed and $\psi \in P$, using the generalized Parseval identity, we obtain the following equalities

$$\begin{aligned} \left\langle T(t) f_n, \psi \right\rangle &= \left\langle \left[\left(\hat{f}_n(k) \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \right)_{k\in Z} \right]^{\vee}, \psi \right\rangle \\ &= 2\pi \left\langle \left(\hat{f}_n(k) \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \right)_{k\in Z}, \tilde{\hat{\psi}} \right\rangle, \end{aligned} \tag{21}$$

$$\begin{aligned} \left\langle T(t) f, \psi \right\rangle &= \left\langle \left[\left(\hat{f}(k) \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \right)_{k\in Z} \right]^{\vee}, \psi \right\rangle \\ &= 2\pi \left\langle \left(\hat{f}(k) \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \right)_{k\in Z}, \tilde{\hat{\psi}} \right\rangle. \end{aligned} \tag{22}$$

From (21) and (22) we get

$$\begin{aligned} &\left\langle T(t) f_n, \psi \right\rangle - \left\langle T(t) f, \psi \right\rangle \\ &= 2\pi \left\{ \left\langle \left(\hat{f}_n(k) \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \right)_{k\in Z}, \tilde{\hat{\psi}} \right\rangle - \left\langle \left(\hat{f}(k) \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \right)_{k\in Z}, \tilde{\hat{\psi}} \right\rangle \right\} \\ &= 2\pi \left\{ \sum_{k=-\infty}^{+\infty} \hat{f}_n(k) \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \tilde{\hat{\psi}}(k) - \sum_{k=-\infty}^{+\infty} \hat{f}(k) \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \tilde{\hat{\psi}}(k) \right\} \\ &= 2\pi \sum_{k=-\infty}^{+\infty} \left\{ \hat{f}_n(k) - \hat{f}(k) \right\} \underbrace{\mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \tilde{\hat{\psi}}(k)}_{\xi_k :=} \to 0 \end{aligned}$$

when $n \to +\infty$, since $\xi := (\xi_k)_{k\in Z} \in S(Z)$ and (20) holds, that is $\left\langle \hat{f}_n - \hat{f}, \xi \right\rangle \to 0$ when $n \to +\infty$.

3) Let $t, r \in I\!R^+$, we will prove that $T(t) \circ T(r) = T(t+r)$. In effect, let $\phi \in P'$,

$$\begin{aligned} T(t+r)\phi &= \left[\left(\hat{\phi}(k) \mathrm{e}^{-i\mu k^m (t+r)} \mathrm{e}^{-\beta k^q (t+r)} \right)_{k\in Z} \right]^{\vee} \\ &= \left[\left(\underbrace{\hat{\phi}(k) \mathrm{e}^{-i\mu k^m r} \mathrm{e}^{-\beta k^q r}} \cdot \mathrm{e}^{-i\mu k^m t} \mathrm{e}^{-\beta k^q t} \right)_{k\in Z} \right]^{\vee}. \end{aligned} \tag{23}$$

Since $\phi \in P'$, using (3) we have that

$$\left(\hat{\phi}(k) \mathrm{e}^{-i\mu k^m r} \mathrm{e}^{-\beta k^q r} \right)_{k\in Z} \in S'(Z), \quad \forall r \in [0, +\infty). \tag{24}$$

Then, taking the inverse Fourier transform, we get:

$$\left[\left(\hat{\phi}(k) \mathrm{e}^{-i\mu k^m r} \mathrm{e}^{-\beta k^q r} \right)_{k\in Z} \right]^{\vee} \in P', \quad \forall r \in [0, +\infty).$$

Thus, we define:

$$g_r := \left[\left(\hat{\phi}(k)\mathrm{e}^{-i\mu k^m r}\mathrm{e}^{-\beta k^q r}\right)_{k\in Z}\right]^{\vee} \in P'.$$

That is,

$$g_r := T(r)\phi. \tag{25}$$

Taking the Fourier transform to g_r we get:

$$\hat{g}_r = \left(\hat{\phi}(k)\mathrm{e}^{-i\mu k^m r}\mathrm{e}^{-\beta k^q r}\right)_{k\in Z},$$

that is,

$$\hat{g}_r(k) = \hat{\phi}(k)\mathrm{e}^{-i\mu k^m r}\mathrm{e}^{-\beta k^q r}, \quad \forall k \in Z. \tag{26}$$

Using (26) in (23) and from (25) we have:

$$\begin{aligned}
T(t+r)\phi &= \left[\left(\hat{g}_r(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\right)_{k\in Z}\right]^{\vee} \in P' \\
&= T(t)g_r \\
&= T(t)\left(T(r)\phi\right) \\
&= \left[T(t)\circ T(r)\right](\phi), \quad \forall t,r \in I\!R^+.
\end{aligned}$$

So we have proven,

$$T(t+r) = T(t)\circ T(r), \quad \forall t,r \in I\!R^+. \tag{27}$$

If $t=0$ or $r=0$ then equality (27) is also true, with this we conclude the proof of

$$T(t+r) = T(t)\circ T(r), \quad \forall t,r \in [0,+\infty). \tag{28}$$

4) Let $f \in P'$, we will prove that:

$$T(t)f \xrightarrow{P'} f \text{ when } t \to 0^+.$$

That is, we will prove that

$$\langle T(t)f,\varphi\rangle \to \langle f,\varphi\rangle \text{ when } t\to 0^+, \quad \forall \varphi \in P.$$

In effect, for $t>0$ and $\varphi \in P$, we have

$$\begin{aligned}
H_t &:= \langle T(t)f,\varphi\rangle - \langle f,\varphi\rangle \\
&= \lim_{n\to+\infty}\left\{\left\langle \sum_{k=-n}^{n}\hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\phi_k,\varphi\right\rangle - \left\langle \sum_{k=-n}^{n}\hat{f}(k)\phi_k,\varphi\right\rangle\right\} \\
&= \lim_{n\to+\infty}\left\langle \sum_{k=-n}^{n}\hat{f}(k)\left(\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}-1\right)\phi_k,\varphi\right\rangle \\
&= \lim_{n\to+\infty}\sum_{k=-n}^{n}\hat{f}(k)\left(\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}-1\right)\langle\phi_k,\varphi\rangle \\
&= \lim_{n\to+\infty}2\pi\sum_{k=-n}^{n}\hat{f}(k)\left(\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}-1\right)\hat{\varphi}(-k) \\
&= 2\pi\sum_{k=-\infty}^{+\infty}\hat{f}(k)\left(\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}-1\right)\hat{\varphi}(-k).
\end{aligned} \tag{29}$$

Since $t>0$, from (7) we get

$$\left|e^{-i\mu k^m t}e^{-\beta k^q t}-1\right|\le\left\{\mu|k|^m+\beta|k|^q\right\}t. \tag{30}$$

From (30) we obtain

$$\left|e^{-i\mu k^m t}e^{-\beta k^q t}-1\right|\le\left\{\mu|k|^m+\beta|k|^q\right\}t,\quad \forall t\in[0,+\infty). \tag{31}$$

From (31) with $0<t<1$, we have

$$\left|e^{-i\mu k^m t}e^{-\beta k^q t}-1\right|\le\mu|k|^m+\beta|k|^q. \tag{32}$$

Then using (32) and that $f\in P'$, we obtain

$$\begin{aligned}&\sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|e^{-i\mu k^m t}e^{-\beta k^q t}-1\right|\left|\hat{\varphi}(-k)\right|\\ &\le C\left\{\mu\sum_{k=-\infty}^{+\infty}|k|^{N+m}\left|\hat{\varphi}(\underbrace{-k}_{=J})\right|+\beta\sum_{k=-\infty}^{+\infty}|k|^{N+q}\left|\hat{\varphi}(\underbrace{-k}_{=J})\right|\right\}\\ &=C\left\{\mu\sum_{J=-\infty}^{+\infty}|J|^{N+m}\left|\hat{\varphi}(J)\right|+\beta\sum_{J=-\infty}^{+\infty}|J|^{N+q}\left|\hat{\varphi}(J)\right|\right\}<\infty\end{aligned}$$

since $\hat{\varphi}\in S(Z)$.

Using the Weierstrass M-Test we conclude that the $\mathcal{H}_t$ series converges absolute and uniformly. So,

$$\lim_{t\to0^+}\mathcal{H}_t=2\pi\sum_{k=-\infty}^{+\infty}\hat{f}(k)\hat{\varphi}(-k)\underbrace{\lim_{t\to0^+}\left\{e^{-i\mu k^m t}e^{-\beta k^q t}-1\right\}}_{=0}=0.$$

Thus, we have proved

$$\lim_{t\to0^+}\langle T(t)f,\varphi\rangle=\langle f,\varphi\rangle.$$

□

Theorem 3.3 *For each* $f\in P'$ *fixed and the family of operators* $\{T(t)\}_{t\ge0}$ *from Theorem 3.2, then the application*

$$\begin{aligned}\zeta:[0,+\infty)&\to P'\\ t&\to T(t)f\end{aligned}$$

is continuous in $[0,+\infty)$. That is,

$$T(t+h)f\xrightarrow{P'}T(t)f\quad\text{when } h\to0,\forall t\in[0,+\infty). \tag{33}$$

(is the continuity at *t*).

That is, (33) tell us that for each $t\in(0,+\infty)$ fixed, the following is satisfied

$$\langle T(t+h)f,\psi\rangle\to\langle T(t)f,\psi\rangle,\quad\text{when } h\to0,\forall\psi\in P,$$

and if $t=0$, we have the continuity of ζ at 0 on the right, which is item 4) of Theorem 3.2.

Proof.- Let $t>0$, arbitrary fixed and $f\in P'$, then $g:=T(t)f\in P'$, using item 4) of Theorem 3.2 we have that $T(h)g\xrightarrow{P'}g$ when $h\to0^+$. That is,

$$\underbrace{\underbrace{T(h)\left(T(t)f\right)}_{=[T(h)\circ T(t)]f}}_{=T(h+t)f}\xrightarrow{P'}T(t)f\quad\text{when } h\to0^+,$$

where we use item 3) of Theorem 3.2. With this we have proved that

$$T(t+h)f \xrightarrow{P'} T(t)f \quad \text{when } h \to 0^+, \forall t \in (0,+\infty). \tag{34}$$

Now, we will prove that $T(t+\nu)f \xrightarrow{P'} T(t)f$ when $\nu \to 0^-$. That is, we will demonstrate

$$\langle T(t-h)f,\varphi\rangle \to \langle T(t)f,\varphi\rangle \text{ when } h \to 0^+, \quad \forall \varphi \in P. \tag{35}$$

In effect, for $t > h > 0$ and $\varphi \in P$, we have

$$\begin{aligned}
\mathcal{L}_{t,h} &:= \langle T(t-h)f,\varphi\rangle - \langle T(t)f,\varphi\rangle \\
&= \lim_{n\to+\infty}\left\{\left\langle \sum_{k=-n}^{n} \hat{f}(k)\mathrm{e}^{-i\mu k^m(t-h)}\mathrm{e}^{-\beta k^q(t-h)}\phi_k,\varphi\right\rangle \right. \\
&\quad \left. -\left\langle \sum_{k=-n}^{n} \hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\phi_k,\varphi\right\rangle\right\} \\
&= \lim_{n\to+\infty}\left\langle \sum_{k=-n}^{n} \hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\left(\mathrm{e}^{i\mu k^m h}\mathrm{e}^{\beta k^q h}-1\right)\phi_k,\varphi\right\rangle \\
&= \lim_{n\to+\infty}\sum_{k=-n}^{n} \hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\left(\mathrm{e}^{i\mu k^m h}\mathrm{e}^{\beta k^q h}-1\right)\langle\phi_k,\varphi\rangle \\
&= \lim_{n\to+\infty} 2\pi\sum_{k=-n}^{n} \hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\left(\mathrm{e}^{i\mu k^m h}\mathrm{e}^{\beta k^q h}-1\right)\hat{\varphi}(-k) \\
&= 2\pi\sum_{k=-\infty}^{+\infty} \hat{f}(k)\mathrm{e}^{-i\mu k^m t}\mathrm{e}^{-\beta k^q t}\left(\mathrm{e}^{i\mu k^m h}\mathrm{e}^{\beta k^q h}-1\right)\hat{\varphi}(-k).
\end{aligned} \tag{36}$$

In the series (36), we need to delimit the expression $\mathrm{e}^{-i\mu k^m h}\mathrm{e}^{-\beta k^q h}-1$. So, we have

$$\begin{aligned}
\mathrm{e}^{i\mu k^m h}\mathrm{e}^{\beta k^q h}-1 &= \int_0^h \left[\mathrm{e}^{(i\mu k^m+\beta k^q)s}\right]' \mathrm{d}s \\
&= \left(i\mu k^m+\beta k^q\right)\int_0^h \mathrm{e}^{(i\mu k^m+\beta k^q)s}\mathrm{d}s.
\end{aligned} \tag{37}$$

Taking norm to equality (37) and using: $\int_0^h \mathrm{e}^{(\beta k^q)s}\mathrm{d}s \le \mathrm{e}^{(\beta k^q)h}h$ for $h > 0$, we obtain

$$\begin{aligned}
\left|\mathrm{e}^{i\mu k^m h}\mathrm{e}^{\beta k^q h}-1\right| &\le \left|i\mu k^m+\beta k^q\right|\int_0^h \mathrm{e}^{(\beta k^q)s}\mathrm{d}s \\
&\le \left\{\mu|k|^m+\beta|k|^q\right\}\mathrm{e}^{(\beta k^q)h}\cdot h \\
&\le \left\{\mu|k|^m+\beta|k|^q\right\}\mathrm{e}^{(\beta k^q)h}
\end{aligned} \tag{38}$$

whenever $0 < h < 1$.

Using inequality (38) and $\mathrm{e}^{-\beta k^q(t-h)} \le 1$ for $0 < h < t$ with $h \ll 1$, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|\mathrm{e}^{-i\mu k^m t}\right|\mathrm{e}^{-\beta k^q t}\left|\mathrm{e}^{i\mu k^m h}\mathrm{e}^{\beta k^q h}-1\right|\left|\hat{\varphi}(-k)\right| \\
&\le \sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\mathrm{e}^{-\beta k^q(t-h)}\left\{\mu|k|^m+\beta|k|^q\right\}\left|\hat{\varphi}(-k)\right| \\
&\le \sum_{k=-\infty}^{+\infty}\left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right|\left\{\mu|k|^m+\beta|k|^q\right\}
\end{aligned}$$

$$\begin{aligned}
&\le \mu \sum_{k=-\infty}^{+\infty} \left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right||k|^m + \beta \sum_{k=-\infty}^{+\infty} \left|\hat{f}(k)\right|\left|\hat{\varphi}(-k)\right||k|^q \\
&\le C\left\{\mu \sum_{k=-\infty}^{+\infty} |k|^{N+m}\left|\hat{\varphi}(-k)\right| + \beta \sum_{k=-\infty}^{+\infty} |k|^{N+q}\left|\hat{\varphi}(-k)\right|\right\} \qquad (39) \\
&\le C\left\{\mu \sum_{J=-\infty}^{+\infty} |J|^{N+m}\left|\hat{\varphi}(J)\right| + \beta \sum_{J=-\infty}^{+\infty} |J|^{N+q}\left|\hat{\varphi}(J)\right|\right\} < \infty
\end{aligned}$$

since $\hat{\varphi} \in S(Z)$.

Using the Weierstrass M-Test we obtain that the series $L_{h,t}$ is absolute and uniformly convergent.

Then, we can take limit and get:

$$\lim_{h\to 0^+} \mathcal{L}_{h,t} = 2\pi \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{-i\mu k^m t} e^{-\beta k^q t} \underbrace{\lim_{h\to 0^+}\left\{e^{i\mu k^m h} e^{\beta k^q h} - 1\right\}}_{=0} \hat{\varphi}(-k) = 0,$$

with this (35) is proved.

From (34) and (35) we conclude that

$$T(t+h) f \xrightarrow{P'} T(t) f \quad \text{when } h \to 0, \forall t \in (0,+\infty). \qquad (40)$$

□

Remark 3.1 *The results obtain in Theorems* 3.2 *and* 3.3 *are also valid for the family of operators* $\{S(t)\}_{t\ge 0}$, *defined as*

$$\begin{aligned}
&S(t): P' \to P' \\
&f \to S(t) f := \left[\left(e^{i\mu k^m t} e^{-\beta k^q t} \hat{f}(k)\right)_{k\in Z}\right]^{\vee},
\end{aligned}$$

for $t \in [0,+\infty)$. Its proof is similar.

3.3. Version of Theorem 3.1 Using the Family $\{T(t)\}_{t\ge 0}$

We improve the statement of theorem 3.1, using a family of weakly continuous Operators $\{T(t)\}_{t\ge 0}$.

Theorem 3.4 *Let* $f \in P'$ *and the family of operators* $\{T(t)\}_{t\ge 0}$ *from Theorem* 3.2, *defining* $u(t) := T(t) f \in P'$, $\forall t \in [0,+\infty)$, *then* $u \in C([0,+\infty), P')$ *is the unique solution of* ($P_{m,q}$). *Furthermore, u continuously depends on f. That is, given* $f_n, f \in P'$ *with* $f_n \xrightarrow{P'} f$ *implies* $u_n(t) \xrightarrow{P'} u(t)$, $\forall t \in [0,+\infty)$, *where* $u_n(t) := T(t) f_n$, $\forall t \in [0,+\infty)$ (*that is,* u_n *is a solution of* ($P_{m,q}$) *with initial data* f_n).

Proof.- It is analogous to the proof of Theorem 3.1.

□

Corollary 3.2 *Let* $f \in P'$ *be fixed and the family of operators* $\{T(t)\}_{t\ge 0}$ *from Theorem* 3.4, *then* $\exists \partial_t T(t) f$, $\forall t \in (0,+\infty)$ *and the mapping*

$$\begin{aligned}
&\eta: (0,+\infty) \to P' \\
&t \to \partial_t T(t) f = i\mu \partial_x^m T(t) f + \beta \partial_x^q T(t) f
\end{aligned}$$

is continuous at $(0,+\infty)$. That is,

$$\partial_t T(t+h) f \xrightarrow{P'} \partial_t T(t) f \quad \text{when } h \to 0, \quad \forall t \in (0,+\infty). \tag{41}$$

(41) tells us that for each $t \in (0,+\infty)$ fixed, it holds:

$$\langle \partial_t T(t+h) f, \varphi \rangle \to \langle \partial_t T(t) f, \varphi \rangle \text{ when } h \to 0, \quad \forall \varphi \in P.$$

Proof.- Indeed,

$$\begin{aligned}
&\langle \partial_t T(t+h) f, \varphi \rangle - \langle \partial_t T(t) f, \varphi \rangle \\
&= i\mu \left\{ \langle \partial_x^m T(t+h) f, \varphi \rangle - \langle \partial_x^m T(t) f, \varphi \rangle \right\} + \beta \left\{ \langle \partial_x^q T(t+h) f, \varphi \rangle - \langle \partial_x^q T(t) f, \varphi \rangle \right\} \\
&= i\mu \underbrace{\left\{ \langle T(t+h) f, \varphi^{(m)} \rangle - \langle T(t) f, \varphi^{(m)} \rangle \right\}}_{\to 0} + \beta \underbrace{\left\{ \langle T(t+h) f, \varphi^{(q)} \rangle - \langle T(t) f, \varphi^{(q)} \rangle \right\}}_{\to 0} \\
&\to 0
\end{aligned}$$

when $h \to 0$, due to Theorem 3.3 with $\psi := \varphi^{(r)} \in P$ for $r \in \{m, q\}$.

□

Corollary 3.3 *Let* $f \in P'$ *be fixed and the family of operators* $\{T(t)\}_{t \ge 0}$ *from Theorem* 3.4, *then the solution of* ($P_{m,q}$): $u(t) := T(t) f$, $\forall t \in [0,+\infty)$, *satisfies* $u \in C^1((0,+\infty), P')$.

Proof.- It comes out as a consequence of Corollary 3.2.

□

4. Conclusions

In our study of the generalized Schrödinger equation in the periodic distributional space P', that is, the problem ($P_{m,q}$) with m and q even numbers not multiple of four, we have obtained the following results:

1) We prove the existence, uniqueness of the solution of the problem ($P_{m,q}$) in P'. Thus we also prove the continuous dependence of the solution respect to the initial data in P'. Remember that P' is not a Banach Space.

2) We introduce families of operators in P': $\{T(t)\}_{t \ge 0}$ and we prove that they are linear and weakly continuous in P'. Furthermore, we proved that they form a semigroup of weakly continuous operators in P'.

3) With the family of operators $\{T(t)\}_{t \ge 0}$ we improve Theorem 3.1.

4) Also, note that this is mathematically enriched with the families of the generated operators and their properties.

5) In contrast to what was obtain in [2]: a group of weakly continuous operators, here we obtain a semigroup of weakly continuous operators.

6) Remark that the results obtained will allow us to apply computational methods to determine the solution with a degree of approximation that is required and with a lower error rate.

7) Finally, we must indicate that this technique can be applied to other evolution equations in P'.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Santiago Ayala, Y. (2022) Existencia de solución de un problema distribucional para una ecuación de Schrödinger generalizada. *Selecciones Matemáticas*, **9**, 91-101. https://doi.org/10.17268/sel.mat.2022.01.07

[2] Santiago Ayala, Y. (2023) Group of Weakly Continuous Operators Associated to a Generalized Schrödinger Type Homogeneous Model. *Journal of Applied Mathematics and Physics*, **11**, 919-932. https://doi.org/10.4236/jamp.2023.114062

[3] Santiago Ayala, Y. (2021) Results on the Well Posedness of a Distributional Differential Problem. *Selecciones Matemáticas*, **8**, 348-359. https://doi.org/10.17268/sel.mat.2021.02.11

[4] Tao, T. (2006) Nonlinear Dispersive Equations: Local and Global Analysis. In: *Regional Conference Series in Mathematics*, No. 106, American Mathematical Society, Rhode Island. https://bookstore.ams.org/cbms-106

[5] Kato, T. (1983) On the Cauchy Problem for the (Generalized) KdV Equations. *Applied Mathematics*, *Advances in Mathematics Supplementary Studies,* **8**, 93-128. https://cir.nii.ac.jp/crid/1571135649130864512

[6] Liu, Z. and Zheng, S. (1999) Semigroups Associated with Dissipative System. Chapman Hall/CRC, New York.

[7] Muñoz Rivera, J.E. (2007) Semigrupos e Equações Diferenciais Parciais. LNCC, Petrópolis-RJ.

[8] Li, Y., Liu, J. and Zhu, X. (2020) Blow-Up for a Periodic Two-Component Camassa-Holm Equation with Generalized Weakly Dissipation. *Journal of Applied Mathematics and Physics*, **8**, 2223-2240. https://doi.org/10.4236/jamp.2020.810167

[9] Gao, Y. (2021) Asymptotic Stability of Singular Solution for Camassa-Holm Equation. *Journal of Applied Mathematics and Physics*, **9**, 1505-1514. https://doi.org/10.4236/jamp.2021.97102

[10] Iorio, R. and Iorio, V. (2002) Fourier Analysis and Partial Differential Equations. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511623745

[11] Santiago Ayala, Y. and Rojas, S. (2017) Regularity and Wellposedness of a Problem to One Parameter and Its Behavior at the Limit. *Bulletin of the Allahabad Mathematical Society*, **32**, 207-230.

[12] Santiago Ayala, Y. and Rojas, S. (2021) Existencia y regularidad de solución de la ecuación de Schrödinger no homogénea en espacios de Sobolev Periódico. *Selecciones Matemáticas*, **8**, 37-51. https://doi.org/10.17268/sel.mat.2021.01.04