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How Does Topology Help Solve the Inscribed Rectangle Problem by Proving that Every Jordan Curve Has 4 Vertices that Form a Rectangle?

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Abstract

When we stare into our complex surroundings, we see objects of different shapes and sizes. However, the shape that is always present, regardless of the complexity of the object, is the circle. The circle is arguably the most fascinating shape in the universe. A circle is defined as the set of all points equidistant from a given point, which, therefore, lies at the center of the circle. One of the various properties of circles is that it has infinite inscribed squares. This is because it is a continuous function, therefore if any point in the circle is transitioned by a specific factor, the other related points in the square would be shifted by the same factor. An ellipse is a similar shape with several inscribed squares. But does any closed curve have an inscribed square? This question was proposed by Otto Toeplitz in 1911 and to this day it is not answered. Another version of this problem is the inscribed rectangle problem which will be discussed in this paper.

Keywords

Inscribed Rectangle Problem, Topology, Jordan Curve, Mobius Strip

1. Introduction

Geometry is a very famous and well taught field in every math course. However, I noticed that it deals with the same three or four shapes: circles, squares, triangles etc. Therefore, I tried to research a mathematical field that covered all the shapes that geometry simply ignored. That's when I found topology. Topology is also known as "rubber-sheet geometry". It studies all the deformed shapes that were twisted, stretched or condensed in 2D or 3D planes. Therefore, significant information in geometry like the distance between two points is not considered in the world of topology. Consequently, the only thing that makes the two shapes different is the number of holes in them. This approach helps us solve problems like the inscribed rectangle problem.

This investigation aims to prove that there are two pairs of points that share a midpoint and are equidistant from their pair, in every Jordan Curve. Consequently, the inscribed rectangle problem is solved. Throughout this investigation, many topological and graphical skills would be used which make this problem worthy of solving.

2. Body

2.1. Investigating

Theorem: For every smooth Jordan curve γ and rectangle R in the Euclidean plane, there exists a rectangle similar to R whose vertices lie on γ.

A Jordan Curve was first proposed as a theorem by Camille Jordan in 1887 [1]. For a curve to be a Jordan Curve it needs to satisfy specific conditions. Firstly, it needs to be a non-self-intersecting loop that separates a plane into two distinct sections, such that for a line to cross from one section to the other, it needs to pass through the curve. Secondly, the Jordan curve must not be an injective function. This means that for every x value, there must be more than one y value that corresponds to it. This ensures that the Jordan Curve has a curve and isn't just a straight line. In addition, it needs to be a smooth and continuous curve with no hard edges. These conditions allow the Jordan Curve to be considered the unit circle of topology as the curve can be stretched or condensed back into a perfect circle as both shapes have one hole.

In 1906, the mathematician Arthur Moritz Schönflies extended the Jordan Curve theorem into 3D space. This introduced the concept of homeomorphism. This concept states that a Jordan Curve mapped in 3D space would give a Mobius Strip, in which the function itself and the inverse of the function are continuous and can be moulded into one another in the world of topology. When shapes are homeomorphic, one shape can be changed into another by deformation. For example, if a Mobius strip is inverted inside out, its shape and function are still consistent because it only contains one side. If the Mobius strip is reflected onto the x-axis or the y-axis, it would still be the same shape.

One of the most important factors that helped solve the inscribed rectangle problem is identifying the difference between ordered pairs and unordered pairs of points. There are two types of spatial planes in math: Cartesian and Euclidean. These planes have one significant difference, allowing the problem we are handling to be solved. Cartesian planes define points in space using a letter and a number. For example, a point may be given the letter A to distinguish it from another point, given the letter B. Each point is unique and cannot be exchanged with another point, this results in the ordered pairs of points $(x, y) \neq (y, x)$. On the Euclidean plane, points are defined as zero-dimensional objects, meaning that if any point is switched with another point on the zero-dimensional planes, there will be no difference; this results in the unordered pairs of points $(x, y) = (y, x)$. The inscribed rectangle problem can only be solved on the Euclidean plane [2].

The Mobius Strip is one of the most significant shapes in topology. It was discovered by August Mobius in 1858, and since then it has dazzled mathematicians, engineers and artists [3]. It can be created by giving a flat sheet of paper a half twist and glueing its ends together **Figure 1**. The Mobius strip is an example of a non-orientable shape. A non-orientable shape is defined as a shape where orientation is inconsistent [4]. For example, a clockwise direction will turn into a counterclockwise direction due to the loops present in the shape **Figure 2**. An orientable shape is one where it has two distinct sides and the orientation is consistent within each one, for example, a cylinder or a torus. The Mobius strip is the only shape with one side, and it also has one hole.

The Euler characteristic is one of the key properties that define certain shapes in topology. This characteristic can be calculated using multiple formulas. The most common one is:

$$x(g) = 2 - 2g, \tag{1}$$

where g stands for the genus, which is the number of holes in the object [5]. Because the Mobius strip has only 1 hole, it is one of the few shapes with an Euler characteristic of 0. This characteristic is useful in helping classify different objects without any complex information needed.

The Mobius Strip formula in the parametric form is [6]:

$$\begin{aligned} x(u,v) &= (R + v/2\cos u/2)\cos u \\ y(u,v) &= (R + v/2\cos u/2)\sin u \\ z(u,v) &= v/2\sin u/2 \end{aligned} \tag{2}$$

where: 1) R is the radius of the loop

2) u is the polar angle of each point, $0 \le u \le 2\pi$

3) v is the width of the strop, $-1 \le v \le 1$

4) $u2$ describes the number of half-twists

The Mobius strip parametric formula is derived through a series of 3D transformations. We start with: $x' = s$, $y' = 0$ and $z' = 0$, which is just a 1D shape. This is followed by 3 transformations:

Rotation of the 1D shape at an angle t2 in the vertical plane, creating a half-twist.

Shift the origin along the x-axis by a distance $R > 0$.

Rotation of the curve at an angle t in the horizontal plane

There are many forms of equations in the mathematics world. The implicit form is the most familiar and it is used to test if one point is present on the surface of the geometric shape. It takes the form $F(x) = 0$. This means that x is only possible if the function is true. For example, the unit circle is represented as:

$$x^2 + y^2 - 1 = 0 \tag{3}$$

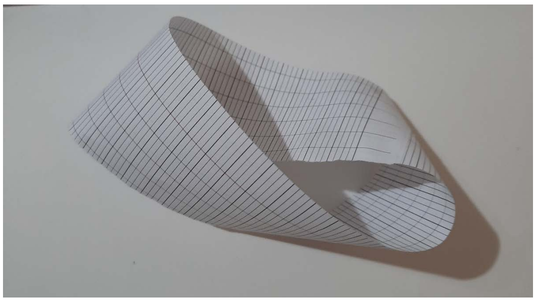

Figure 1. Mobius Strip made out of paper.

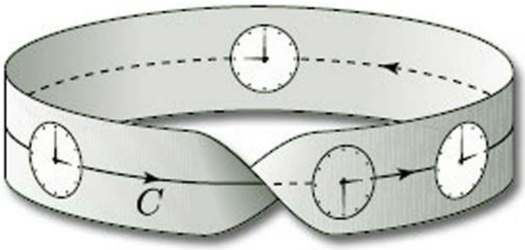

Figure 2. Mobius strip showing clock directions.

This form is non-regenerate. This means that there is no way to systematically generate consecutive points after the first one is found. Implicit forms have the following properties:

$F(x) > 0$, if x is "above" the surface

$F(x) = 0$, if x is on the surface

$F(x) < 0$, if x is "below" the surface

For example, the implicit equation of a sphere is

$$x^2 + y^2 + z^2 = r^2 \quad (4)$$

If $x^2 + y^2 + z^2 - r^2 > 0$ then the point lies outside the sphere, but if $x^2 + y^2 + z^2 - r^2 < 0$, then the point is inside the sphere [7].

The other form is the parametric form, which is the form used to give the Mobius strip formula. The parametric form is useful because it can generate multiple points from the same formula drawing the whole surface. This takes the form $x = F(u, v)$. Using the same example of the sphere, its formula can be written in spherical coordinates:

$$x = r\sin\varphi\cos\theta,\ y = -r\cos\varphi,\ z = -r\sin\varphi\sin\theta \quad (5)$$

θ: the longitude positive rotation around the y-axis, $0 \le \theta \le 2\pi$

φ: the latitude positive rotation around the z-axis, $0 \le \varphi \le \pi$

The parametric form is effective in the description of the Mobius strip as it presents a system to generate several points in 3D space.

Many contributors helped solve this problem. Many proofs were built onto the one before. Three proofs will be discussed in this section.

Lemma 1: Mark J. Nielsen and S.E. Wright were the first to try to solve this problem. They found a conditional proof, which laid a concrete ground for others to build on. They proposed that: Every simple closed curve that is symmetric about the origin has an inscribed square [8].

Step 1: Draw a Jordan Curve (γ) symmetrical about the origin.

Step 2: Choose a point on the graph where the point itself and its negative coordinates both lie on the curve. For example, $P(3, 4)$ and $P'(-3, -4)$ lie on the Jordan Curve.

Step 3: Rotate the Jordan Curve 90˚ about the origin to draw $f(\gamma)$.

Step 4: The pair of points P and $-P$ on the Jordan Curve will result in another pair of points on the new rotations. These four points make an inscribed square **Figure 3**.

Lemma 2: In 1977, H. Vaughan stated that if the Jordan curve is mapped onto 3D space (x, y, z) in the Euclidean plane, where (x, y) are points on the Jordan Curve and (z) is the distance between x and y ($x \leftrightarrow y$), the Mobius Strip would be homeomorphic to the 3D graph obtained from the Jordan curve, where any points of self-intersection on the Mobius Strip are the midpoints of inscribed rectangles on the Jordan Curve.

Because this 3D graph is a representation of the Jordan Curve, the distance between x and y is significant as it specifies the height of the graph.

This can be broken down into several steps:

Step 1: If we cut the Jordan Curve at a specific point and straighten the edge of the curve, we can get a straight line that we can label in coordinates. Because the Jordan Curve is a 2D shape, we can draw all the unordered pairs of points placed on the curve on a 2D graph.

Step 2: Because the pairs of points on the Jordan Curve are unordered ($A = A'$), we could easily fold the graph on the $y = x$ line. For example, point $A(1, 2)$ while point $A'(2, 1)$, because both points correspond to the same point on the Jordan Curve, we can fold the graph on itself. This would result in a triangle where its hypotenuse are points (x, x) e.g. (6, 6), **Figure 4**.

Step 3: If we glue the non-hypotenuse edges of the triangle onto one another, taking into consideration their separate directions, and therefore adding a half-twist to the shape, we get the Mobius strip. The edge of the Mobius strip also corresponds to the (x, x) points.

Step 4: As the Mobius strip and the Jordan Curve have one hole, they are topologically identical, therefore they can be glued to one another, with the edges of the Mobius strip being the edges of the Jordan Curve, forcing the Mobius strip to self-intersect.

Step 5: Note down all the points of self-intersection as they are the midpoints of the inscribed rectangles.

Aspect ratio (r) is one of the important concepts in this problem. Aspect ratio is defined as the ratio between the length and the width of a shape. The aspect ratio of a square is $r = 1$. Vaughan's theory didn't specify the aspect ratio of the rectangle that may be inscribed in the Jordan curve. However, further research into the knot theory resulted in additional information that specifies the inscribed rectangle problem.

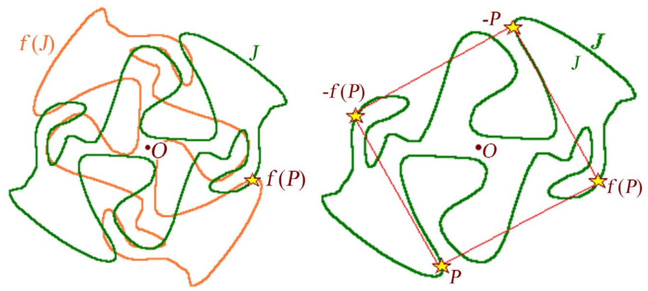

Figure 3. A rotation of the Jordan Curve presented in orange, followed by a square drawn by specific points.

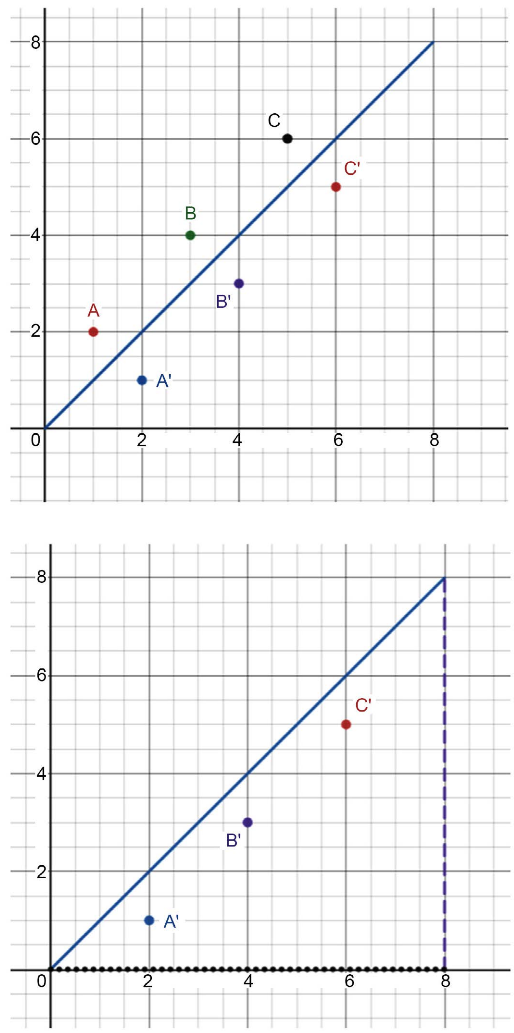

Figure 4. Three points reflected along the $y = x$ line are shown, followed by a fold onto the $y = x$ line where the points and their reflections are equal.

Lemma 3: In 2018, Cole Hugelmeyer came up with a theory that all rectangles inscribed in a Jordan curve have an aspect ratio of 3.

Hugelmeyer took this problem to the 4th dimension and applied aspect ratio properties.

Notations:

n = set of real numbers; $n \geq 3$*

$k = \{1, \ldots, n-1\}$

μ = Mobius strip

γ = Jordan curve

*If $n < 3$ then there is no self-intersection of the Mobius strip in 4D space which doesn't serve our problem.

If the rectangle (w, x, y, z) is set on the Jordan curve then,

A) If we set the point of (x, y) on the Mobius strip, it will have the following function:

$$\mu(x,y) = \left(\gamma(x)+\gamma(y)/2\right), \left(\gamma(y)-\gamma(x)\right)^{2n} \tag{6}$$

This formula defines the coordinates of (x, y) as two sets of information (A, B) where A represents the midpoint between x and y, while B represents the distance between x and y to the power of n.

B) Given that $\mu(x,y) = \mu(w,z)$, then the pairs of points share a midpoint m, and the angle between x and w must be a multiple of π/n

C) Given that the Jordan curve is not an injective function with no vanishing derivative, the aspect ratio(r) of its inscribed rectangle is $\tan(\pi k/2n)$

D) Given that $n = 3$, then the angle between the x and w is $\pi/3$

E) Given that $n = 3$, we can use the aspect ratio formula to give two results, when $k = 1$ and $k = 2$.

$$r = \tan(\pi/6) = 1/\sqrt{3} \tag{7}$$

$$r = \tan(2\pi/6) = 3 \tag{8}$$

Because the aspect ratio is only defined up to reciprocals, eg. rectangle (1:3) = rectangle (3:1). We can conclude that all rectangles inscribed in Jordan curves have an aspect ratio of 3.

2.2. Comparison

The two mathematicians that contributed to solving this problem have used advanced mathematical approaches. However, there are some defining differences. Firstly, Vaughan focused on solving this problem in 3D only and used mainly graphical skills and drawings in order to visualize the Mobius Strip and how it's glued onto the Jordan Curve. Meanwhile, Hugelmeyer raised this problem to 4D and used functions in order to reach the specific ratio that the inscribed rectangle must be. Hugelmeyer's contribution was crucial as it gives us all the information needed to prove that there is an inscribed rectangle in every Jordan Curve with a fixed ratio.

2.3. Testing

1) Using a known example, I will apply the Vaughan and Hugelmeyer methods **Figure 5**. Placed a known loop on Geogebra.

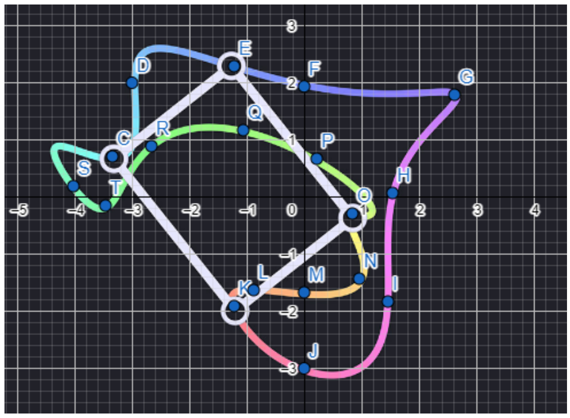

Figure 5. An example of a Jordan Curve on a line graph where random points are drawn in blue.

2) Set the picture as the background

3) Placed points C to T on loop

4) Graph the unordered coordinates of the points on a separate graph Figure 6.

5) Fold the graph onto the $y = x$ graph Figure 7.

6) Glue the red lines together, forming a Mobius Strip

7) Glue the Mobius strip back on the loop, forcing it to intersect.

8) The midpoint of the rectangle is (−1.22, 0.2).

9) Therefore the pairs of points that create the inscribed triangles are points C, O and E, K, Figure 8.

10) Other than the rectangle shown in the picture there is another rectangle that was revealed when the Mobius strip self-intersected.

11) The other midpoint is (0.42, −2.2).

12) This proves that this Jordan curve has more than one inscribed rectangle.

2.4. Proving

Using an unknown new loop, I will apply the Vaughan and Hugelmeyer methods in hardware Figure 9. In the first example, I began by following Vaughan's method. I drew a random loop on graph paper. I then built a Mobius strip out of paper. I tried to glue the edges of the Mobius strip onto the loop, however, it would not stand as a 3D graph. Afterwards, I tried a different method which is holding down the edges of the Mobius strip with pins. Halfway through the loop, I reached a point of self-intersection, which I noted down as the midpoint of the inscribed rectangle. After I reached this point, I could not go further as the Mobius strip got stuck. This is expected as I am working in a 3D space, however, in 4D, I would be able to continue glueing the Mobius strip after the intersection. To avoid this problem, I made sure to start glueing the Mobius strip from different places and that is what lead me to the second point of self-intersection. In this example, the top shape is a rectangle while the bottom one is a square. This shows that one loop can have multiple different quadrilaterals. Following Hugelemeyer's theory, the top rectangle has an aspect ratio of 2 instead of 1.7 ($\sqrt{3}$).

This error may be due to human inaccuracy in ruler measurements, the uncertainty degree of the ruler (±0.1) or the pins not held at the far edge of the Mobius strip, which is almost impossible on a piece of paper. The aspect ratio of the square is one, which is the same as the theoretical data.

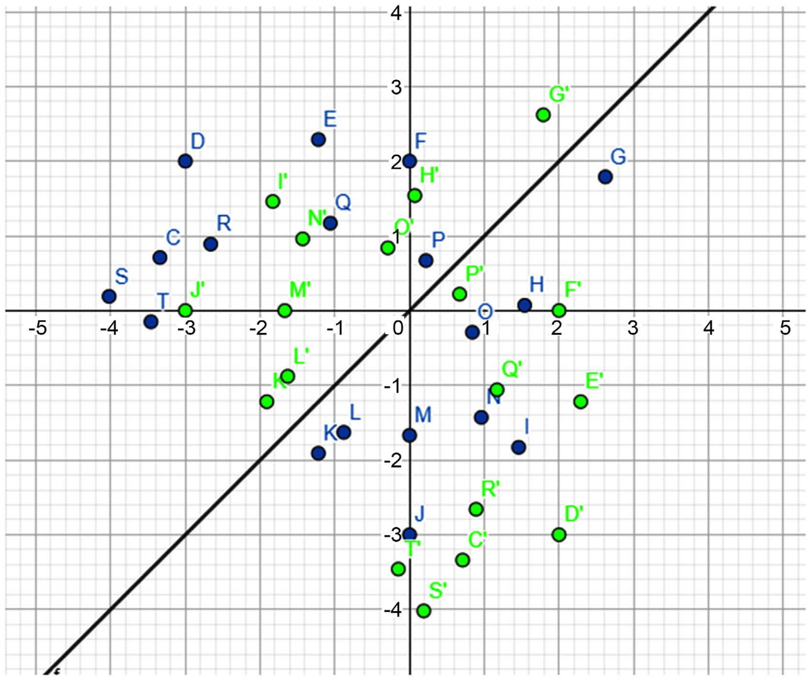

Figure 6. The Jordan Curve is removed and the original points in blue are reflected along the y = x line, creating the points in green.

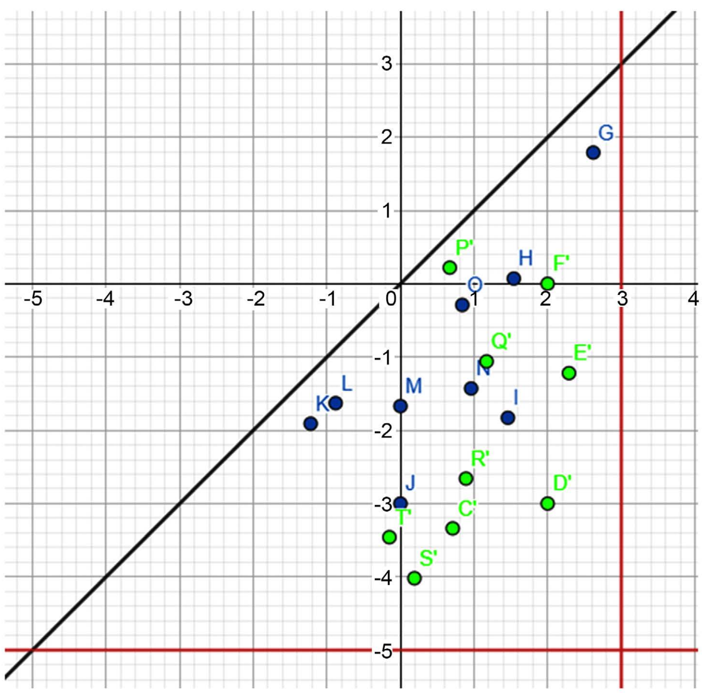

Figure 7. The graph is folded onto the y = x line.

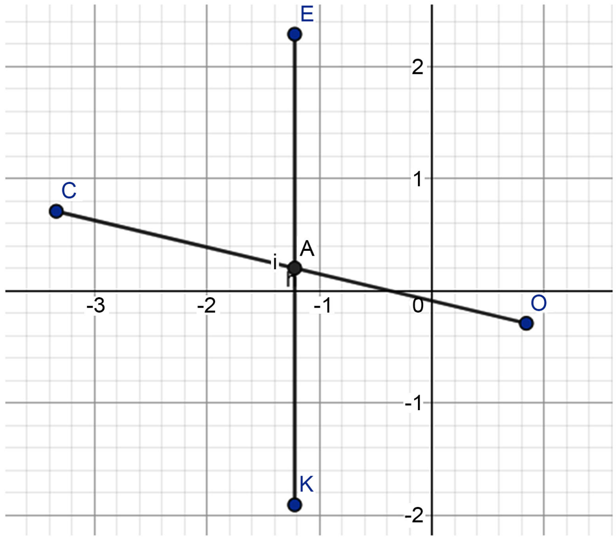

Figure 8. The four points that make a square are shown on the graph in blue with midpoint A in grey.

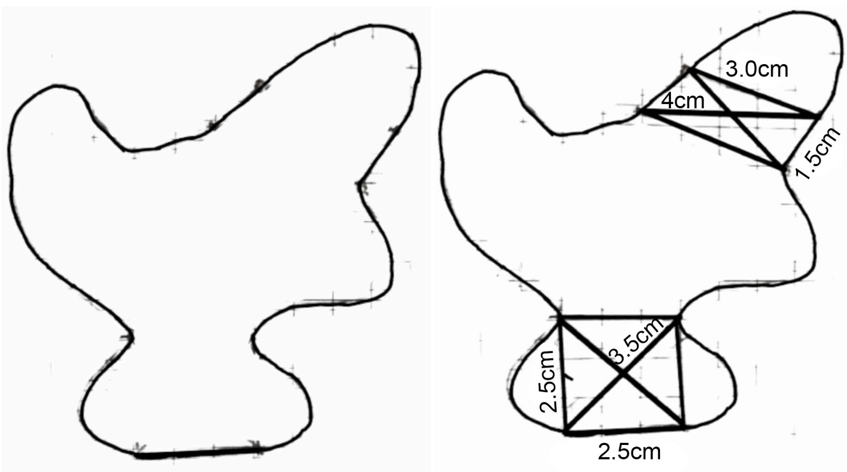

Figure 9. A plain drawing of the Jordan Curve, followed by a drawing of the two rectangles that are inscribed on the curve.

In the second example, **Figure 10** I drew a completely different loop. Having variety in the shapes of the loop would ensure that I am covering different aspects before proving that the theoretical methods work. Similar to the previous steps, I pined the Mobius strip onto the loop and noted down the midpoint of the rectangle. The aspect ratio in this example is 1.25. Similar errors may have led to a result with lower accuracy.

More examples **Figure 11** were done to ensure there is sufficient data to show that the inscribed rectangle problem is solved. In example 4, the Mobius strip is self-intersected at a midpoint of a square. This shows that Vaughan's method is not limited to rectangles only, but it is also true for squares. This contributed to solving the inscribed square problem. The aspect ratio of example 3 is 1.2 while in example 4, the aspect ratio of a square is always 1.

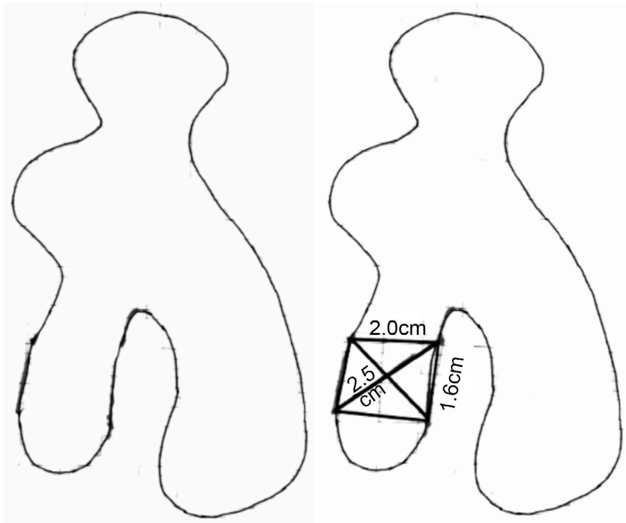

Figure 10. A plain Jordan Curve followed by a drawing showing the inscribed rectangle.

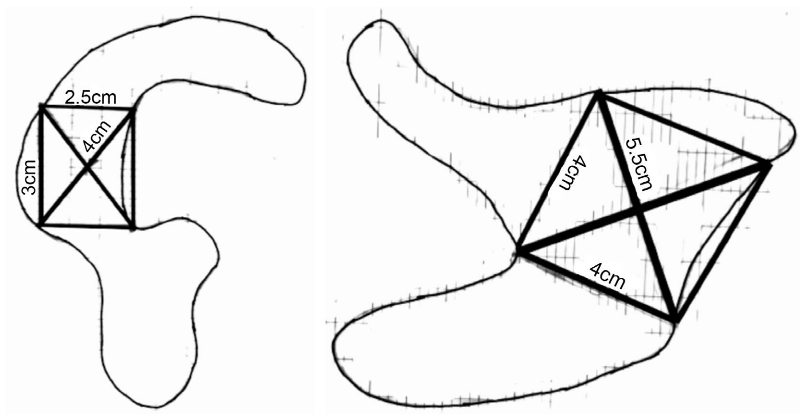

Figure 11. Two other examples showing inscribed rectangles.

2.5. Justifying

If the steps are this easy, then why did the problem remain unsolved for a long time? Topology is a field that is not very commonly studied in high schools or universities. Attempting such a question would require a high level of mathematical skills but more importantly a high level of creativity and problem-solving. Moreover, taking the problem to the unexplored area of 4D space, makes the problem more unapproachable. Several systematic and random errors have happened, but all of them are justifiable.

3. Conclusions

In conclusion, the inscribed rectangle problem has been solved through the use of topology. Many mathematicians like Vaughan and Hugelmeyer have contributed and developed the method by which it is solved. The Mobius strip can indicate the midpoint of any inscribed rectangle or square in a Jordan curve as it self-intersects itself. This problem smoothly transitions from a 2D loop on paper to a 3D model to a 4D drawing that goes beyond our world's ability.

The complexity of this problem can inspire further research into solving this problem on other mathematical planes. Currently, it is only solvable on the Euclidean plane, therefore, it's an opportunity to deepen the research and solve this problem on the Cartesian plane or even the Argand plane, which includes complex numbers and vectors.

Other than contributing to solving the inscribed square problem which hasn't been solved yet, this problem can have beneficial real-life applications.

Some mathematicians tried to optimize the biggest rectangle that can be obtained within the Jordan curve. Topology optimization is defined as maximizing the spatial capacity of the distribution of material within a given design space. This has crucial applications in the lightweight design of car mechanics and the aerospace industry. **Figure 12** They are specifically used to minimize the materials used while sustaining the same mechanical strength [9]. Earlier they used topology optimization for simple ratios like stiffness to weight; to optimize the function of the product, however, now as 3D printing is developing, the output of the topology optimization can be directly put into the 3D printer which minimizes the time needed to design and implement such products. Topology optimization usually required an engineer's initial model which it can optimize, however, it has been further developed into a new concept called "Generative design". This tool creates multiple optimized models using limited data that includes the external forces and design space, even without a preliminary model, which makes the possibilities endless.

4. Evaluation

One of the strengths of this investigation is that I included many mathematicians' perspectives, which strengthens the method of solving this problem. Moreover, I created my own loop and solved it which shows how I can apply the theoretical steps, both in Geogebra and in hardware.

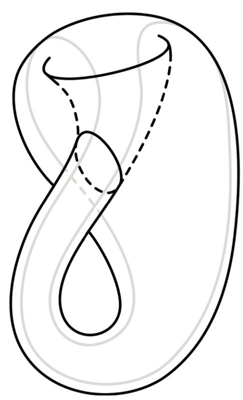

Figure 12. A 3D diagram showing the materials needed for an Audi car design developed using topology.

One of the weaknesses of this investigation is not reaching a general formula that could be applied to any curve. This shows how each loop is unique and must be solved individually even if there is information on the aspect ratio and the angle of the inscribed rectangle.

If I would repeat this investigation, I would look at a wide variety of resources as videos and scholarly articles were hard to find. Most of the discoveries are extremely recent, therefore, have not been reported or revised properly. Returning to this investigation again would ensure more reliable resources.

This investigation could be extended to include how to graph a continuous loop on an online tool and build a 3D graph onto the loop, as this was out of the limited scope of resources available. Moreover, more methods used by mathematicians could be discussed whilst comparing which one is the easiest and most useful in real-life applications.

5. Beyond the Problem

One of the numbers that we took for granted without much explanation is that $n \geq 3$ for Hugelmeyer's theory to work. This number is derived from the knot theory. This is another interesting field of topology that relates to loops. A knot is a closed curve in space. A knot is called a trivial if it can deform into a normal circle with no self-intersections. In 4D space, a knot cannot exist as there are too many dimensions for the shape to self-intersect. Andreas Floer discovered a series of invariant knots in 3D space that builds onto Donaldson's theory proved in 1983, in differential topology. One of the knots that Floer worked on is the Mobius strip and how it unknots in 4D space. He found specific dimensions that enable the Mobius strip to self-intersect even in 4D, these calculations lead to the $n \geq 3$ rule.

Differential topology is a specific field for the study of the topological properties of differentiable manifolds [10]. Differential manifolds are types of topological spaces that are similar enough to a vector space to allow calculus laws to be true. A collection of 2D charts that represent one manifold is called an atlas **Figure 13**.

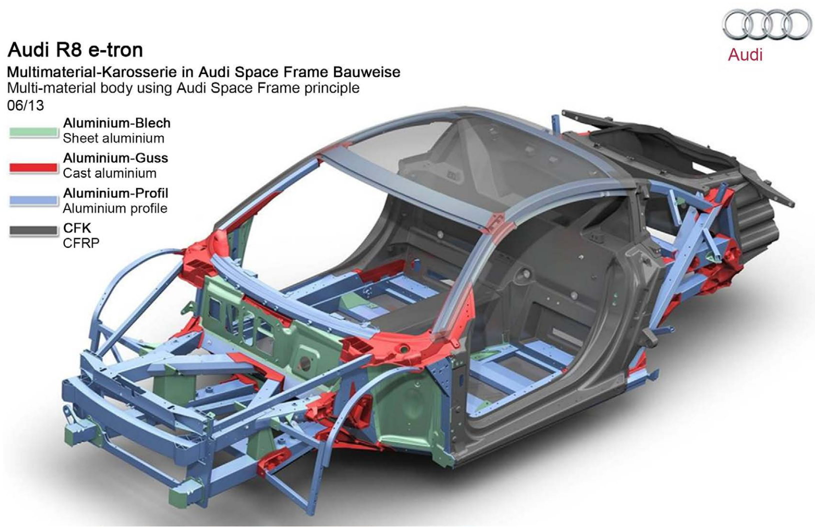

Figure 13. An image showing how differential manifolds are transferred onto two graphs.

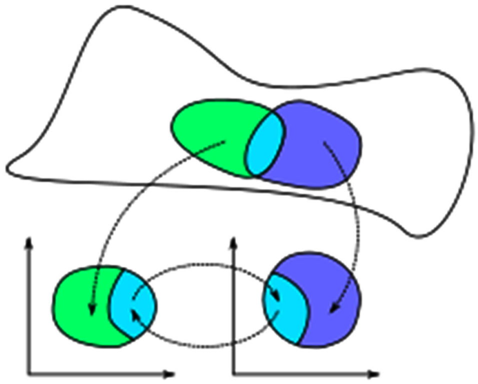

Figure 14. A 3D representation of a Klein bottle.

One of the most beneficial examples of differential manifolds is the world map. Each continent is not geometrically defined (in triangles, circles, or squares), therefore, topology is used to extract necessary data. For example, the globe can be presented in 3 different representations that correspond to different angles of the globe. Each representation is not identical to the other and some continents may have different dimensions from each angle in 2D.

Because the globe is a differential manifold, it allows us to shift the topological space into vector space, where the rules of distance and angles apply. This helps us to identify the distance between countries, and the area of oceans and helps in aerial transportation.

As this problem is solved in 4D space, it is interesting to look at Klein bottles. A Klein bottle is named after the mathematician who discovered it Felix Klein [11]. It is a non-orientable shape, similar to the Mobius strip, that has only one side and doesn't intersect itself in 4D space. Klein bottles are made by glueing two Mobius strips together **Figure 14**. So far, Klein bottles have no real-world applications.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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