Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2023, 11, 884-890
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Matrix Boundary Value Problem on Hyperbola

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How to cite this paper: Fan, S.H. (2023) Matrix Boundary Value Problem on Hyperbola. *Journal of Applied Mathematics and Physics*, **11**, 884-890.
https://doi.org/10.4236/jamp.2023.114059

Received: March 6, 2023
Accepted: April 15, 2023
Published: April 18, 2023

Abstract

We study a special class of lower trigonometric matrix value boundary value problems on hyperbolas. Firstly, the pseudo-orthogonal polynomial on hyperbola is given in bilinear form and it is shown that it is the only one. Secondly, a special boundary value problem of lower triangular matrix is presented and transformed into four related boundary value problems. Finally, Liouville theorem and Painlevé theorem and pseudo-orthogonal polynomials are used to give solutions.

Keywords

Hyperbola, Matrix Boundary Value Problem, Orthogonal Polynomial

1. Introduction

In some references [1] [2] [3], the boundary value problem (Riemann-Hilbert problem) of analytic functions on finite curves is discussed, but the research on infinite curves is not deep enough. In [4], the author discusses the Riemann boundary value problem on the positive real axis and generalizes the concept of the generalized principal part.

The Riemann-Hilbert method is a brand-new method for studying orthogonal polynomials formed in recent 20 years. In 1992, FoKas A S, Its A R and Kitaev A V constructed a matrix-valued Riemann-Hilbert boundary value problem in [5], the only solution of which is the orthogonal polynomial on the real axis. In 1993, Deift P and Zhou X introduced the Riemann-Hilbert boundary value problem of oscillatory type in [6], and applied it to the study of orthogonal polynomials. Therefore, the Riemann-Hilbert method was formed [6].

2. Preliminary

In this paper, the right branch of the Hyperbola $x^2 - y^2 = 1$ is denoted by default to *L*, which is regarded as the image of the function $x = \varphi(y) = \sqrt{y^2 + 1}$, and L is oriented from top to bottom.

Denote by l_a the point $\varphi(a)+ia$ and $\infty^{\pm}$ respectively its upper and lower infinite ends. Then $\mathbb{C}$ consists of two connected components, the right part S^+ and the left part S^-.

We use bilinear form to replace inner product on hyperbola, which is a common way. For example, [Lu J K, 1993] gives the solvable condition of singular integral equation by this way; for example, Delft P. defined a polynomial group similar to orthogonal polynomials in bilinear form in [7], and we studied similar polynomial groups on hyperbola:

Let $w(t)$ be a nonzero weight function. We introduce bilinear form in polynomial space Π_n with degree no more than n:

$$(f,g)=\int_L w(t)f(t)g(t)\mathrm{d}t,\quad f,g\in\Pi_n \tag{1}$$

Take a group of bases $1,t,t^2,\cdots,t^n$ in Π_n and make Schmidt orthogonalization on this group of bases, then we have

$$p_0(z)=\frac{1}{\left(\int_L w\mathrm{d}t\right)^{\frac{1}{2}}}$$

$$p_1(z)=\frac{t-\frac{(t,p_0)}{(p_0,p_0)}p_0}{\left(t-\frac{(t,p_0)}{(p_0,p_0)}p_0,t-\frac{(t,p_0)}{(p_0,p_0)}p_0\right)^{\frac{1}{2}}}$$

$$\vdots$$

$$p_n(z)=\frac{t^n-\frac{(t^n,p_{n-1})}{(p_{n-1},p_{n-1})}p_{n-1}-\cdots-\frac{(t^n,p_1)}{(p_1,p_1)}p_1-\frac{(t^n,p_0)}{(p_0,p_0)}p_0}{(A_n,A_n)}$$

where $A_n=t^n-\frac{(t^n,p_{n-1})}{(p_{n-1},p_{n-1})}p_{n-1}-\cdots-\frac{(t^n,p_1)}{(p_1,p_1)}p_1-\frac{(t^n,p_0)}{(p_0,p_0)}p_0$, If (p_n,p_n) is always not zero, then this process can always be carried out. Finally, we get a pseudo-orthogonal polynomial group with a weight function of $p_0,p_1(z),\cdots,p_n(z)$ on L:

$$P_k(z)=\frac{1}{\alpha_k}p_k(z),k=0,1,\cdots,n, \tag{2}$$

where α_k is the first coefficient of $p_k(z)$, then $P_k(z)$ is a pseudo-orthogonal polynomial of degree k with the first coefficient of 1. Obviously, the pseudo-orthogonal polynomial group $P_0,P_1(z),\cdots,P_n(z)$ is unique.

Definition 1. Let f is defined on L, if there is some positive real number a, such that

$$\left|f(t')-f(t'')\right|\le M\left|\frac{1}{t'}-\frac{1}{t''}\right|^{\mu},\quad t',t''\in\widehat{l_{\infty^+}l_a}\cup\widehat{l_a l_{\infty^-}} \tag{3}$$

where M and $0<\mu\le 1$ are definite constants, then denoted by $f\in\hat{H}^{\mu}(\infty)$, and if $f \in H^{\mu}(L)$, then denoted by $f \in \hat{H}^{\mu}(L)$. If $f \in \hat{H}^{\mu}(\infty)$ and $f(\infty)=0$, then denoted by $f \in \hat{H}_0^{\mu}(\infty)$, or $f \in \hat{H}_0(\infty)$. Moreover, if $t^{\lambda} f \in \hat{H}_0(\infty)$, then denoted by $f \in \hat{H}_{\lambda,0}(\infty)$.

Definition 2 Let *f* is a function defined on *L*. There exists $t \to \infty$ such that

$$f(t)=\frac{f^{*}(t)}{t^{v}},$$

where v is a real number and f^{*} is a bounded function, then denoted by $f \in O^{v}(\infty)$.

Definition 3 If *F* is holomorphic in the complex plane cut by the Hyperbola, then denoted by $F \in A(\mathbb{C} \setminus L)$.

Definition 4 Let *f* be a locally integrable function on *L*. If

$$(C[f])(z)=\frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - z}\mathrm{d}\tau,\ \ z \in \mathbb{C}\setminus L \tag{4}$$

is integrable, it is called the Cauchy-type integral with kernel density *f* on *L*, and the Cauchy principal value integral with kernel density *f* is defined by

$$(C[f])(t)=\frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - t}\mathrm{d}\tau = \lim_{r\to 0^{+}}\frac{1}{2\pi i}\int_{|y-a|>0}\frac{f\left(\left(\varphi(y)+iy\right)\right)\left(\varphi'(y)+iy\right)}{\varphi(y)+iy-t}\mathrm{d}y \tag{5}$$

where $t=\varphi(a)+ia \in L$, if the integral exists.

Ref [Wang Ying, 2017], below we introduce the concept of a generalized main part.

Definition 5 Let $F \in A(\mathbb{C} \setminus L)$. If there exists an entire function $E(z)$ such that

$$\lim_{z\to\infty}\left[F(z)-E(z)\right]=0. \tag{6}$$

and then $E(z)$ is called the generalized principal part of $F(z)$ at ∞, denoted by $\mathrm{G.P}[F,\infty]$.

Reference [8] proves the generalized principal part of Cauchy integral at infinity and Plemelj formula.

Theorem 1 [8] If $f \in H(L)\cap O^{v}(\infty)(v>0)$ is locally integrable on *L*. Then

$$\mathrm{G.P}\left[C[f],\infty\right]=0. \tag{7}$$

Theorem 2 [8] If $f \in \hat{H}^{\mu}(L)$, then the boundary values of the Cauchy-type integral $C[f]$ exist and have the following Plemelj formula:

$$(C[f])^{\pm}(t)=\pm\frac{1}{2}f(t)+\frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - t}\mathrm{d}\tau. \tag{8}$$

3. Matrix Value Riemann Boundary Value Problem

In this paper, we consider the Riemann boundary value problem of lower trigonometric matrix on hyperbola.

Let

$$\Phi(z)=\begin{pmatrix}\Phi_{1,1}(z) & \Phi_{1,2}(z)\\ \Phi_{2,1}(z) & \Phi_{2,2}(z)\end{pmatrix} \tag{9}$$

be a matrix-valued function defined on subset Ω of the complex plane $\mathbb{C}$, and each element $\Phi_{j,k}$ be a function defined on Ω. If every element $\Phi_{j,k}$ of Φ satisfies the same property, then Φ is said to have its corresponding property, such as $\Phi \in A(\mathbb{C}\setminus L), \mathrm{G.P}[\Phi,\infty](z), \Phi \in H(L)$.

Problem (boundary value problem of lower trigonometric matrix value function) Find the matrix-valued partitioned holomorphic function Φ with L as the jump curve, such that

$$\begin{cases} \Phi^{+}(t) = \begin{pmatrix} 1 & 0 \\ w(t) & 1 \end{pmatrix} \Phi^{-}(t), t \in L, \\ \mathrm{G.P}[\Xi\Phi,\infty](z) = I, \end{cases} \tag{10}$$

where

$$\Xi(z) = \begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n} \end{pmatrix}, \tag{11}$$

I is the identity matrix of 2 × 2, $w \in H^{\mu}(L) \cap \hat{H}_{2n,0}(\infty)$.

We can convert (10) into four related Riemann boundary value problems:

$$\begin{cases} \Phi_{1,1}^{+}(t) = \Phi_{1,1}^{-}(t), \quad t \in L, \\ \mathrm{G.P}\left[z^{-n}\Phi_{1,1},\infty\right] = 1, \end{cases} \tag{12}$$

$$\begin{cases} \Phi_{1,2}^{+}(t) = \Phi_{1,2}^{-}(t), \quad t \in L, \\ \mathrm{G.P}\left[z^{-n}\Phi_{1,2},\infty\right] = 0, \end{cases} \tag{13}$$

$$\begin{cases} \Phi_{2,1}^{+}(t) = \Phi_{2,1}^{-}(t) + w(t)\Phi_{1,1}^{-}(t), \quad t \in L, \\ \mathrm{G.P}\left[z^{n}\Phi_{2,1},\infty\right] = 0, \end{cases} \tag{14}$$

$$\begin{cases} \Phi_{2,2}^{+}(t) = \Phi_{2,2}^{-}(t) + w(t)\Phi_{1,2}^{-}(t), \quad t \in L, \\ \mathrm{G.P}\left[z^{n}\Phi_{2,2},\infty\right] = 1. \end{cases} \tag{15}$$

Obviously, (12) is a Liouville problem. It is known from Painlevé theorem that $\Phi_{1,1}(z)$ is analytic over the entire complex plane. Because $\mathrm{G.P}\left[z^{-n}\Phi_{1,1},\infty\right] = 1$, it is known from the generalized Liouville theorem that

$$\Phi_{1,1}(z) = P_n(z), \tag{16}$$

where $P_n(z)$ is a polynomial with a leading coefficient of 1 and a degree of n.

By (16), we have

$$\begin{cases} \Phi_{2,1}^{+}(t) = \Phi_{2,1}^{-}(t) + w(t)P_n(t), \quad t \in L, \\ \mathrm{G.P}\left[z^{n}\Phi_{2,1},\infty\right] = 0, \end{cases} \tag{17}$$

Obviously (17) is a jump problem with L as the jump curve. Let

$$\psi(z) = C[wP_n](z) = \frac{1}{2\pi i}\int_L \frac{w(\tau)P_n(\tau)}{\tau - z}\mathrm{d}\tau, \quad z \in L, \tag{18}$$

by $w \in H^{\mu}(L) \cap \hat{H}_{2n,0}(\infty)$,

$$wP_n \in H^{\mu}(L) \cap \hat{H}_{n,0}(\infty). \tag{19}$$

Therefore, by Plemelj formula (8) and Theorem 1, we can know that $\psi(z)$ is a partitioned holomorphic function with L as the jump curve, and satisfies:

$$\begin{cases} \psi^{+}(t) = \psi^{-}(t) + \omega(t) P_n(t), t \in L, \\ \text{G.P}[\psi, \infty](z) = 0, \end{cases} \tag{20}$$

let $F(z) = \Phi(z) - \psi(z)$, then F is a partitioned holomorphic function with L as the jump curve and satisfies:

$$\begin{cases} F^{+}(t) = F^{-}(t), t \in L, \\ \text{G.P}[F, \infty] = 0, \end{cases} \tag{21}$$

Obviously problem (21) is a zero-order Liouville problem, its solution is $F(z) = 0$, so

$$\Phi_{2,1}(z) = C[wP_n](z) = \frac{1}{2\pi i} \int_L \frac{w(\tau) P_n(\tau)}{\tau - z} \mathrm{d}\tau, \ z \in \mathbb{C} \setminus L \tag{22}$$

if and only if condition $\text{G.P}\left[z^n \Phi_{2,1}, \infty\right] = 0$ is satisfied. By

$$\begin{aligned} z^n \Phi_{2,1}(z) &= \frac{1}{2\pi i} \int_L \frac{w(\tau) P_n(\tau)(z^n - \tau^n)}{\tau - z} \mathrm{d}\tau + \frac{1}{2\pi i} \int_L \frac{w(\tau) P_n(\tau) \tau^n}{\tau - z} \mathrm{d}\tau \\ &= -\sum_{k=0}^{n-1} \frac{z^k}{2\pi i} \int_L w(\tau) P_n(\tau) \tau^{n-1-k} \mathrm{d}\tau + \frac{1}{2\pi i} \int_L \frac{w(\tau) P_n(\tau) \tau^n}{\tau - z} \mathrm{d}\tau, \end{aligned} \tag{23}$$

and Theorem 1 and (19), it can be seen that $\text{G.P}\left[z^n \Phi_{2,1}, \infty\right] = 0$ is equivalent to

$$\frac{1}{2\pi i} \int_L w(\tau) P_n(\tau) \tau^k \mathrm{d}\tau = 0, k = 0, 1, \cdots, n-1. \tag{24}$$

Obviously (13) is the Liouville problem, similar to (12) we have

$$\Phi_{1,2}(z) = q_{n-1}(z) \tag{25}$$

where $q_{n-1}(z)$ is a polynomial of order not exceeding $n-1$.

By (16), we have

$$\begin{cases} \Phi_{2,2}^{+}(t) = \Phi_{2,2}^{-}(t) + w(t) q_{n-1}(t), \quad t \in L, \\ \text{G.P}\left[z^n \Phi_{2,2}, \infty\right] = 1. \end{cases} \tag{26}$$

Obviously, (26) is a fixed-order jump problem, similar to (15). It can be seen that its solution is

$$\Phi_{2,2}(z) = C[wq_{n-1}](z) = \frac{1}{2\pi i} \int_L \frac{w(\tau) q_{n-1}(\tau)}{\tau - z} \mathrm{d}\tau, \ z \in \mathbb{C} \setminus L \tag{27}$$

if and only if condition

$$\begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_L w(\tau) q_{n-1}(\tau) \tau^k \mathrm{d}\tau = 0, k = 0, 1, \cdots, n-2, \\ \dfrac{1}{2\pi i} \displaystyle\int_L w(\tau) q_{n-1}(\tau) \tau^{n-1} \mathrm{d}\tau = -1, \end{cases} \tag{28}$$

is satisfied.

Let $q_{n-1} = \lambda P_{n-1}$, then

$$\frac{1}{2\pi i} \int_L \omega(\tau) \lambda P_{n-1} P_{n-1} = -1, \tag{29}$$

that is,

$$\lambda = \frac{-2\pi i}{\int_L \omega(\tau) P_{n-1}^2(\tau)\,\mathrm{d}\tau} \tag{30}$$

then q_{n-1} is a pseudo-orthogonal polynomial of degree $n-1$ on L with respect to the weight function w.

Definition 6

$$f^*(z) = \frac{1}{2\pi i}\int_L \frac{\omega(\tau) f(\tau)}{\tau - z}\mathrm{d}\tau,\quad z \notin L, \tag{31}$$

we call it the companion function of f with respect to the weight function w.

Theorem 3 If $w \in H^{\mu}(L) \cap \hat{H}_{2n,0}(\infty)$, then the lower triangular matrix-valued Riemann boundary value problem (10) has a solution, and its solution has the following form:

$$\Phi(z) = \begin{pmatrix} P_n(z) & \lambda P_{n-1}(z) \\ P_n^*(z) & \lambda P_{n-1}^*(z) \end{pmatrix}, \tag{32}$$

where $P_n(z)$ is a polynomial with a leading coefficient of 1 and a degree of n, and P_n^* is the companion function of P_n with respect to the middle weight function w.

Proof: If (10) has a solution, it can be seen from the previous discussion that its solution is of the form (32).

Conversely, the polynomial with pseudo-orthogonal and leading coefficient 1 is unique, and by reversing each previous step, we get that Φ is the solution of (10), that is, (10) has and only one set of solutions (32).

The matrix-valued boundary value problem (10) is characterized by the pseudo-orthogonal polynomial P_n on L with respect to the weight function w and the leading coefficient is 1. Therefore, we call this problem the Riemann-Hilbert characteristic characterization of the orthogonal polynomial of the weight function w on hyperbola, or P_n is the characteristic orthogonal polynomial of the matrix-valued boundary value problem (10), please refer to [Deift P, 2011] for details.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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