Journal of Applied Mathematics and Physics, 2023, 11, 332-359
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Application of Thermomechanical Dynamics (TMD) to the Analysis of Nonequilibrium Irreversible Motion and a Low-Temperature Stirling Engine

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How to cite this paper: Uechi, H., Uechi, L. and Uechi, S.T. (2023) The Application of Thermomechanical Dynamics (TMD) to the Analysis of Nonequilibrium Irreversible Motion and a Low-Temperature Stirling Engine. *Journal of Applied Mathematics and Physics*, **11**, 332-359.
https://doi.org/10.4236/jamp.2023.111019

Received: November 15, 2022
Accepted: January 28, 2023
Published: January 31, 2023

Abstract

We applied the method of Thermomechanical Dynamics (TMD) to a low-temperature Stirling engine, and the *dissipative equation of motion* and time-evolving physical quantities are self-consistently calculated for the first time in this field. The thermomechanical states of the heat engine are in Non-equilibrium Irreversible States (NISs), and time-dependent thermodynamic work $W(t)$, internal energy $\mathcal{E}(t)$, energy dissipation or entropy $Q_d(t)$, and temperature $T(t)$, are precisely studied and computed in TMD. We also introduced the new formalism, $Q(t)$-picture of thermodynamic heat-energy flows, for consistent analyses of NISs. Thermal flows in a long-time uniform heat flow and in a short-time heat flow are numerically studied as examples. In addition to the analysis of time-dependent physical quantities, the TMD analysis suggests that the concept of force and acceleration in Newtonian mechanics should be modified. The acceleration is defined as a continuously differentiable function of Class C^2 in Newtonian mechanics, but the thermomechanical dynamics demands piecewise continuity for acceleration and thermal force, required from physical reasons caused by frictional variations and thermal fluctuations. The acceleration has no direct physical meaning associated with force in TMD. The physical implications are fundamental for the concept of the macroscopic phenomena in NISs composed of systems in thermal and mechanical motion.

Keywords

Thermomechanical Dynamics (TMD), The Dissipative Equation of Motion,

$Q(t)$-Picture of Thermodynamic Heat-Energy Flows, Temperature of a Nonequilibrium Irreversible State, A Low-Temperature Stirling Engine (LTSE)

1. Introduction

Heat and energy are indispensable for the prosperity of human societies and ecological systems on Earth, and thermomechanical work is derived from mechanical, electric, and thermal energy sources and is extensively used for social living activities by sustaining the natural environment. Technologies for renewable energies have been actively investigated in the modern world. Heat engines are one of the energy-sustainable clean technologies, which converts thermal energy into mechanical energy and electric energy. It is important to understand that the mechanism of energy conversion goes through a nonequilibrium irreversible state, and so, it is necessary to have a theoretical model or method to study nonequilibrium irreversible states. The Thermomechanical Dynamics (TMD) is a theoretical model proposed for the first time by the authors of this paper to study Nonequilibrium Irreversible States (NISs), which is constructed from the TMD analysis of a drinking bird [1] [2]. The drinking bird's equation of motion, the *dissipative* equation of motion, is consistently solved, producing time-evolving thermodynamic quantities, such as internal energy, work, entropy, and temperature.

A drinking bird's motion is a manifestation of transitions from thermodynamic equilibrium to a nonequilibrium irreversible state and vice versa. The analysis of a drinking bird system directed us to study emergent phenomena: the coupling or decoupling between mechanics and thermodynamics. It is never a simple, complementary mechanism explained by adding mechanics and thermodynamics. It demands the transition from a time-symmetric state to a time-symmetry-broken state and vice versa. Both mechanics and thermodynamics will break down, and new states emerge as NISs. In other words, the time-symmetry of Hamiltonian or Lagrangian is broken, resulting in the non-conservation of mechanical energy, but instead, the total thermal energy is conserved in the time range of NISs. This is the physical reason why we call the drinking bird's equation of motion as the *dissipative* equation of motion, indicating no Lagrangian or Hamiltonian exists during NISs.

Mathematically speaking, the equation of motion for the mechanical equilibrium of a drinking bird's system is given by a nonlinear differential equation with constant coefficients, and when the drinking bird system progresses to a nonequilibrium irreversible state, the equation of motion changes to the same nonlinear differential equation with time-dependent parameters. The nonlinear differential equation with time-dependent parameters has independent solutions not derivable from the identical nonlinear differential equation with constant parameters. The transition from a nonlinear differential equation with constant parameters to time-dependent parameters corresponds to a mathematical way of expressing a transition to a new emergent physical phase: a nonequilibrium irreversible state. Therefore, the nonlinear differential equations with constant and time-dependent parameters must be mathematically categorized respectively as a different class of nonlinear equations. This is one of the important mathematical properties discovered in the TMD analysis of a drinking bird system [1].

The physical indication of the *dissipative* equation of motion would be more fundamental than that of mathematics. In order to determine the dissipative equation of motion consistent with thermal states, it is required to introduce a driving force produced by a piecewise continuous force caused by frictional variations and thermal fluctuations. The mechanical system of heat engines is driven by heat flows, but frictional variations of the working fluid, and friction in every internal part of the engine accelerate or decelerate the motion of heat engines. This means that angular accelerations for the flywheel rotation in heat engines cannot be determined continuously in time. The acceleration is a continuously differentiable function (Class C^2) in Newtonian mechanics. A piecewise continuous function is differentiable in its subdomain, but is not continuous on the entire domain. Hence, it is not possible to obtain a continuously differentiable function in the analysis of heat engines. If one dares to numerically differentiate a piecewise continuous line to obtain accelerations, one obtains spiny, *hedgehog*-like lines, which may be considered to be a realization of frictional and viscous variations, and the hedgehog-like accelerations produce wavy lines when velocity is numerically calculated. The results are shown in Section 6.

The hedgehog-like accelerations or decelerations are produced by the way of sudden changes in friction, the viscosity of fluids, and thermal fluctuations. These phenomena are observed and shown numerically in the analysis of a drinking bird system and also in the current analysis of a low-temperature Stirling engine. The velocity appears as a wiggly line depending on how intense the viscous and frictional fluctuations are affecting the entire motion. The resultant trajectory or line of motion becomes a continuous smooth line. The impact of hedgehog-like spiny variations on physical values is numerically shown in the current analysis of a low-temperature Stirling engine. The time-dependent physical quantities in NISs are explicitly evaluated. The results are interesting and showing the TMD analysis is self-consistent for thermal internal energy, work, heat, and temperature progressing from thermodynamic equilibrium to NISs and vice versa.

The TMD method and results are applied to a mechanoelectrical power conversion of a low-temperature Stirling engine, and a new type of heat-electricity conversion technique is developed, which is tentatively termed as the Disc-Magnet Electromagnetic Induction (DM-EMI) technique, and electric power can be extracted from boiled water (50 - 100), even from tiny drinking bird oscillations. It is usually understood that a turbine must be accelerated to a very high speed in order to extract a large amount of energy. However, the DM-EMI technique shows that there exists an optimal speed of rotation (rotation per minute, rpm) to produce electric power to a low-temperature Stirling engine. This is one of the important results for constructing a thermoelectric conversion generator [3] [4]. It is possible to improve the DM-EMI technique to a high-temperature heat flow, such as geothermal and thermal power plants to produce kW-level intermediate energy for realistic applications.

The triad propositions in TMD are reviewed in Section 2. The first proposition demands the *dissipative* equation of motion for a low-temperature Stirling engine. The dissipative equation of motion is discussed in Section 3, which has a piecewise continuous driving force coming from frictional variations and thermal fluctuations. The second proposition demands that all heat-energy thermal flows are defined consistently with the thermodynamic conservation law. We introduce $Q(t)$-picture of thermodynamics in order to be precise and prudent, which is explained in Section 4. The dissipative equation of motion for a low-temperature Stirling engine is introduced in Section 5. The motion and thermodynamic quantities of a low-temperature Stirling engine are solved by the TMD model. The piecewise continuous acceleration or deceleration in TMD *vs* the continuously differentiable acceleration in Newtonian mechanics is discussed in Section 6. The numerical simulations of time-progressing thermodynamic work, internal energy, heat flows, and temperature are shown in the case of a homogeneous slow heat flow and a local ignition-type rapid heat flow in Section 7. Conclusions and perspectives are discussed in Section 8.

2. The Triad Propositions of Thermomechanical Dynamics (TMD)

The method of TMD consists of three propositions for constructing the equation of motion (the *dissipative* equation of motion), the conservation law for the total energy flow, and the definition of measure for temperature in NISs. The method of TMD is based on the analysis of NISs in the drinking bird system [2], and it is different from probability theory and distribution function approach in kinetic theories. A low-temperature Stirling engine is a mechanical system of heat engines driven by the cyclic compression and expansion of gas in order to convert thermal energy into mechanical work. In the current paper, we apply the TMD model to a low-temperature Stirling engine [5] in order to study thermomechanical motion and time-progressing internal thermodynamic quantities.

The three propositions in the TMD method are [1]:

1) The dissipative equation of motion: In the case that mechanical and thermal states coexist (thermomechanical states), the dissipative equation of motion for work must be constructed by considering phenomenological effects of frictional variations, time-dependent changes of physical quantities, thermal conductivity and efficiency. It would be useful to make use of Hamiltonian or Lagrangian, however, one should note that the time-symmetry is broken in NISs.

2) The total energy-flow conservation law at time t: The thermodynamic work $\mathrm{d}W_{th}(t)$, the internal energy $\mathrm{d}\mathcal{E}(t)$ and the total entropy $T(t)\mathrm{d}\mathcal{S}(t)$ are related to each other by the energy conservation law. Equivalently, it is essential in TMD that the time-dependent total energy-flow maintains,

$$\mathrm{d}\mathcal{E}(t)/\mathrm{d}t = T(t)\mathrm{d}\mathcal{S}(t)/\mathrm{d}t + \mathrm{d}W_{eq}(t)/\mathrm{d}t = \mathrm{d}Q(t)/\mathrm{d}t + \mathrm{d}W_{th}(t)/\mathrm{d}t. \quad (1)$$

The expression of heat flow (entropy flow), $T(t)\mathrm{d}\mathcal{S}(t)/\mathrm{d}t = \mathrm{d}Q(t)/\mathrm{d}t$, is explicitly used in the analysis of heat engines. The time-dependent thermodynamic power in a nonequilibrium irreversible state, $\mathrm{d}W_{th}(t)/\mathrm{d}t$, is essentially different from mechanical work, because of the constraint from the first law of thermodynamics and energy dissipation phenomena. Thermodynamic equilibrium is defined by $\mathrm{d}W_{eq}(t)/\mathrm{d}t = 0$: no thermodynamic power exists in thermodynamic equilibrium [2].

3) Temperature in a nonequilibrium irreversible state: The measure of a nonequilibrium irreversible state is defined by the ratio of entropy-flow against energy-flow:

$$\tau(t) = \frac{T(t)\mathrm{d}\mathcal{S}(t)/\mathrm{d}t}{\mathrm{d}\mathcal{E}(t)/\mathrm{d}t} = \frac{\mathrm{d}Q(t)/\mathrm{d}t}{\mathrm{d}\mathcal{E}(t)/\mathrm{d}t}. \quad (2)$$

The value of $\tau(t)$ is a dimensionless, positive-definite function, $\tau(t) > 0$. The *temperature* in NISs is defined by,

$$\tilde{T}(t) \equiv T\tau(t), \quad (3)$$

where T is the initial equilibrium temperature. When $\tau(t) = 1$ holds identically with respect to time *t*, it defines thermodynamic equilibrium, which shows no work exists, $\mathrm{d}W_{th}(t)/\mathrm{d}t = 0$, at thermodynamic equilibrium. The conditions of near equilibrium states, local equilibrium, linearity of fluxes and forces of transport processes [6] [7] [8] [9] [10] are examined by the condition, $\tau(t) = T(t)(\mathrm{d}\mathcal{S}/\mathrm{d}t)/(\mathrm{d}\mathcal{E}/\mathrm{d}t) \sim 1$ in the TMD model. When $\tau(t) = 0$, it leads to a contradiction that thermodynamic power $\mathrm{d}W_{th}(t)/\mathrm{d}t$ exists without heat conduction, which is proven by $\tau(t) = 0$ and Equation (1). The values of $\tau(t) > 1$, or $1 > \tau(t) > 0$, correspond respectively to a high or a low-temperature state in a nonequilibrium irreversible state.

The nonequilibrium irreversible state of a drinking bird's motion is rigorously solved and consistently explained by TMD [1] [2], which is the example of the case: $1 > \tau(t) > 0$. The drinking bird motion in an initial thermodynamic equilibrium develops to a low-temperature nonequilibrium irreversible state, which is a local, thermal equilibrium with temperature $\tilde{T}(t) = T\tau(t)$. The drinking bird repeats back and forth oscillations from thermal equilibrium to NISs and then to thermal equilibrium, and so, it is important to understand that the equation of motion of the drinking bird expresses transitions from NISs to thermodynamic equilibrium. Without water to keep the drinking bird' head wet, the system eventually progresses to thermodynamic equilibrium with $\tau(t) \to 1$ as $t \to \infty$.

3. The Classification of Thermodynamic Processes of a Low-Temperature Stirling Engine

The mechanical rotations of a low-temperature Stirling engine are ideally explained by four approximate thermodynamic processes: isothermal expansion, isovolumetric (isochoric) heat-removal, isothermal compression and isovolumetric heat-addition. The system of a heat engine is physically driven by mechanical and thermal energies induced by heat-energy flows. Therefore, it is fundamental to examine the physical meanings of four processes of a heat engine in order to apply the method of TMD. The analysis of four processes of a low- temperature Stirling engine is essential to determine the dissipative equation of motion.

A low-temperature Stirling engine is a thermomechanical rotation-motor, or a motion-converter from reciprocating motion to rotary motion using heat flows, schematically shown in **Figure 1**. It converts the reciprocating motion of a piston and a displacer into mechanical work (the rotation of the flywheel). The system is composed of:

1) Heat source: A homogeneous heat flow, such as from boiled water (40˚C - 100˚C) and geothermal heat, $\cdots$. The heat flow coming in the system is defined by $\mathrm{d}Q(t)/\mathrm{d}t > 0$.

2) Heat exchangers: The power piston is used to improve the heat flow and the flywheel rotation affected by friction losses.

3) Regenerator: The internal mechanism of heat exchangers between a hot plate and a cold plate. The thermomechanical conversion for work (mechanical rotations) depends on thermal efficiency, heat transfer, viscous pumping and friction losses.

4) Heat sink: The temperature difference between a hot plate and a cold plate is needed for internal heat flows.

5) Displacer: The thermal heat flow from a hot plate to a cold plate exerts vertical oscillations of the displacer. The efficiency of displacer to maintain appropriate heat dissipations is essential for mechanical rotations of the flywheel.

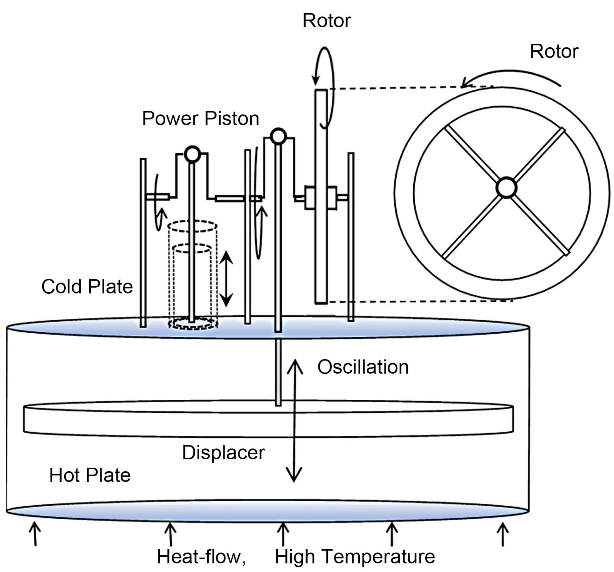


Figure 1. A schematic structure of a Low-Temperature Stirling Engine (LTSE) [11].

It is important that mechanical energies are extracted according to time-dependent variations of internal energy, thermodynamic force, work and entropy. Temperature variations and thermal fluctuations come out together with energy losses via internal frictions of a mechanical system and thermal conduction. The driving force inside cylinder comes from heat flows, and in order to continue the rotational motion, the displacer room inside cylinder must be returned to a local-equilibrium state or an approximately similar thermal state. In other words, the expansion-space and heat exchanger should be maintained at a thermal temperature so that the displacer can maintain oscillations in near-isothermal states by absorbing and releasing heat from the hot plate to the cold plate. To understand the relations among thermodynamic processes, heat flows and thermodynamic work (the flywheel rotation) is essential to solve thermomechanical motion of heat engines by way of the TMD method.

4. The Total Heat-Energy Conservation Law, Conditions and Constraints in $Q(t)$-Picture

In traditional explanation, vertical oscillations of a power piston and a displacer are explained by pressure, compression and expansion of air in the container, and they are smoothly connected so that air can move freely between the hot plate and cold plate, mixing heated and cooled air in the cylinder container. However, the concept of motion is changed in TMD such that the cyclic vertical oscillations of a power piston, a displacer in the container and a flywheel are driven by the incoming and outgoing heat flows, thermal conduction and dissipation of heat, which is supposed to maintain thermal states in the container for continuous vertical oscillations.

The fundamental relation among motions and kinetic work, associating heat dissipation and internal energy and associating heat flows should be carefully studied. Thermodynamic work and internal energy respectively have their portions of heat for mechanical motion and dissipation. In other words, thermomechanical work, $Q_w(t)$, is assumed completely partitioned into portions of heat contributing to mechanical work and heat for dissipation. All mechanical motions, internal energy, heat dissipations, and other heat-energy losses must maintain the total heat-energy conservation law.

The second proposition of TMD (the conservation law of energy flows) must be carefully established to the total heat-energy flows constrained by frictional variations, heat conduction and thermal efficiency of the mechanical structure of heat engines. We introduce $Q(t)$-picture to carefully manage time-dependent variations of thermodynamic energies. For example, internal energy is expressed as $\mathcal{E}(t) \to Q_\varepsilon(t)$ in calorie unit, and thermodynamic work as $W_{th}(t) \to Q_w(t)$, the total entropy dissipation $TS(t) \to Q_d(t)$ and so forth. The expression becomes convenient to use when thermal energies progress in time. We assume that thermal energy is decomposed into two parts as:

$$Q(t) = \text{thermally conserved energy} + \text{thermally dissipating energy}, \tag{4}$$

for time-dependent internal energy, work and other thermodynamic quantities. The entropy is given by the sum of all thermally dissipating energies. The $Q(t)$ -picture is useful for checking and maintaining thermodynamic consistency, and it is explicitly defined and used in the following discussions.

The integrated expression of the heat-energy conservation law (1) in the TMD Proposition 2) is written in the following form:

$$\Delta\mathcal{E}(t) = Q_{in}(t) + W_{th}(t), \tag{5}$$

where $Q_{in}(t)$ denotes the total heat coming in the system at time *t*. The heat $Q_{in}(t)$ is positive and the heat flow-in is, $\mathrm{d}Q_{in}(t)/\mathrm{d}t > 0$, to operate a heat engine and has a minimum value to produce work. It is determined by external conditions and constraints, depending on thermal efficiency and structure of machines. The time-change of internal energy is written as, $\Delta\mathcal{E}(t) = \mathcal{E}(t) - \mathcal{E}_0$, with $\mathcal{E}(0) = \mathcal{E}_0$, measured from the initial value $\mathcal{E}_0$. Thermodynamic work, $W_{th}(t)$, consists of the energy of the flywheel rotation, power piston, and associating heat dissipation. The initial conditions are set as: $Q_{in}(0) = 0$, $W_{th}(0) = 0$, $\Delta\mathcal{E}(0) = 0$.

Let us denote thermodynamic work as $W_{th}(t) \equiv Q_w(t)$ and internal energy as $\mathcal{E}(t) \equiv Q_\varepsilon(t)$. The heat used for thermodynamic work $Q_w(t)$ is written by

$$Q_w(t) = Q_{wk}(t) + Q_{wd}(t). \tag{6}$$

The heat $Q_{wk}(t)$ is thermal energy used for the kinetic energy of the flywheel rotations, the displacer and power piston oscillations, and $Q_{wd}(t)$ is the associating heat dissipation. Similarly, the total heat used for internal energy process $Q_\varepsilon(t)$ is written by:

$$Q_\varepsilon(t) = Q_{\varepsilon i}(t) + Q_{\varepsilon d}(t). \tag{7}$$

The heat $Q_{\varepsilon i}(t)$ is thermal energy used to change internal energy (*thermal internal energy*), and $Q_{\varepsilon d}(t)$ is the associating heat dissipation.

The total heat into the system of heat engine is denoted by $Q_{in}(t)$, and the heat-energy conservation law (5) is now rewritten as:

$$\begin{aligned} Q_{in}(t) &= Q_\varepsilon(t) + Q_w(t) \\ &= Q_{\varepsilon i}(t) + Q_{\varepsilon d}(t) + Q_{wk}(t) + Q_{wd}(t) \\ &= Q_{\varepsilon i}(t) + Q_{wk}(t) + Q_d(t). \end{aligned} \tag{8}$$

The total heat dissipation with internal friction losses is defined by $Q_d(t) = Q_{\varepsilon d}(t) + Q_{wd}(t)$, with $Q_d(0) = 0$. The differential form of time-dependent total energy-flow of (1) is written in $Q(t)$ -picture as:

$$\mathrm{d}Q_\varepsilon(t)/\mathrm{d}t = \mathrm{d}Q_d(t)/\mathrm{d}t + \mathrm{d}Q_w(t)/\mathrm{d}t. \tag{9}$$

Similarly, temperature $\tau(t)$ in (2), in $Q(t)$ -picture is expressed as:

$$\tau(t) = \frac{T(t)\,\mathrm{d}\mathcal{S}(t)/\mathrm{d}t}{\mathrm{d}\mathcal{E}(t)/\mathrm{d}t} = \frac{\mathrm{d}Q_d(t)/\mathrm{d}t}{\mathrm{d}Q_{\varepsilon i}(t)/\mathrm{d}t}. \tag{10}$$

Now, the total heat-energy conservation law or the Carnot cycle in NISs is expressed by:

$$Q_{in}(t) - Q_d(t) = Q_{\varepsilon i}(t) + Q_{wk}(t), \tag{11}$$

which must be valid from the initial thermodynamic equilibrium state to a thermal equilibrium. The kinetic heat energy, $Q_{wk}(t)$, is interpreted as the flywheel energy of rotations at time *t*:

$$Q_{wk}(t) = \frac{I_0}{2}\theta'(t)^2. \tag{12}$$

In $Q(t)$-picture, thermodynamic variables such as volume, pressure, work, internal energy and total entropy are completely suppressed in $Q_w(t)$, $Q_\varepsilon(t)$ and $Q_{in}(t) - Q_d(t)$. One should note that the time-change of internal energy and work are explicitly included in the Carnot cycle in NISs.

The heat-in $Q_{in}(t)$ and the total dissipation heat $Q_d(t)$ are related with the definition of thermal efficiency:

$$\eta_{th}(t) = \frac{Q_{in}(t) - Q_d(t)}{Q_{in}(t)} = 1 - \frac{Q_d(t)}{Q_{in}(t)}. \tag{13}$$

The amount of heat, $Q_{in}(t) - Q_d(t) = \eta_{th}(t) Q_{in}(t)$, is used for internal energy and mechanical work as $Q_{\varepsilon i}(t)$ and $Q_{wk}(t)$ in (11). The thermal efficiency is usually introduced as a constant, for example $\eta_{th} \sim 20\%$, constrained by machine structure and thermal efficiency of mechanical system. However, it can change according to the time-variation given by (13) and becomes time-dependent, $\eta_{th} \to \eta_{th}(t)$. The thermal efficiency at time *t* in NISs is numerically determined to be consistent with (13) with the initial condition $\eta_{th}(0) = \eta_{th}$ and the constraint $\eta_{th}(t) = \eta_{th}$ at thermal equilibrium when the heat engine reaches an internal thermal equilibrium.

The time-dependent thermal efficiency is closely related to temperature in NISs inside heat engines, $\tilde{T}(t) = T_0 \tau(t)$. Temperature at $\tilde{T}(0) = T_0$ changes abruptly at the beginning and becomes a constant, $\tilde{T}(t) = \tilde{T}$, at a thermal equilibrium. In numerical simulations, time-retardation of heat transfers among right terms in (11) is neglected; in other words, the upper gas and the lower gas of the displacer in the cylinder room are supposed to be mixed uniformly and simultaneously during the cycle of oscillations (see, **Figure 1**). The time-retardation and spatial thermal conduction mechanism will be included in TMD in the future work.

5. The Dissipative Equation of Motion for a System of a Low-Temperature Stirling Engine

The vertical oscillations of the displacer are driven by thermal conduction and converted to rotations of the flywheel, and the thermomechanical motions are related to thermal efficiency of a hydraulic gas in the container. Therefore, thermodynamic accelerations and decelerations of motion by heat flows and friction of working fluid and all crankshafts of the Stirling engine must be considered to construct the *dissipative* equation of motion for the flywheel. One should note that the equation of motion is not derivable from Lagrangian or Hamiltonian because of the time-dependent change of entropy and internal energy. The dissipative equation of motion must be determined self-consistently with Propositions 2) and 3).

The variable to express thermal work of a low-temperature Stirling engine is the angle θ, used for rotations of the flywheel shown in **Figure 2** and chosen from the vertical axis (starting from *z*-axis as $\theta = 0$). The rotation of the flywheel is connected to vertical oscillations of the displacer exerting the resultant rotational force (torque). In the TMD model, it is essential to recognize that the rotation of the low-temperature Stirling engine is produced by heat flows and friction of working fluid, and so, the coupling among rotational force, heat flows and frictional variations must be carefully considered to construct the dissipative equation of motion.

Based on the discussions so far, one could start from an equation of motion for a simple pendulum or a drinking bird [2]. The starting mechanical equation of motion may be chosen as:

$$I_0\ddot{\theta}(t) + c\dot{\theta}(t) + glm\sin\theta(t) = 0, \tag{14}$$

where $c\dot{\theta}$ is a friction term; the length, l, may be considered as the radius of the flywheel, $l \sim 10.0$ (cm); $g = 980$ (cm/s^2). The masses of the flywheel, $m_1 = 50$ (g), $m_2 = 200$ (g): the width $d = 3.0$ cm. The moment of inertia of the flywheel in **Figure 2** is:

$$I_0 = m_2 l^2/2 + (2m_1/3 + m_2/2) l^2 (1 - d/l)^2 . \tag{15}$$

The moment of inertia of a power piston and frictional effects can be effectively included in (15) as mass of displacer portion × (displacer amplitude)^2. However, the effective moment of inertia hardly changes numerical results compared with the flywheel-only calculations. The angle $\theta = \pi$ is mechanically adjusted as the point of the lowest potential energy, and the gravitational force, *g*, frictional forces from the working gas in the container change respectively for and against the *θ*-direction, in the range, $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$, resulting in a rapid convergence to $\theta = \pi$. Hence, the equation of motion (14) cannot reproduce the motion of the heat engine, but it gives ideas to construct a correct dissipative equation of motion.

Now, it is essential to consider carefully about driving forces of the heat engine. The vertical oscillation of displacer receives friction from mechanical connections and working fluid (gas inside cylinder) and thermal fluctuations. The frictions of working fluid work always against the *θ*-direction and heat flows are directed in the positive direction defined by the heat flows of heat-in and heat-out. In other words, the heat flows are working against the motion in the range, $0 \le \theta \le \pi$ and for the motion in the range, $\pi \le \theta \le 2\pi$. The directions of frictional forces of working fluids change suddenly against *θ*-direction in the range $0 \le \theta \le \pi$ (upper direction) and $\pi \le \theta \le 2\pi$ (lower direction). The sudden changes of frictional forces on the displacer produced by mechanical rotations, working fluid and heat flows are not correctly expressed in Equation (14).

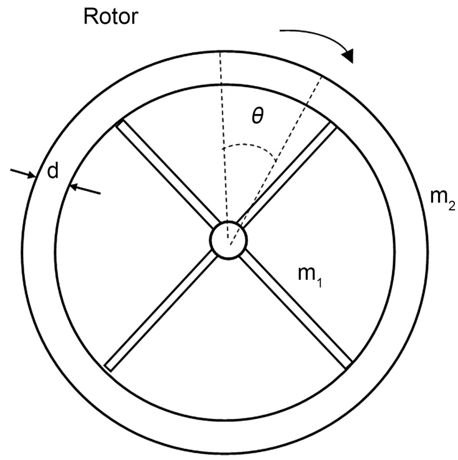


Figure 2. The rotational angle, θ, starting from the vertical axis.

Therefore, it is important to account for sudden changes of frictional forces, resulting in a discontinuity of acceleration or deceleration in oscillations of the displacer. The characteristics of discontinuous forces against θ-directions in vertical oscillations can be mathematically defined by introducing a piecewise continuous function to express sudden changes of frictional forces. The function is piecewise continuous and differentiable on the corresponding intervals, but it is not continuous on the entire domain as it contains jump discontinuities. The piecewise continuity of motion is required in thermomechanical motion as well as natural phenomena.

The well-known example of piecewise continuous change of motion would be Brownian motion of pollen through the microscope, and the random Brownian motion is produced by the external driving force of the intermediate-microscopic interactions between a pollen and particles, resulting in the random fluctuations of a pollen's position in a medium [12]. Brownian motion of a relatively larger particle (dust particle) than molecules of medium moves with different random trajectories, velocities and accelerations.

The external, thermal driving force in heat engines produces piecewise continuous force (torque) to oscillations for the displacer. Therefore, we introduce the appropriate and phenomenological thermal driving force by employing the piecewise continuous function in the form $\left|\sin\theta(t)\right|$. We assume that the piecewise continuous driving force couples to thermodynamic work, $Q_w(t)$, and the driving force produces mechanical rotations of the flywheel and power-piston with associating heat dissipation in the form (6). As the first requirement of TMD, the *dissipative* equation of motion for a low-temperature Stirling engine is defined by:

$$I_0\theta''(t)+c\theta'(t)-\lambda_w Q_w(t)\left|\sin\theta(t)\right|=0, \tag{16}$$

where λ_w is a dimensionless coupling constant for heat and mechanical work. The coupling constant, λ_w and heat for work $Q_w(t)$, mainly determine the magnitude of angular velocity and number of rotations, and so, they are carefully adjusted in computer simulations. One should also be careful that the (cgs) and calorie unit are used in the numerical calculations. The initial value of heat for work $Q_w(t)$ is chosen as $Q_w(t) = \eta_w Q_{in}(t)$ (η_w is an arbitrary small number) and the computation is repeated until one obtains solutions: the number of rotations, $\theta(t)/2\pi$ and a stable maximum angular velocity, $\theta'(t)/2\pi$, rotations per minute (rpm).

Although the fundamental equation of motion (16) seems simple, its mathematical and physical consequences are profound. The piecewise continuous driving force in (16) immediately indicates that the acceleration is not defined as differentiable and continuous quantity as supposed in Newtonian mechanics. The acceleration cannot be determined as the second-order derivative derived from the trajectory of motion, because the driving force contains jump discontinuities in the entire domains of motion. In TMD, the concept of force is physical, and force only changes directions of motion or velocities of particles, but not associated with mass × acceleration. The jump discontinuities are not avoidable, and they naturally emerge from friction and viscosities of working fluid, sheer stress and machine structure, temperature and thermal fluctuations. Macroscopic motions of fluids or particles are all exposed in sudden and unexpected thermal disturbances. In the TMD model, the trajectory and velocity are continuous and differentiable, but the acceleration is only piecewise continuous, and when numerical differentiation is executed to an angular velocity, one obtains spiny hedgehog-like lines.

The observability of macroscopic motion indicates that the jump discontinuities or frictional and thermal disturbances must be small enough so that velocities and trajectories or reciprocated motion of heat engines are able to be observed as a continuous motion. The concept of indeterminacy of acceleration in the macroscopic disturbances, or piecewise continuity in the mathematical sense must be manifest in the dissipative equation of motion (16) as physical constraints required by natural phenomena. The concept required by TMD is new and different from methods in Newtonian mechanics and thermodynamics.

The dissipative equation of motion for a low-temperature Stirling engine and piecewise continuous changes of accelerations are explicitly examined by numerical simulations in the following section.

6. The Solutions to the Dissipative Equation of Motion and the Piecewise Continuous Force

Numerical simulations of a low-temperature Stirling engine are performed by solving (16) with initial conditions, for instance, $\theta(0) = 0$ and $\theta'(0) = 0.1\pi$. The unit of angle $\theta(t)$ is expressed by (radian) or the number of revolutions $\theta(t)/2\pi$. Similarly, the speed of rotation is expressed by $\theta'(t)$ (rad/s), or revolutions per second $\theta'(t)/2\pi$ (revolutions/s). The numerical calculations of angular accelerations, $\theta''(t)$ (rad/s^2), are formally executed, resulting in hedgehog-like spiny lines.

The thermomechanical coupling constant, λ_w, is a small constant, and the appropriate initial starting value should be chosen arbitrarily and adjusted in self-consistent calculations, because the speed of rotations apparently depends on λ_w, affecting values of internal energy, entropy, work and temperature. The thermal efficiency, $\eta_{th}(t)$, is a device-dependent external constraint, and so, we supposed $\eta_0 = 20\%$ at thermodynamic equilibrium and thermal equilibrium of the heat engine. The numerical simulations are performed in two cases: a slowly-decreasing long-time uniform heat flow and a rapidly-decreasing short-time uniform heat flow.

6.1. Thermodynamic Work in a Slowly-Decreasing, Long-Time Uniform Heat Flow

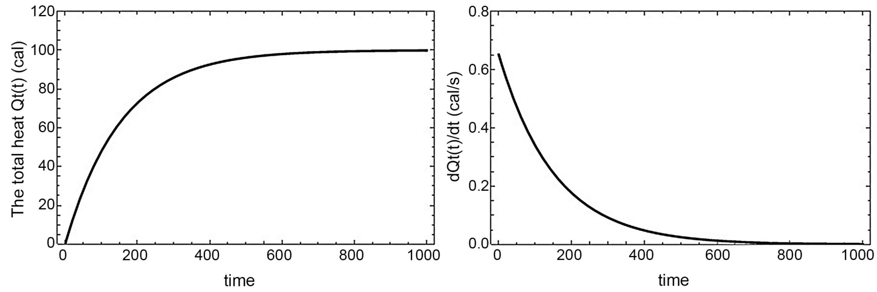
Two types of flywheel rotations driven by (A) a slowly-decreasing, long-time uniform heat flow and (B) a rapidly-decreasing, short-time heat flow are computed as examples. The constant heat source like a solar heat collector, a low-temperature hot-spring, heat exchanger and waste heat from industrial systems may correspond to examples for (A), and an internal combustion engine, reciprocating heat engines (piston engines) may be regarded as examples for (B), which should be respectively investigated in detail in the future. The rotations, velocities and accelerations are calculated specifically in the following analyses.

The heat flow-in, $Q_{in}(t)$, is defined by:

$$Q_{in}(t) = Q_{1H}\left(1.0 - \mathrm{e}^{-\xi_1 t}\right). \tag{17}$$

The constants are fixed in the case of (A) as, $Q_{1H} = 100.0$ (cal) and $\xi_1 = 6.51 \times 10^{-3}$ (1/s); the total heat $Q_{in}(t)$ and heat flow $\mathrm{d}Q_{in}(t)/\mathrm{d}t$ are shown in **Figure 3**. The heat $Q_{in}(t)$ slowly increases to its maximum value Q_{1H}, and accordingly, $\mathrm{d}Q_{in}(t)/\mathrm{d}t$ decreases in the time range $0 < t < 1000$ (s), but the motion continues to over 2400 (s). The constant ξ_1 would be related to thermal conductivity of working fluid and materials used in a heat engine, and a delicate task is necessary by adjusting parameters in computer simulations. The initial value of heat used for thermomechanical work (the flywheel rotations), $Q_w(t)$ in (6), should be supplied, for example, as $Q_w(t) = \eta_{1w} Q_{in}(t)$, and η_{1w} is a small number at the beginning and an appropriate value has to be determined consistently by repeating numerical calculations. The mechanical and thermal responses of the heat engine are reasonably controlled in numerical simulations by changing values of λ_{1w}, ξ_1 and η_{1w}. The constants are adjusted to obtain the angular velocity $60 \times \theta'(t)/2\pi$ (rpm), compatible with experimental values of a low-temperature heat engine.

The maximum value of angular velocity is set around, $30 < 60 \times \theta'(t)/2\pi < 40$ (rpm), in the simulation (A), and the dissipative equation of motion (16) is solved to obtain the solution, $\theta(t)$ and $\theta'(t)$. It is better to note that the optimum angular velocity for thermoelectric energy conversion should exist in $30 < 60 \times \theta'_{\max}(t)/2\pi < 40$ (rpm) in the low-temperature Stirling engine, and it is



(a) (b)

Figure 3. The total heat $Q_{in}(t)$ and the heat flow $\mathrm{d}Q_{in}(t)/\mathrm{d}t$ for a slowly-decreasing, long-time uniform heat flow. (a) The total heat $Q_{in}(t)$ in time range $0 < t < 1000$ (s). (b) The heat flow $\mathrm{d}Q_{in}(t)/\mathrm{d}t$ in time range $0 < t < 1000$ (s).

discussed in detail as DM-EMI thermoelectric conversion to extract electric energy from boiled water ($T \lesssim 100$), using the current type of rotations of a low-temperature Stirling engine [3] [4]. Thermodynamic work of the flywheel rotations at a time t is given by:

$$Q_{wk}(t) = \frac{I_0^*}{2}\theta'(t)^2. \tag{18}$$

The effective inertial mass of moment, $I_0^* = I_0 + m_{dp}a^2$ (m_{dp} for the mass of displacer and power piston, a for the rotational amplitude of the displacer), is used to include effectively the displacer and piston oscillations into flywheel rotations, as explained in Section 5. When thermomechanical work $Q_{wk}(t)$ is obtained, the associating heat dissipation, $Q_{wd}(t)$, is determined by $Q_{wd}(t) = Q_w(t) - Q_{wk}(t)$ in (6).

The number of rotations $\theta(t)/2\pi$ and the angular velocity $\theta'(t)$ (revolutions/s) of the flywheel are respectively shown in **Figure 4(a)** and **Figure 4(b)**. One can check that the trajectory of motion, $\theta(t)/2\pi$, changes continuously. Thermal force exerted by heat flows from a cold plate to a hot plate accelerates the angular velocity of rotations in the beginning, but mechanical motions reach a plateau, a relatively stable level of the angular velocity, as shown in **Figure 4(b)**. The stable maximum angular velocity seems constant, but one can notice that the angular velocity in **Figure 4(b)** has tiny fluctuations along the ($\theta'(t)/2\pi$)-solution. The tiny fluctuations are caused by frictional variations and thermal fluctuations between the displacer and working fluid, which can be naturally examined in the experiment of a low-temperature Stirling engine. The realistic flywheel thermal motion is produced reasonably well by the dissipative equation of motion (16). When heat exchangers and regenerators work properly, the heat engine persists in a long period of time.

It is important to compare the corresponding values between the heat flow, $\mathrm{d}Q_{in}(t)/\mathrm{d}t$ in **Figure 3(b)** and the angular velocity, $\theta'(t)/2\pi$ in **Figure 4(b)**, after they reached a stable value ($400 \lesssim t$). The heat flow becomes small and very slowly decreasing with time, but the flywheel keeps a stable maximum angular

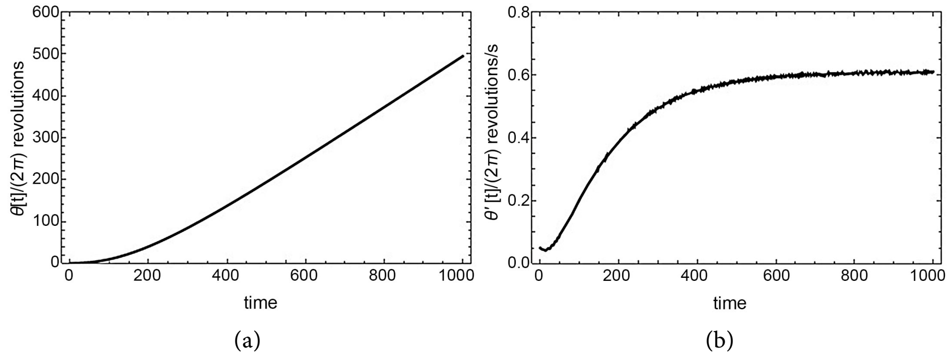


Figure 4. The number of revolutions and angular velocity (revolutions/s) for a slowly-decreasing, long-time uniform heat flow. (a) The number of revolutions, $\theta(t)/2\pi$, in the time range $0<t<1000$ (s). (b) The angular velocity, $\theta'(t)/2\pi$ (revolutions/s), in the time range $0<t<1000$ (s). Note the tiny fluctuations along the angular velocity.

velocity for a long time period, which physically indicates that the flywheel can keep its maximum angular velocity for a long time with very small amount of heat flows $Q_{in}(t)$. In other words, once the flywheel rotation reaches a maximum stable angular velocity, thermodynamic work of flywheel can be maintained with a very little addition of rotational energy or rotational force (torque). It confirms the empirical fact of inertia to keep the flywheel rotations, which is specifically shown for the first time in the TMD method. The fact and analyses would be helpful for heat-energy conversion and flywheel-storage technologies.

As discussed in Section 5, the fact that thermal disturbances and frictional forces create jump discontinuities on the angular velocity can be clearly observed in numerical calculations, and it is important to recognize that wiggly lines of angular velocity and hedgehog-like spiny lines of angular accelerations become visible when lines are magnified. One can notice that the angular velocity, $\theta'(t)/2\pi$, changes as a small wiggly line, shown explicitly in **Figure 5(a)** by changing the scale and expanding the line segment of **Figure 4(b)**, and the numerical values of angular acceleration derived from the angular velocity in **Figure 5(a)** are shown in **Figure 5(b)**, and they are solutions restricted in the time range $0<t<100$ (s) in a magnified fashion. The results come from abrupt changes of frictional braking effects against up-down oscillations of the displacer. Mathematically speaking, the sinuous small oscillations of angular velocity are produced by numerical integration of piecewise-continuous angular acceleration, $\theta''(t)$. In case that oscillations caused by friction are too small to be observed in a macroscopic energy-time scale, angular acceleration, velocity and trajectory would be regarded as smooth, continuous and differentiable quantities.

Thermodynamic work (rotational energy of the flywheel and power piston), $I_0^*\theta'(t)^2/2$ (Joule), is shown in **Figure 6(a)**. The rotational energy reaches a maximum stable value, which has a continuous, tiny-wiggly line because of $\theta'(t)$. The stable rotational velocity of the heat engine produced by the dissipative equation of motion should not be considered smooth and constant as perceived and supposed in human sense, which is essentially related to frictional variations and internal thermal fluctuations.

The TMD thermomechanical approach to physical phenomena demands fundamental changes regarding the concept of thermodynamic force and work, $W_{th}(t)$. Mechanical work is based on continuity and differentiability of motion, which is integrated in changes of velocity and trajectory of particles. In other words, it is essential to recognize that modifications of mechanical motion caused by friction, wear, deformation and thermal fluctuations generate the fundamental change to the concept of differentiability of physically observable quantities. The trajectory $\theta(t)$ and angular velocity $\theta'(t)$ are continuous and differentiable, whereas the angular acceleration, $\theta''(t)$, is piecewise continuous and has finite numbers of jump discontinuities in a finite interval. The whole view of acceleration results in an assembly of hedgehog-like spiny lines as shown

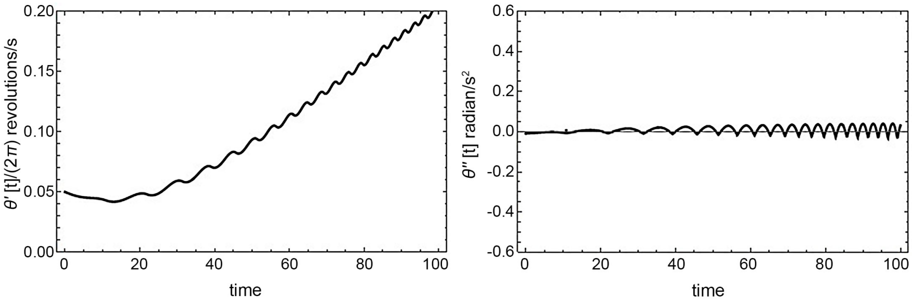


Figure 5. (a) The continuous and differentiable angular velocity (the line-segment expanded view), and (b) the corresponding piecewise continuous angular acceleration derived from the time-derivative of (a). (a) The angular velocity $\theta'(t)/2\pi$ (rotations/s) is continuous and differentiable. The scale is changed in a magnified fashion to show wiggly lines of the angular velocity line, $0 < t < 100$ (s) in Figure 4(b). (b) The piecewise continuous angular acceleration, $\theta''(t)$ (rad/s^2), $0 < t < 100$ (s), derived from the angular velocity in Figure 5(a).

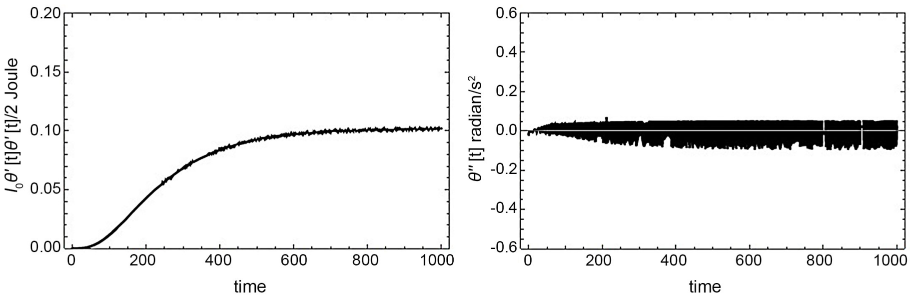


Figure 6. Thermodynamic work (rotational energy of the flywheel and power piston) and the angular acceleration for a slowly-decreasing, long-time uniform heat flow. (a) Thermodynamic work (rotational energy of the flywheel and power piston), $I_0^*\theta'(t)^2/2$ (Joule), in the time range $0 < t < 1000$ (s). (b) The piecewise continuous angular acceleration, $\theta''(t)$ (rad/s^2), $0 < t < 1000$ (s), derived from the velocity in Figure 4(b).

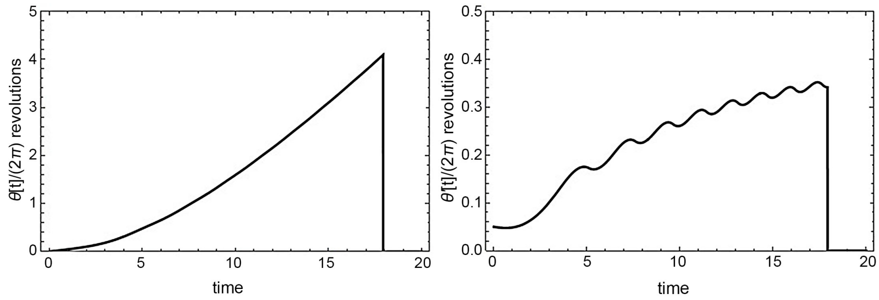
in **Figure 6(b)**. The angular acceleration cannot be determined in the mechanical sense of differentiable continuous functions of Class C^2, but force remains as a physical entity related to change velocities and positions of particles and matter, which is one of the fundamental consequences in TMD.

The numerical results of the dissipative equation of motion should be considered as a reflection of physical macroscopic phenomena, which consists of irreversible heat-energy dissipations, thermal fluctuations, or random fluctuations in a mathematical term. Thermal fluctuations become larger as temperature, pressure and thermal efficiency change while a thermal process, as well as motions of wheels, power pistons, coupler, and crank, fluctuates about mechanically given conditions and states. When dissipative changes on motion are suppressed small in the macroscopic heat-energy scale, thermomechanical trajectory and motion, thermal work, internal energy and heat would be considered smooth and continuous as classical functions in Class C^2. The dissipative equation of motion is successful for producing thermomechanical flywheel rotations, and it is also useful to apply to the thermoelectric energy conversion [3].

The wiggly line of angular velocities and spiny hedgehog-like accelerations are also found in the dissipative equation of motion for a drinking bird system, which is a reasonable experimental device to study NISs. The time-dependent progression of internal energy, work, entropy and temperature is solved and studied consistently, which is the origin of the TMD method [1] [2]. The low-temperature Stirling engine is the second example solved consistently by the TMD method. The applications of the TMD method to other types of α, β, γ Stirling engines and piston engines as well as internal combustion engines in general would be expected, and also, the applications are useful for thermomechanical and thermoelectric energy-conversion technologies [4]. The construction of the dissipative equation of motion based on mechanical equations of motion is a key to solve thermal and macroscopic mechanical systems, which is one of the interesting open questions.

6.2. Thermodynamic Work in a Rapidly-Decreasing, Short-Time Uniform Heat Flow

The flywheel rotations in the case (B) is computed in this section, and it is defined by (17) with $Q_{2H} = Q_{1H}$ and $\xi_2 = 100 \times \xi_1$. The flywheel rotations and angular velocities are slow and small, and the heat flow rapidly ends within 18 seconds. Other constants, η_{2w} and λ_{2w}, are adjusted accordingly in numerical simulations, considering applications to an ignition mechanism of heat engines and internal combustion engines. The numerical result of the flywheel revolutions is shown in **Figure 7(a)**, and the corresponding angular velocity is shown in **Figure 7(b)**. The short-time uniform heat flow exhibits that the heat flow ends very rapidly and produces only several rotations (4 rotations, in **Figure 7(a)**), and the angular velocity is very slow shown in **Figure 7(b)**. The heat $Q_{in}(t)$ soon reaches its maximum value Q_{2H}, and accordingly, the flywheel rotation ends at $t \sim 18$ (s). In reality, a low-temperature Stirling engine continues



(a) (b)

Figure 7. The number of revolutions and angular velocity (revolutions/s) for a repidily-decreasing, short-time uniform heat flow. (a) The number of revolutions, $\theta(t)/2\pi$ (revolutions) in the time range $0<t<20$ (s). The flywheel rotations stop at $t\sim 18$. (b) The angular velocity, $\theta'(t)/2\pi$ (revolutions/s), in the time range $0<t<20$ (s). The angular velocity ends at $t\sim 18$.

some back-and-force small swingings before it stops completely, but the last small flywheel swingings are completely ignored in computer simulations, because they do not make a rotation, resulting in the sudden stop of rotation in the simulation.

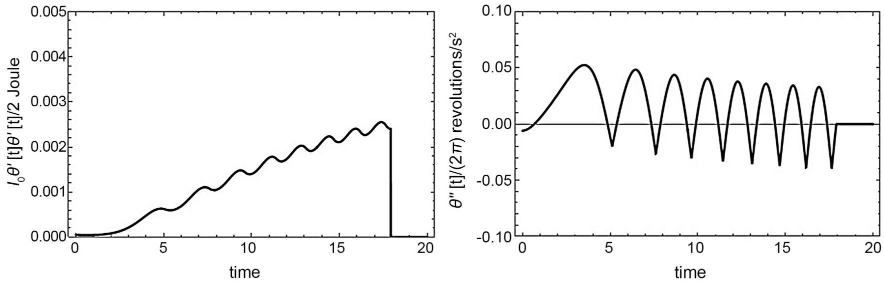
The flywheel rotation-time is flexibly adjusted by changing ξ_2, and a faster angular velocity is produced by assuming a larger heat energy, Q_{2H}. The short-time ignition-like experiment could be feasible by supplying the heat, $Q_{in}(t)$, discretely in time as:

$$Q_{in}(t)=Q_{in}(t,t_1)+Q_{in}(t,t_2)+Q_{in}(t,t_3)+\cdots,\quad (t_1<t_2<t_3\cdots). \tag{19}$$

where $Q_{in}(t,t_i)$ $(i=1,2,\cdots)$ could be taken as rectangular functions. This is possible to calculate in the current TMD approach by setting the discrete ignition-time, which may be applied as a model to scrutinize complicated mechanism of piston engines or internal combustion engines in general.

The magnified views of numerical results are useful to clearly examine mechanical quantities in NISs. The angular velocity in **Figure 7(b)** shows a slowly increasing and wiggly angular velocity. It is compatible with empirically observed motion of the flywheel rotations when the flywheel rotates slowly, and the wiggly motion can be specifically checked by the flywheel experiments, which is produced by frictional variations of working fluid and mechanism of the heat engine. The rotations and angular velocities $\theta(t)$ and $\theta'(t)$, should be compared with **Figure 4** to understand how frictional variations affect work of heat engines.

The angular acceleration and thermal kinetic energy of the flywheel rotation are shown in **Figure 8(a)** and **Figure 8(b)** in a magnified fashion. The angular acceleration is a piecewise continuous quantity induced by frictional, mechanical and thermal disturbances to the system of displacer and flywheel. One can specifically observe the small and slow wiggling angular velocity $\theta'(t)/2\pi$, in an experiment of a low-temperature Stirling engine, showing that the angular velocity



(a) (b)

Figure 8. Thermodynamic work (rotational energy of the flywheel and power piston) and the angular acceleration for a rapidly-decreasing, short-time uniform heat flow. (a) Thermodynamic work (rotational energy of the flywheel and power piston), $I_0^*\theta'(t)^2/2$ (Joule), in the time range $0<t<20$ (s). The angular velocity ends at $t\sim 18$. (b) The piecewise continuous angular acceleration, $\theta''(t)$ (rad/s^2), $0<t<20$ (s), corresponding to the velocity in Figure 7(b).

is approximately constant: $\theta'(t)\approx\omega$, with tiny, wiggly changes. The dissipative equation of motion and the thermomechanical method to the Stirling engine produce motion of the flywheel rotation reasonably well.

The energy of the flywheel rotations and angular accelerations (Figure 6 and Figure 8), should be respectively compared. The results help us understand how we perceive motion in human macroscopic scale. When the flywheel angular velocity is relatively fast to human conception of motion, the angular velocity with tiny-wiggly line would not be recognized and lead to the assumption that the angular acceleration be assumed smooth and differentiable. The theoretical and numerical analyses reveal clearly that accelerations are not directly useful in NISs, but the concept of force is fundamental for changing velocities and directions of motion.

The thermomechanical relations among quantities, $\theta(t),\theta'(t),\theta''(t)$ and $Q_{wk}(t)$ in (18), are consistently solved by the dissipative equation of motion. Applications to reciprocating engines or piston engines to convert high-temperature and pressure heat flows to a rotating motion, as well as thermoelectric conversions are expected [3] [4] These are also interesting open questions. Based on the thermomechanical results, thermodynamic relations among temperature, internal energy and entropy flows, thermodynamic work in NISs are studied in the next section.

7. Internal Energy, Dissipations of Heat and Temperature for a Low-Temperature Stirling Engine

The thermal energy flows produce the time-progress of thermomechanical work $Q_{wk}(t)$, internal energy $Q_{\varepsilon}(t)$ and associating dissipations of heat, $\mathrm{d}Q_{wd}(t)/\mathrm{d}t$, $\mathrm{d}Q_{\varepsilon d}(t)/\mathrm{d}t$, the total dissipation $\mathrm{d}Q_{d}(t)/\mathrm{d}t$, and temperature $\tilde{T}(t)=T\tau(t)$. These dynamical quantities must maintain thermodynamic consistency, (9) and (10) as Requirements 2) and 3) in the TMD hypothesis. The time-dependent physical quantities derived from the solutions of the dissipative equation of motion are specifically examined in the section.

7.1. Thermodynamic Quantities in a Slowly-Decreasing, Long-Time Uniform Heat Flow

The dissipation of heat from thermomechanical work, $Q_{wd}(t)$ (the flywheel rotations), and the total heat dissipation, $Q_d(t) = Q_{\varepsilon d}(t) + Q_{wd}(t)$, are shown in Figure 9(a) and Figure 9(b). The heat dissipation from work $Q_{wd}(t)$ increases for a while at the beginning, but when the heat-kinetic energy conversion gradually becomes effective and smooth with the inertia of flywheel and power piston, the total heat dissipation, $Q_{wd}(t)$, gradually reaches a minimum and stable value; it is consistent with the empirically known performance of piston engines in general, which is theoretically produced for the first time in the TMD method.

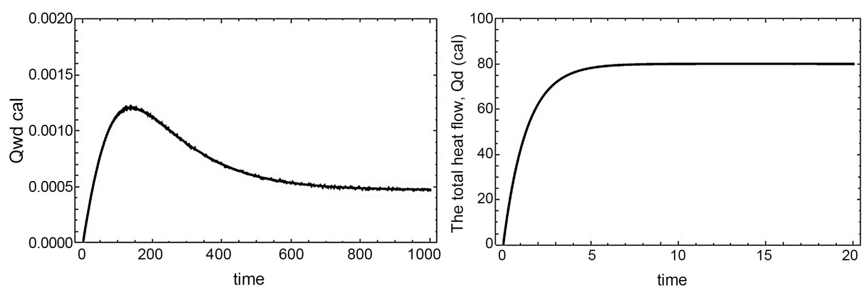
The kinetic work $Q_{wk}(t) = \left(I_0^*/2\right)\theta'(t)^2$ and the associated heat-flow $Q_{wd}(t)$ give the thermomechanical work $Q_w(t) = Q_{wk}(t) + Q_{wd}(t)$. The heat engine efficiency is an externally given constraint defined in (13), taken as $\eta_{th} \sim 20\%$ in the computer simulation. Therefore, the residual heat, $Q_{in}(t) - Q_w(t)$, is used for the internal energy $Q_{\varepsilon i}(t)$ and the associating dissipation $Q_{\varepsilon d}(t)$. The total internal energy is given by $Q_{\varepsilon}(t) = Q_{in}(t) - Q_w(t)$, and so, as the overall machine condition, the heat efficiency confines, $Q_{\varepsilon i}(t)$ (thermal internal energy) as:

$$Q_{\varepsilon i}(t) = \eta_{th}\left(Q_{in}(t) - Q_w(t)\right), \tag{20}$$

and the associating heat dissipation of internal thermal energy as:

$$Q_{\varepsilon d}(t) = \left(1 - \eta_{th}\right)\left(Q_{in}(t) - Q_w(t)\right), \tag{21}$$

and the total internal energy $Q_{\varepsilon i}(t) + Q_{\varepsilon d}$, is consistent with $Q_{in}(t) - Q_w(t)$ (see discussions in Section 4). The thermal efficiency is a machine-dependent value and fixed at the beginning and at thermal equilibrium as $\eta_{th} \sim 20\%$, but it changes in time as $\eta_{th}(t)$ in NISs. We used a time-dependent function to satisfy the condition, $\eta_{th} = 20\%$, for $\eta_{th}(t)$ as,



(a) (b)

Figure 9. The dissipation of heat from thermodynamic work, $Q_{wd}(t)$ and the total dissipation of heat, $Q_d(t)$. (a) The dissipation of heat, $Q_{wd}(t)$, from thermodynamic work of the flywheel and displacer rotations. (b) The total dissipation of heat, $Q_d(t) = Q_{\varepsilon d}(t) + Q_{wd}(t)$.

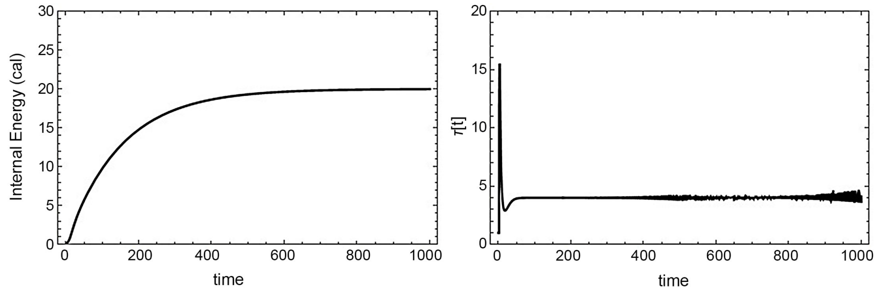
$$\eta_{th}(t) = \eta_{th}\left(1 - \eta_0\left(1.0 - e^{-\eta_0 t}\right) / \left(1.0 + e^{\eta_0 t}\right)\right), \tag{22}$$

where η_0 is a suitable constant determined by computer simulations. It should be consistently adjusted with other parameters, the number of revolutions, angular velocities and thermal quantities in computer simulations.

The total heat dissipation $Q_d(t)$ reaches a stable maximum value shown in **Figure 9(b)**, corresponding to the stable minimum dissipation $Q_{wd}(t)$. The results indicate that heat-mechanical energy conversions are optimally performed in the stable region, indicating that the mechanical energy of rotations can be optimally extracted in the stable region. The time-retardation of the flywheel-rotation performance can be observed by comparing **Figure 9(a)** and **Figure 9(b)**. In other words, a system of heat engines demands certain excessive heat before it reaches an optimal angular speed, and the heat dissipation is minimum at an optimal stable angular velocity. This is a well known empirical phenomenon, and the performance of engine is calculated explicitly in theoretical calculations in TMD. Thermomechanical work $Q_{wk}(t)$ and associated heat dissipation $Q_{wd}(t)$ maintain the relation (6), and $Q_{wk}(t)$ is small compared to $Q_{wd}(t)$. The ratio is $Q_{wk}(t)/Q_{wd}(t) \sim 0.05$ for the time region of a stable angular velocity in the computer simulation. The results depend on combinations of empirical parameters, λ_w, ξ, η_w, c and thermal efficiency of heat engines, and so, the ratio is useful to study the heat engine performance.

The angular velocity, $\theta'(t)$, reaches a stable maximum value and persists for a long time until $t \sim 2400$ (s). It is noteworthy that the maximum stable angular velocity $\theta'(t)$ can be maintained for a long time, although the heat flow, $\mathrm{d}Q_{in}(t)/\mathrm{d}t$, becomes very small for $500 < t$ as shown in **Figure 3(b)**, indicating that the maximum angular velocity can be maintained by supplying a very small amount of external heat flow. In other words, heat losses are suppressed minimum at a maximum stable rotation, and the energy restorations and extractions are performed suitable in the time range of a stable maximum rotation. The result is useful for the analysis of engine-performance and applications to the flywheel energy storage [13] and energy conversion mechanism [4].

The internal energy $Q_\varepsilon(t)$ and the measure of NISs, $\tau(t)$ are respectively shown in **Figure 10(a)** and **Figure 10(b)**. The time-dependent change of internal energy $Q_\varepsilon(t)$ is similar to that of $Q_d(t)$ and soon reaches a stable maximum value. The derivatives of internal energy and entropy (the total heat dissipation) are fundamentally important as Requirement 3) in TMD, which defines $\tau(t)$ or temperature $\tilde{T}(t)$ in NISs. The time-dependent temperature $\tilde{T}(t)$ in a nonequilibrium irreversible state of a low-temperature Stirling engine is shown in **Figure 10(b)** for the first time, in the theoretical calculation. The temperature, $\tilde{T}(t) = T\tau(t)$, (T is an initial temperature defined by $\tau(0) = 1$ at $t = 0$) changes rapidly within $0 < t \lesssim 50$ (s) and then, approaches a stable temperature, $\tilde{T}(t)$, at thermal equilibrium, which indicates that the system of heat engine needs certain amount of excess heat at the beginning of motion. The effect of wiggly deviations from frictional and thermal fluctuations appears at some times in



(a) (b)

Figure 10. The time variation of internal energy, $Q_\varepsilon(t)$ and the measure of NISs, $\tau(t)$. (a) The total internal energy, $Q_\varepsilon(t) = Q_{\varepsilon i}(t) + Q_{\varepsilon d}(t)$ (cal). (b) The measure of nonequilibrium irreversible states, $\tau(t)$. Note the tiny-wiggly line of $\tau(t)$. Temperature is defined by $\tilde{T} = T\tau(t)$.

$\tilde{T}(t)$, especially about the end of rotation; otherwise, temperature $\tilde{T}(t)$ inside heat engines is quite stable against uniform heat flows.

The sudden change of initial temperature would be consistent with an empirically known engine-response, which is reproduced naturally by the TMD method. In case of Stirling engines, the system needs certain amount of large heat-energy to initially activate the displacer and flywheel, which is consistent with the initial heat-demand indicated in **Figure 9(a)**, but it is interesting that $\tilde{T}(t)$, inside heat engines is quite stable during the time-progression against uniform heat flows. Thermal temperature depends on mechanical structure and thermal efficiency of heat engines, and so, the $\tilde{T}(t)$-analysis is useful to understand thermodynamic state inside heat engines, responses and efficiency of engines. It is interesting to know if the current model can be extended to ignition, detonation and internal combustion engine mechanism, which is a reason why we apply the model to a rapidly-decreasing, short-time uniform heat flow.

7.2. Thermodynamic Quantities in a Rapidly-Decreasing, Short-Time Uniform Heat Flow

Thermodynamic quantities derived from the result in Section 6.2 by solving the dissipative equation of motion in case (B) are employed to examine time-dependent thermal quantities. The associated heat-flow $Q_{wd}(t)$ from thermomechanical work $Q_{wk}(t)$ (the flywheel rotations) and the total dissipation of heat, $Q_d(t)$, are shown in **Figure 11(a)** and **Figure 11(b)**. As explained in the previous section, the flywheel rotation ends at $t \sim 18$ (s), and the last back-and-force small swingings of the flywheel are completely ignored in computer simulations, resulting in a sudden brake of motion in **Figure 11(a)**. The heat dissipation increases abruptly to a stable maximum value and ends at $t \sim 18$ (s). The total heat of dissipation Q_d, internal energy $Q_\varepsilon(t)$ and temperature $\tilde{T} = T\tau(t)$ should be understood in the time range $0 < t \lesssim 18$ (s).

The thermal dissipations progress rapidly to respective values of thermal equilibrium and a maximum stable state. It is noticeable that the ignition-like, short-time heat flows and time-progresses of thermal state are reproduced self-consistently. The results would indicate applicability to study heat engines by introducing the external discrete heat flows in (19). The heat dissipations change rapidly in time and reaches their maximum values, and the heat dissipation, $Q_{wd}(t)$, suddenly breaks off at $t \sim 18$ (s), because the flywheel rotation, $\theta'(t)/2\pi$, stops, shown in Figure 7(b) in the model simulation. However, the rotation of flywheel can be readily maintained by adding a small amount of heat flow, $\mathrm{d}Q_{in}(t)/\mathrm{d}t$, as explained in Section 6, and in this sense, the rapidly-decreasing, short-time calculations may be used to analyze one ignition or detonation mechanism. The theoretical analysis in a rapidly-decreasing, short-time uniform heat flow may suggest that the TMD analysis is applicable to study piston engines or internal combustion engines that use one or more reciprocating pistons to convert complicated heat flows into mechanical motion.

Internal energy $Q_\varepsilon(t)$ and a measure of NISs, $\tau(t)$, are shown in Figure 12(a) and Figure 12(b). The total internal energy $Q_\varepsilon(t)$ changes in time similar to $Q_d(t)$, and the measure, $\tau(t)$ is directly calculated by (2) with the initial condition $\tau(0)=1$ and $\tilde{T}(t)=T\tau(t)$ at thermal equilibrium. Internal energy $Q_\varepsilon(t)$ rapidly reaches a stable maximum value, and heat flows of $Q_\varepsilon(t)/\mathrm{d}t$ and $Q_d(t)/\mathrm{d}t$ produce a stable thermal temperature. The sudden jump from thermodynamic equilibrium temperature $T(\tau(0)=1)$ to thermal equilibrium temperature $\tilde{T}(t)$ should be taken qualitatively, since thermal conduction mechanism characterized by thermal conductivity of working fluid and mechanical structure is not included in the current calculations. The inclusion of thermal conduction mechanism would make $\tau(t)$ smooth and continuous. Time-retardation effects of physical quantities could be realized by including Fourier thermal conduction mechanism in the TMD formalism, The analysis of continuous progress from thermodynamic equilibrium to NISs by employing Fourier's law of thermal conduction, and the thermal analysis of ignition and detonation mechanism is a physically interesting open question, which should be considered in the future work.

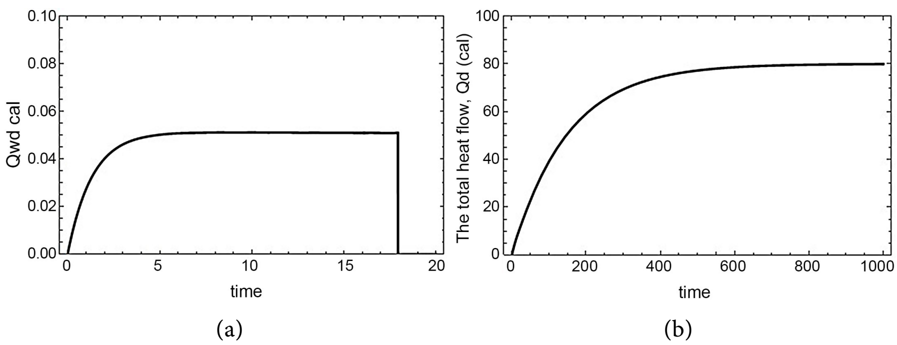
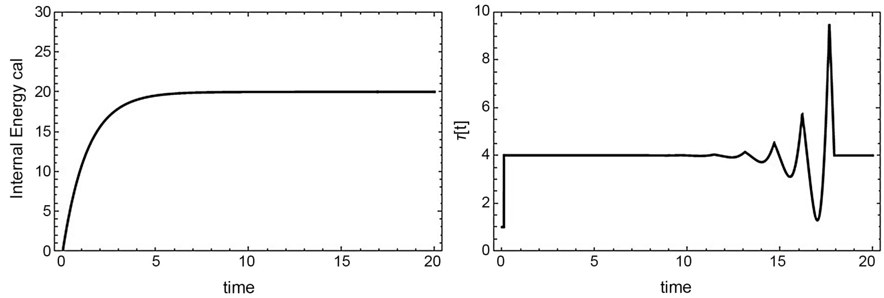


Figure 11. The associating dissipation of heat from thermal work, $Q_{wd}(t)$ and the total heat of dissipation, $Q_d(t)$. (a) The associating dissipation of heat from thermal work, $Q_{wd}(t)$ in the process $Q_w(t)$. The flywheel rotations end at $t \sim 18$ (s). (b) The total dissipation of heat, $Q_d(t) = Q_{\varepsilon d}(t) + Q_{wd}(t)$.



(a) (b)

Figure 12. The change of internal energy, $Q_\varepsilon(t)$ and the measure of NISs, $\tau(t)$. (a) The internal energy, $Q_\varepsilon(t) = Q_{\varepsilon i}(t) + Q_{\varepsilon d}(t)$. (b) The measure of nonequilibrium irreversible states, $\tau(t)$. Note that temperature is defined by $\tilde{T} = T\tau(t)$.

The thermal temperature $\tilde{T}(t)$ is stable, but it deviates abruptly about the end of rotations. It would be a physically reasonable phenomenon, because the flywheel oscillation becomes very slow and the displacer is not able to dissipate heat properly, resulting in accumulation of heat and the high-temperature deviation. The fact indicates that the dissipative heat $Q_d(t)$ must first accumulate before thermal temperature $\tilde{T}(t)$ increases. In other words, thermal temperature increases slowly than heat-flow does and the fact could be useful to apply the TMD method to the law of Fourier thermal conduction. Thermal temperature inside cylinder shows the existence of a steady state thermal state, which is assumed in a thermal conductivity measurement. The existence of a stable thermal temperature is shown in Figure 10(b) and Figure 12(b). The stable thermal temperature $\tilde{T}(t)$ at thermal equilibrium and properties at the beginning and end of rotations would depend on parameters of thermal conduction, and so, thermal temperature, $\tilde{T}(t)$, is an important quantity to analyze properties of internal energy, thermodynamic work and entropy in thermal states inside machines, theoretically and experimentally. Thermomechanical phenomena in NISs in a low-temperature heat engine are reasonably and consistently studied by the TMD method, and it is expected to be applied to other types of sophisticated heat engines as well as piston engines.

8. Conclusions and Perspectives

One of the fundamental results in TMD is that the concept of acceleration is no longer decisive for trajectories of thermomechanical motion, while it is fundamental in Newtonian mechanics. The common concept of force between TMD and Newtonian mechanics is confined to the fact that force can cause an object to change its velocity and direction. The force is rather important in TMD as external piecewise-continuous force coupled to the heat of work $Q_w(t)$ (kinetic energy + associated entropy) as shown in the dissipative equation of motion and time-progressing physical quantities. In other words, Newtonian mechanics is rigorous in the world without friction and thermal fluctuations, which is declared at the outset in Newton's assumption that excludes friction in the principle of inertia as the first law, and mathematically speaking, the mechanical motion is defined as continuous and differentiable functions of Class C^2. However, in the TMD model, thermomechanical motion and physical quantities are confined to functions of Class C^1.

The nonequilibrium irreversible motion and time-dependent work, internal energy, and entropy are first solved and examined for the first time by authors, and the theoretical analysis of the drinking bird system led us to the TMD model [1] [2]. The low-temperature Stirling engine is the second example solved consistently by the TMD method, and heat engines are excellent experimental devices to study NISs. Thermomechanical velocities in a thermal state are fundamentally different from Newtonian mechanics when mass is affected by time-dependent variation of frictions and thermal fluctuations caused by heat flows. In the nonequilibrium irreversible state, the concept of force is only effective to change the direction of motion, and force is not necessarily associated with mass × acceleration. The time-dependent thermal quantities, such as internal energy, thermal work, entropy, and temperature are self-consistently obtained by solving the dissipative equation of motion and the triad propositions in TMD. It is remarkable that the fundamental requirements for a scientific theory, *reproducibility*, *testability,* and *self-consistency* explained in the paper [2], are maintained in calculations. The self-consistent results suggest that the thermal conduction mechanism, for example, Fourier's law of conduction could be incorporated into the TMD method to improve analyses of thermal conduction.

When the dissipations of energy caused by frictional variations and thermal fluctuations do not affect the motion of particles so much, the trajectory, velocity, and acceleration of particles are well described by Newtonian mechanics. Statistical mechanics is the description of states, which fluctuates about average values and is characterized by probability distribution. However, the fundamental requirements of the TMD method are different from Newtonian mechanics, statistics and probability approaches. The frictional variations only appear in physical quantities by way of velocities. The dissipative equation of motion for heat engines in the TMD method consists of the external driving force composed of thermal work and frictional variations, which indicates that heat engines cannot operate without friction and viscosity of the working fluid. In other words, heat engines cannot function without friction and viscosity, which is a positive and scientific way of understanding the ability of friction.

The trajectory, velocity, and acceleration of Brownian motion are random and piecewise continuous, which indicates that velocity and acceleration have no direct physical meaning in the thermal world. However, the trajectory and velocity are physical in the TMD model; the trajectory and velocity of motion are only meaningful as Class C^1 continuously differentiable functions. The external driving forces are meaningful to cause an object to change its velocity and direction, but the derivation of acceleration is not meaningful, resulting in spiny hedgehog-like lines. The fundamental assumptions in TMD are general and simple and yield maximum knowledge from a system of minimum basic laws. When the assumptions in TMD are adapted, the nonequilibrium irreversible states of a drinking bird and a low-temperature Stirling heat engine are solved consistently [1] [2] [3] [4]. Applications are expected, for example, in the fields of irreversible processes such as complicated piston engines, ignition, combustion, and detonation mechanisms and systems [14] [15] [16], quantum heat engines [17] [18] [19], solar-powered, high and low-temperature Stirling engines, quantum thermodynamic systems and technologies for sustainable energies [20] [21] [22] [23] [24].

The analysis in TMD for thermomechanical phenomena becomes useful in realistic applications for clean and sustainable technologies by way of thermoelectric energy conversions. In conventional applications of heat engines and energy generations, high-temperature pressurized steam for high rotations of the turbine is assumed to produce much electric energy, resulting in the massive and sturdy structure of heat engines. The method of thermoelectric generation produces, for example, a different technological device, which we call the DM-EMI low-temperature thermoelectric generator [4]. The very high-temperature pressurized steam is not necessarily required to obtain electric energy in thermoelectric energy conversions. A low-temperature thermoelectric heat generator can be developed by way of thermoelectric energy conversion of hot water about 40˚C - 100˚C [3] [4]. The TMD method is useful for practical applications and can be applied to resolve energy and environmental problems for clean and sustainable energy.

Acknowledgements

The research is supported by Kansai Research Foundation (KRF) for Technology Promotion, Osaka, Japan (2023).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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