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Application of HAM for Nonlinear Integro-Differential Equations of Order Two

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Abstract

In this work, we consider the second order nonlinear integro-differential Equation (IDEs) of the Volterra-Fredholm type. One of the popular methods for solving Volterra or Fredholm type IDEs is the method of quadrature while the problem of consideration is a linear problem. If IDEs are nonlinear or integral kernel is complicated, then quadrature rule is not most suitable; therefore, other types of methods are needed to develop. One of the suitable and effective method is homotopy analysis method (HAM) developed by Liao in 1992. To apply HAM, we firstly reduced the IDEs into nonlinear integral Equation (IEs) of Volterra-Fredholm type; then the standard HAM was applied. Gauss-Legendre quadrature formula was used for kernel integrations. Obtained system of algebraic equations was solved numerically. Moreover, numerical examples demonstrate the high accuracy of the proposed method. Comparisons with other methods are also provided. The results show that the proposed method is simple, effective and dominated other methods.

Keywords

Integral-Differential Equations, Homotopy Analyses Method, Iterative System, Algebraic Equations, Gauss-Legendre Quadrature Formula

1. Introduction

In the recent literature, there is a growing interest to solve IDEs, because many problems in mathematical physics, theory of elasticity, visco-dynamics fluid and mixed problems of mechanics of continuous media reduce to the integro-differential Equation (IDEs) (Volterra or Fredholm type) of the first or second kind. Many methods are elaborated as a numerical tool to solve IDEs with initial and boundary conditions, for instance, Adomian decomposition method (ADM) (proposed by Adomian [1] [2]). It has been shown that ADM yields a rapid convergence of the solution series to linear and nonlinear deterministic and stochastic equations. Then, this has been extended by Wazwaz [3] [4] to Volterra integral equation and to boundary value problems for higher-order integro-differential equations. There are many other methods developed by different researchers for linear and nonlinear IDEs with initial, boundary or mixed conditions, for instance homotopy analysis method (HAM) developed by Liao [5] [6] [7], modified HAM [8], q-HAM [9], new development of HAM [10], homotopy perturbation method (HPM) developed by Ji-Huan He [11] [12], HPM for nonlinear differential-difference equations [13], HPM for nth-Order Integro-Differential Equations [14], collocation method [15], new boundary element method [16], Linear Programming Method [17], Laplace Decomposition Algorithm [18], polynomial approximations [19], Wavelet Galerkin method [20], and so on.

In this paper, we consider nonlinear Fredholm-Volterra integro-differential Equations (FVIDEs) of the order two in the form:

$$\sum_{k=0}^{2} c_k(t)u^{(k)}(t) = g(t)z(u(t)) + \mu_1\int_a^t K_1(t,s)G_1[u(s),u'(s)]\mathrm{d}s + \mu_2\int_a^b K_2(t,s)G_2[u(s),u'(s)]\mathrm{d}s \tag{1}$$

with the initial conditions

$$u(a) = b_0,\ u'(a) = b_1, \tag{2}$$

where $u^{(k)}(t)$ is the k^{th} derivative of the unknown function $u(x)$ that needs to be determined, $K_1(t,s)$ and $K_2(t,s)$ are the kernels of the equation, $c_k(t)$ and $g(t)$ are known analytic functions, G_1 and G_2 are nonlinear functions of u,u', and μ_1,μ_2,b_0,b_1 are real constants.

Primarily, to solve nonlinear IDEs (1) and (2), we have applied a integral transformation to reduce it to nonlinear integral Equations (NIEs) of Volterra-Fredholm type; then, we applied the standard HAM together with Gauss-Legendre quadrature Formulas (GLQFs). Once solving nonlinear IEs, the inverse transformation is used to restore the original solutions of the problem (1) and (2). The results obtained are compared with other methods at the same number of iterations with a different number of node points.

2. Gauss-Legendre Quadrature Formula

In Eshquvatov *et al.* [19], for the kernels of Fredgholm and Volterra integrals on the interval $[a,b]$ the Gauss-Legendre QF are developed

$$Q_1(s) = \int_a^b K_1(s,t)x(t)\mathrm{d}t = \frac{b-a}{2}\sum_{j=1}^{n+1} W_{1,j}(s)x(t_{1,j}) + R_{n+1,1}(s), \tag{3}$$

$$Q_2(s) = \int_a^s K_2(s,t)x(t)\mathrm{d}t = \frac{s-a}{2}\sum_{j=1}^{m+1} W_{2,j}(s)x(s_{2,j}) + R_{n+1,2}(t), \tag{4}$$

where

$$W_{1,j}(s)=K_1(s,t_{1,j})w_j,\ t_{1,j}=\frac{b-a}{2}r_j+\frac{b+a}{2},\ j=\{1,\cdots,n+1\}$$

$$W_{2,j}(s)=K_2(s,t_{2,j})w_j,\ t_{2,j}=\frac{b-s_m}{2}r_j+\frac{b+s_m}{2},\ n\ge m\ge 2,$$

with

$$w_j=\frac{2}{(1-r_j^2)\left[P_n'(r_j)\right]^2},\ j=1,2,\cdots,n+1,\quad \sum_{k=1}^{n+1}w_k=2, \tag{5}$$

with r_j are the roots of the Legendre polynomial $P_{n+1}(r)$, *i.e.*

$$P_{n+1}(r_j)=0,\ j=1,2,\cdots,n+1,\ r_j\in[-1,1] \tag{6}$$

Application of Gauss-Legendre QF for the nonlinear integral is as follows

$$\begin{aligned}\int_a^s\frac{K_1(s,\tau)}{c_2(s)}G_1[u(\tau),u'(\tau)]\mathrm{d}\tau&=\frac{b-s}{2}\sum_{k=1}^{m+1}W_{1,k}(s)G_1[u(\tau_k),u'(\tau_k)]+R_{n+1,1}(s),\\ \int_a^b\frac{K_2(s,\tau)}{c_2(s)}G_2[u(\tau),u'(\tau)]\mathrm{d}\tau&=\frac{b-a}{2}\sum_{k=1}^{n+1}W_{2,k}(s)G_2[u(\tau_k),u'(\tau_k)]+R_{n+1,2}(s),\end{aligned} \tag{7}$$

where

$$s_m=a+mh,\ h=\frac{b-a}{n+1},$$

and

$$\begin{aligned}W_{k,1}(t)&=\frac{K_1(s,\tau_{k,1})}{c_2(s)}w_k,\quad \tau_{k,1}=\frac{b-s_m}{2}r_k+\frac{b+s_m}{2},\ n\ge m\ge 2,\\ W_{k,2}(t)&=\frac{K_2(s,\tau_{k,2})}{c_2(s)}w_k,\quad \tau_{k,2}=\frac{b-a}{2}r_k+\frac{b+a}{2},\end{aligned} \tag{8}$$

here w_k and r_k are defined by (5) and (6) respectively.

Quadrature Formulas (7) and (8) are used in the evaluation of kernel integrals when integrals in (9) and (11) have no antiderivative functions.

3. Homotopy Analysis Method (HAM)

Let us rewrite Eq. (1) and (2) in the form

$$\begin{aligned}&Lu+\sum_{k=0}^{1}\frac{c_k(t)}{c_2(t)}u^{(k)}(t)\\ &=\frac{g(t)}{c_2(t)}z(u(t))+\mu_1\int_a^t\frac{K_1(t,s)}{c_2(t)}G_1[u,u']\mathrm{d}s+\mu_2\int_a^b\frac{K_2(t,s)}{c_2(t)}G_2[u,u']\mathrm{d}s,\end{aligned} \tag{9}$$

where $Lu=\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t)$ is the second order differential operator.

Acing inverse operator $L^{-1}(\bullet)=\int_a^t\int_a^{t_1}(\bullet)\mathrm{d}\tau\mathrm{d}t_1=\int_a^t(t-s)(\bullet)\mathrm{d}s$ on both sides of Eq. (9) and taking into account initial conditions (2), we obtain

$$
\begin{aligned}
& u(t)-\sum_{k=0}^{1}\int_a^t (t-s)c_k^*(s)u^{(k)}(s)\mathrm{d}s-\int_a^t (t-s)g^*(s)z(u(s))\mathrm{d}s \\
& +\mu_1\int_a^t (t-s)G_1^*\left[u(s),u'(s)\right]\mathrm{d}s+\mu_2\int_a^t (t-s)G_2^*\left[u(s),u'(s)\right]\mathrm{d}s=b_0+b_1(t-a)
\end{aligned} \tag{10}
$$

where $c_k^*(t)=\dfrac{c_k(t)}{c_2(t)}, k=\{0,1\}, g^*(t)=\dfrac{g(t)}{c_2(t)}$, and

$$
\begin{aligned}
G_1^*\left[u(s),u'(s)\right] &= \int_a^s \frac{K_1(s,\tau)}{c_2(s)}G_1\left[u(\tau),u'(\tau)\right]\mathrm{d}\tau, \\
G_2^*\left[u(s),u'(s)\right] &= \int_a^b \frac{K_2(s,\tau)}{c_2(s)}G_2\left[u(\tau),u'(\tau)\right]\mathrm{d}\tau.
\end{aligned} \tag{11}
$$

Writing Eq. (10) in the form,

$$
N\left[\phi(t,q)\right]=P(t), \tag{12}
$$

where $P(t)=b_0+b_1(t-a)$ and

$$
\begin{aligned}
N\left[\phi(t,q)\right] = \phi(t,q) & -\sum_{k=0}^{1}\int_a^t (t-s)c_k^*(s)\phi^{(k)}(s,q)\mathrm{d}s \\
& -\int_a^t (t-s)g^*(s)z(\phi(s,q))\mathrm{d}s \\
& +\mu_1\int_a^t (t-s)G_1^*\left[\phi(s,q),\frac{\partial}{\partial s}\phi(s,q)\right]\mathrm{d}s \\
& +\mu_2\int_a^t (t-s)G_2^*\left[\phi(s,q),\frac{\partial}{\partial s}(s,q)\right]\mathrm{d}s
\end{aligned} \tag{13}
$$

We apply HAM. To do this end search solution of (14) in the series form

$$
\phi(t,q)=\sum_{m=0}^{\infty}u_m(t)q^m=u_0(t)+\sum_{m=1}^{\infty}u_m(t)q^m \tag{14}
$$

For the sake of clarity, we will first present a brief description of the standard HAM proposed by Liao ([5], 1992). He constructed the so-called zeroth-order deformation equation

$$
(1-q)L\left[\phi(t,q)-u_0(t)\right]=q\hbar H(t)\left\{N\left[\phi(t,q)\right]-P(t)\right\} \tag{15}
$$

Since $h\neq 0, H(t)\neq 0, \forall t\in[a,b]$ then $q=0$ and $q=1$, leads

$$
\phi(t,0)=u_0(t),\quad N\left[\phi(t,1)\right]=P(t),
$$

It is known that m-th order deformation equation is

$$
L\left[u_m(t)-\chi_m u_{m-1}(t)\right]=\hbar R_m(\bar{u}_{m-1}), \tag{16}
$$

where $\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$ and

$$
R_m(\bar{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[N(\phi(t,q))-P(t)\right]\bigg|_{q=0} \tag{17}
$$

On the basis of Equations (18) and (19), we find sequence of solutions $u_1(t), u_2(t), u_3(t), \cdots, u_n(t), \cdots$ and substitute it into (16) at $q=1$ we obtain approximate solution of the form

$$u(t)=\phi(t,1)=\sum_{m=0}^{n}u_m(t)=u_0(t)+\sum_{m=1}^{n}u_m(t). \tag{18}$$

To find iterative solution $u_m(t)$ of the problem (14) and (15), let us choose initial guess $u_0(t)$ and for $m=1$, we have

$$\begin{aligned}
u_1(t) &= \hbar\left[N(\phi(t,q))-P(t)\right]\Big|_{q=0} \\
&= \hbar\left\{\phi(t,q)-\sum_{k=0}^{1}\int_a^t (t-s)c_k^*(s)\phi^{(k)}(s,q)\,\mathrm{d}s\right. \\
&\quad -\int_a^t (t-s)g^*(s)z(\phi(s,q))\,\mathrm{d}s \\
&\quad +\mu_1\int_a^t (t-s)G_1^*\left[\phi(s,q),\frac{\partial}{\partial s}\phi(s,q)\right]\mathrm{d}s \\
&\quad \left.+\mu_2\int_a^t (t-s)G_2^*\left[\phi(s,q),\frac{\partial}{\partial s}\phi(s,q)\right]\mathrm{d}s\right\}_{q=0} \\
&= \left[u_0(t)-\sum_{k=0}^{1}\int_a^t (t-s)c_k^*(s)u_0^{(k)}(s)\,\mathrm{d}s\right. \\
&\quad -\int_a^t (t-s)g^*(s)\left[z(\phi(s,q))\right]_{q=0}\mathrm{d}s \\
&\quad +\mu_1\int_a^t (t-s)G_1^*\left[\phi(s,q),\frac{\partial}{\partial s}\phi(s,q)\right]_{q=0}\mathrm{d}s \\
&\quad \left.+\mu_2\int_a^t (t-s)G_2^*\left[\phi(s,q),\frac{\partial}{\partial s}\phi(s,q)\right]_{q=0}\mathrm{d}s-P(t)\right],
\end{aligned} \tag{19}$$

For $m=2$, from (18) and (19) it follows that

$$\begin{aligned}
u_2(t) &= u_1(t)+\hbar\frac{\partial}{\partial q}\left[N(\phi(t,q))-P(t)\right]\Big|_{q=0} \\
&= u_1(t)+\hbar\left\{u_1(t)-\sum_{k=0}^{1}\int_a^t (t-s)c_k^*(s)u_1^{(k)}(s)\,\mathrm{d}s\right. \\
&\quad -\int_a^t (t-s)g^*(s)\frac{\partial}{\partial q}\left[z(\phi(s,q))\right]_{q=0}\mathrm{d}s \\
&\quad +\mu_1\int_a^t (t-s)\frac{\partial}{\partial q}\left[G_1^*\left(\phi(s,q),\frac{\partial}{\partial s}\phi(s,q)\right)\right]_{q=0}\mathrm{d}s \\
&\quad \left.+\mu_2\int_a^t (t-s)\frac{\partial}{\partial q}\left[G_2^*\left(\phi(s,q),\frac{\partial}{\partial s}\phi(s,q)\right)\right]_{q=0}\mathrm{d}s\right\}
\end{aligned} \tag{20}$$

Continue this procedure, we obtain

$$\begin{aligned} u_m(t) &= u_{m-1}(t) + \hbar \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[N(\phi(t,q)) - P(t) \right] \Big|_{q=0} \\ &= u_{m-1}(t) + \hbar \Bigg\{ u_{m-1}(t) - \sum_{k=0}^{1} \int_a^t (t-s) c_k^*(s) u_{m-1}^{(k)}(s) \mathrm{ds} \\ &\quad - \int_a^t (t-s) g^*(s) \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[z(\phi(s,q)) \right]_{q=0} \mathrm{ds} \\ &\quad + \mu_1 \int_a^t (t-s) \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[G_1^*\left(\phi(s,q), \frac{\partial}{\partial s} \phi(s,q) \right) \right]_{q=0} \mathrm{ds} \\ &\quad + \mu_2 \int_a^t (t-s) \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[G_2^*\left(\phi(s,q), \frac{\partial}{\partial s} \phi(s,q) \right) \right]_{q=0} \mathrm{ds} \Bigg\} \end{aligned} \tag{21}$$

Finding all iterations $u_m(t), m = 1, 2, \cdots$ and substituting it into (18) yields approximate solution of the Equation (12) which is equivalent solution of integro-differential Equations (1) with initial conditions (2).

4. Numerical Experiments

Example 1. (*Ahmed Hamoud.et al.* [16]) Consider the following Fredgholm integro-differential equation with initial condition.

$$\begin{aligned} &u'(s) = e^s(1+s) - s + \int_0^1 s u(t) \mathrm{d}t, \\ &u(0) = 0. \end{aligned} \tag{22}$$

The exact solution of (22) is $u(s) = se^s$.

To apply HAM convert Eq. (22) into integral equations of the form

$$u(s) - \frac{s^2}{2} \int_0^1 u(t) \mathrm{d}t = se^s - \frac{s^2}{2}. \tag{23}$$

Let us write Eq. (23) in the operator form

$$N(\phi(s,q)) = P(s), \tag{24}$$

where

$$N(\phi(s,q)) = \phi(s,q) - \frac{s^2}{2} \int_0^1 \phi(t,q) \mathrm{d}t, \quad P(s) = se^s - \frac{s^2}{2}. \tag{25}$$

Choose initial guess as $u_0(s) = s$ then from (21) - (23) it follows that

$$\begin{aligned} u_1(s) &= \hbar \left[N(\phi(s,q)) - P(s) \right] \Big|_{q=0} \\ &= \hbar \left[\phi(s,q) - \frac{s^2}{2} \int_0^1 \phi(t,q) \mathrm{d}t - \left(se^s - \frac{s^2}{2} \right) \right]_{q=0} \\ &= \hbar \left[u_0(s) - \frac{s^2}{2} \int_0^1 u_0(t) \mathrm{d}t - \left(se^s - \frac{s^2}{2} \right) \right] \\ &= \hbar \left[s + \frac{s^2}{4} - se^s \right], \end{aligned} \tag{26}$$

$$\begin{aligned} u_2(s) &= u_1(s) + \hbar \frac{\partial}{\partial q}\left[N(\phi(s,q)) - P(s)\right]\Big|_{q=0} \\ &= u_1(s) + \hbar\left[u_1(s) - \frac{s^2}{2}\int_0^1 u_1(t)\,\mathrm{d}t\right] \\ &= u_1(s)(1+\hbar) + \hbar^2\frac{s^2}{6}, \end{aligned} \tag{27}$$

In general, m-th term of iteration can be computed as

$$\begin{aligned} u_m(s) &= u_{m-1}(s) + \hbar\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[N(\phi(s,q))\right]\Big|_{q=0} \\ &= u_{m-1}(s)(1+\hbar) - \hbar\frac{s^2}{2}\int_0^1 u_{m-1}(t)\,\mathrm{d}t. \end{aligned} \tag{28}$$

So that three-term approximate solution at $\hbar = -1$ is

$$U_2(s)_{\hbar=-1} = \left[u_0(s) + u_1(s) + u_2(s)\right]_{\hbar=-1} = se^s - \frac{s^2}{24} \tag{29}$$

We find $U_3(s)$, $U_5(s)$, and $U_{10}(s)$ for $\hbar = -1$ according to (18)

$$\begin{aligned} U_3(s)_{\hbar=-1} &= se^s - \frac{s^2}{144} \\ U_5(s)_{\hbar=-1} &= se^s - \frac{s^2}{5184} \\ U_{10}(s)_{\hbar=-1} &= se^s - \frac{s^2}{40310784} \end{aligned} \tag{30}$$

Numerical results are summarized in Table 1.

Table 1. HAM for Example 1 at different values of "N".

s	Exact $u(s) = se^s$	Error (N = 3)	Error (N = 5)	Error (N = 10)
0	0	0	0	0
0.2	0.2442805	0.0002777	0.0000077	$9.92\ 10^{-10}$
0.4	0.5967298	0.0011111	0.0000308	$3.96\ 10^{-9}$
0.6	1.0932712	0.0025000	0.0000694	$8.93\ 10^{-9}$
0.8	1.7804327	0.0044444	0.0001234	$1.58\ 10^{-8}$
1	2.7182818	0.0069444	0.0001929	$2.48\ 10^{-8}$

From Table 1, and Eq. (30), we can conclude that the proposed method approaches to exact solution very fast when number of iteration is increased.

Example 2. (*Huseen* [9]) Let us consider non-linear VIEs

$$\begin{aligned} u'(s) &= -1 + \int_0^s u^2(t)\,\mathrm{d}t, \\ u(0) &= 0. \end{aligned} \tag{31}$$

Solution: There is no analytic solution of Equation (31). To solve it by standard HAM, we rewrite it in the operator form

$$N\left(\phi(s,q)\right)=f(s), \tag{32}$$

where

$$\begin{cases} N\left(\phi(s,q)\right)=\dfrac{\partial}{\partial s}\phi(s,q)-\int_0^s\phi^2(t,q)\,\mathrm{d}t, \\ f(s)=-1. \end{cases} \tag{33}$$

It is known that

$$\begin{cases} \phi(s,q)=u_0(s)+\sum_{m=1}^{\infty}q^m u_m(s), \\ \dfrac{\partial}{\partial q}\phi(s,q)\Big|_{q=0}=u_1(s), \\ \dfrac{\partial^{m-1}}{\partial q^{m-1}}\phi(s,q)\Big|_{q=0}=(m-1)!u_{m-1}(s), \\ \dfrac{\partial}{\partial q}\phi^2(s,q)\Big|_{q=0}=2u_0(s)u_1(s),\ \dfrac{\partial}{\partial q}\phi^2(s,q)\Big|_{q=0}=u_0^2(s), \\ \dfrac{1}{(m-1)!}\dfrac{\partial^{m-1}}{\partial q^{m-1}}\phi^2(s,q)\Big|_{q=0}=\sum_{i=0}^{m-1}u_i(s)u_{m-1-i} \end{cases} \tag{34}$$

Applying HAM yields

$$\begin{aligned} L\left[u_1(s)\right]&=\hbar\left[\frac{\partial}{\partial s}u_0(s)-\int_0^s u_0^2(t)\,\mathrm{d}t-f(s)\right], \\ L\left[u_2(s)-u_1(s)\right]&=\hbar\left[\frac{\partial}{\partial s}u_1(s)-2\int_0^s u_0(t)u_1(t)\,\mathrm{d}t\right], \\ L\left[u_m(s)-u_{m-1}(s)\right]&=\hbar\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial s^{m-1}}\left[N\left(\phi(s,q)\right)\right]\Big|_{q=0} \\ &=\hbar\left[\frac{\partial}{\partial s}u_{m-1}(s)-\int_0^s\sum_{i=0}^{m-1}u_i(t)u_{m-1-i}(t)\,\mathrm{d}t\right]. \end{aligned} \tag{35}$$

Since $Lu(s)=\frac{\mathrm{d}}{\mathrm{d}s}u(s)$, by choosing initial guess as $u_0(s)=-s$ it follows that

$$\begin{aligned} u_1(s)&=-\frac{\hbar}{12}\cdot s^4 \\ u_2(s)&=u_1(s)-\frac{\hbar^2}{252}\cdot s^4\left(21+s^3\right) \\ u_3(s)&=u_2(s)+\frac{\hbar^2}{6048}\cdot s^4\left(504(1+\hbar)+24(1+2\hbar)s^3+\hbar s^6\right). \end{aligned} \tag{36}$$

Five terms approximation of the HAM at $\hbar=-1$ is

$$\begin{aligned} U_{5HAM}(s)&=u_0(s)+u_1(s)+u_2(s)+u_3(s)+u_4(s)+u_5(s) \\ &=-s+\frac{1}{12}s^4-\frac{1}{252}s^7+\frac{1}{6048}s^{10}+\frac{1}{157248}s^{13}+\frac{37}{158505984}s^{16}. \end{aligned} \tag{37}$$

Fifth terms approximation of the Adomian decomposition method (ADM) developed in El-Sayed and Abdel-Aziz [21] has the form,

$$U_{5ADM}(s) = u_0(s)+u_1(s)+u_2(s)+u_3(s)+u_4(s)+u_5(s)$$
$$= -s+\frac{1}{12}s^4-\frac{1}{252}s^7+\frac{1}{6048}s^{10}+\frac{1}{157248}s^{13}+\frac{79}{264176640}s^{16}. \quad (38)$$

From (37) and (38), it has almost same terms except last terms. Let us see the numerical comparisons of two methods

From Table 2, it can be seen that HAM is slightly better than ADM. Both methods are highly accurate.

Table 2. HAM and ADM for Example 2 at different values of "$N = 5$".

S	Exact $u(s) = se^s$	HAM	ADM
0.0000	0	0	0
0.0938	−0.0937	−0.0937935	−0.0937935
0.2188	−0.2186	−0.2186089	−0.2186090
0.3125	−0.3117	−0.3117041	−0.3117060
0.4062	−0.4040	−0.4039240	−0.4039390
0.5000	−0.4948	−0.4947605	−0.4947230
0.6250	−0.6124	−0.6123349	−0.6124310
0.7188	−0.6969	−0.6969540	−0.6969410
0.8125	−0.7771	−0.7770340	−0.7770900
0.9062	−0.8520	−0.8519477	−0.8519340
1.0000	−0.9205	−0.9205260	−0.9204760

Example 3. (Majid Khan, *et al.* [18]) Let us consider Fredholm integro-differential equation with initial condition

$$u'(s) = 1-\frac{s}{3}+\int_0^1 stu(t)\,dt, \quad u(0)=0. \quad (39)$$

The exact solution of Eq. (33) is $u(s) = s$.

To apply HAM convert Eq. (33) into integral equations of the form

$$u(s)-\frac{s^2}{2}\int_0^1 tu(t)\,dt = s-\frac{s^2}{6}. \quad (40)$$

Let us write Eq. (29) in the operator form

$$N(\phi(s,q)) = P(s),$$

where

$$N(\phi(s,q)) = \phi(s,q)-\frac{s^2}{2}\int_0^1 t\phi(t,q)\,dt, \quad P(s) = s-\frac{s^2}{6}. \quad (41)$$

Choose initial guess as $u_0(s) = s-\frac{s^2}{12}$ then from Eqs. (19) - (21), it follows that

$$u_1(s) = \hbar\left[N(\phi(s,q)) - P(s)\right]\Big| = -\hbar\left[\frac{7s^2}{96}\right], . \tag{42}$$

$$u_2(s) = u_1(s) + \hbar\frac{\partial}{\partial q}\left[N(\phi(s,q)) - P(s)\right]\Big|_{q=0} = u_1(s)(1+\hbar) - \hbar^2\frac{s^2}{768} . \tag{43}$$

So that two-term approximate solution at $\hbar = -1$ is

$$\begin{aligned} U_2(s)_{\hbar=-1} &= \left[u_0(s) + u_1(s) + u_2(s)\right]_{\hbar=-1} = s - \frac{1}{768}s^2 \\ U_5(s)_{\hbar=-1} &= \left[u_0(s) + u_1(s) + u_2(s) + u_3(s) + u_4(s) + u_5(s)\right]_{\hbar=-1} \\ &= s - \frac{1}{393216}s^2 \end{aligned} \tag{44}$$

Majid Khan, *et al.* [18] have developed the Adomian decomposition method (ADM) and two terms approximation of the Laplace decomposition method (LDM). It is shown that ADM at the initial guess $u_0(s) = s - \frac{s^2}{12}$ has the form

$$U_2(s)_{LDM} = \left[u_0(s) + u_1(s) + u_2(s)\right] = s - \frac{113}{384}s^2 . \tag{45}$$

Numerical comparisons of two methods are given in **Table 3**.

Table 3. LDM [18] and HAM for Example 3 at different values of "*N* = 2".

s	Exact $u(s) = s$	Error LDM [18] (*N* = 2)	Error HAM (*N* = 2)	Error HAM (*N* = 5)
0	0	0	0	0
0.2	0.2	0.011771	0.000260	$5.08 * 10^{-7}$
0.4	0.4	0.047083	0.000520	$1.01 * 10^{-6}$
0.6	0.6	0.105938	0.000781	$1.52 * 10^{-6}$
0.8	0.8	0.188333	0.001041	$2.03 * 10^{-6}$
1	1	0.294271	0.001302	$2.54 * 10^{-6}$

From **Table 3**, it can be concluded that the standard HAM is dominated the LDM [18]. From last column of **Table 3**, we can observe that error term of the HAM decreases drastically at five iterations only.

Example 4. (Manafianheris [22]). Let us consider non-linear FIEs

$$\begin{aligned} &u''(s) = \sinh(s) + s - \int_0^1 s\left(\cosh^2(t) - u^2(t)\right)dt, \\ &u(0) = 0, \quad u'(0) = 1 \end{aligned} \tag{46}$$

The exact solution of Eq. (46) is $u(s) = \sinh(s)$.

Solution: To apply HAM we convert Eq. (46) into integral equations of the form

$$u(s)=\sinh(s)+\frac{s^3}{12}\left(1-\frac{e^4-1}{4e^2}\right)+\frac{s^3}{6}\int_0^1 u^2(t)\,\mathrm{d}t$$

To solve it by standard HAM we rewrite it in the operator form

$$N(\phi(s,q))=f(s),$$

where

$$\begin{cases} N(\phi(s,q))=\dfrac{\partial^2}{\partial^2 s}\phi(s,q)-\dfrac{s^3}{6}\displaystyle\int_0^1\phi^2(t,q)\,\mathrm{d}t, \\ f(s)=\sinh(s)+\dfrac{s^3}{12}\left(1-\dfrac{e^4-1}{4e^2}\right). \end{cases}$$

In view of Equation (34), we obtain

$$L[u_1(s)]=\hbar\left[N(\phi(s,q))-f(s)\right]\Big|_{q=0}=\hbar\left[\frac{\partial}{\partial s}u_{m-1}(s)-\int_0^1\sum_{i=0}^{m-1}u_i(t)u_{m-1-i}(t)\,\mathrm{d}t\right].$$

Since $Lu(s)=\dfrac{\mathrm{d}^2}{\mathrm{d}^2 s}u(s)$, if the initial guess is chosen as $u_0(s)=\sinh(s)$ then the next iteration is

$$L[u_1(s)]=\hbar\left[\frac{\partial^2}{\partial^2 s}\sinh(s)-\frac{s^3}{6}\int_0^1\sinh^2(t)\,\mathrm{d}t-\left(\sinh(s)+\frac{s^3}{12}\left(1-\frac{e^4-1}{4e^2}\right)\right)\right]=0$$

Apparently, the next iterations are as follows

$$u_1(s)=u_2(s)=\cdots=u_n(s)=0\,.$$

So that, $U_m(s)$ is

$$U_m(s)=\left[u_0(s)+u_1(s)+u_2(s)+\cdots+u_m(s)+\cdots\right]\Big|_{\hbar=-1}=\sinh s \tag{47}$$

Thus, for the choice of initial guess $u_0(s)=\sinh(s)$ we got exact solution. To get approximate solution let us choose an initial guess as $u_0(s)=s$. In this case, next iterations has the form

$$\begin{aligned} u_1(s)&=-\hbar\left[\sinh(s)+\frac{s^5}{360}+\frac{s^5}{240}\left(1-\frac{e^4-1}{4e^2}\right)-s\right] \\ u_2(s)&=u_1(s)-\hbar^2\left[\sinh(s)+\frac{s^5}{360}+\frac{s^5}{240}\left(1-\frac{e^4-1}{4e^2}\right)\right. \\ &\quad\left.-\frac{s^5}{60}\left(\frac{1}{e}+\frac{1}{1680}\left(1-\frac{e^4-1}{e^2}\right)-\frac{739}{2520}\right)-s\right] \end{aligned} \tag{48}$$

From Eq. (18), one can find two-terms approximate solution at $\hbar=-1$ in the form

$$U_2(s)=\left[u_0(s)+u_1(s)+u_2(s)\right]_{\hbar=-1}=\sinh s+\frac{s^5}{60}\left(\frac{1}{e}-\frac{319}{2520}+\frac{421}{1680}\left(1-\frac{e^4-1}{4e^4}\right)\right),$$

Numerical results of HAM at two iterations with initial guess $u_0(s)=s$ are given in **Table 4**.

Table 4. LDM, and HAM for Example 3 at different values of "$N = 2$".

S	Exact	HAM ($N = 2$)	Error HAM ($N = 2$)
0	0	0	0
0.2	0.201336	0.20134264	0.00000664
0.4	0.410752	0.41096505	0.00021273
0.6	0.636654	0.63826899	0.00161542
0.8	0.888106	0.89491333	0.00680736
1	1.175201	0.19597561	0.02077443

Table 4 demonstrates that by increasing the number of iteration, error term of HAM decreases gradually. Thus, the proposed method is highly accurate and suitable. Manafianheris [22] have used LDM for Eq. (46) and got exact solution by LDM. In Eq. (47), it is shown that HAM can also give exact solution too when initial guess $u_0(s) = \sinh(s)$.

5. Conclusions

In this work, we have developed HAM for nonlinear Fredholm-Volterra integro-differential equations by combining Gauss-Legendre quadrature formulas. Numerical results (Example 2 and Example 4) reveled that HAM gave exact solution for the suitable choice of initial guess. From Example 1, it follows that HAM approaches to the exact solution very fast by increasing number of iterations. In Example 2, we can see that HAM and ADM are highly accurate and approaches to the exact solution. Example 3 shows that standard HAM is better than standard ADM. In Example 4, Manafianheris [22] found an exact solution using the Laplace transformation together with ADM, and we also found an exact solution by choosing $u_0(s) = \sinh(s)$ as an initial guess. For the another initial guess $u_0(s) = s$, we got a very high accurate solution at $m = 2$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Adomian, G. (1988) A Review of the Decomposition Method in Applied Mathematics. *Journal of Mathematical Analysis and Applications*, **135**, 501-544. https://doi.org/10.1016/0022-247X(88)90170-9

[2] Adomian, G. (1994) Solving Frontier Problems of Physics: The Decomposition Me-

thod. Kluwer Academic Publishers, Boston.

[3] Wazwaz, A. (2001) A Reliable Algorithm for Solving Boundary Value Problems for Higher-Order Integro-Differentiable Equations. *Applied Mathematics and Computation*, **118**, 327-342. https://doi.org/10.1016/S0096-3003(99)00225-8

[4] Wazwaz, A. (2002) A Reliable Treatment for Mixed Volterra-Fredholm Integral Equations. *Applied Mathematics and Computation*, **127**, 405-414. https://doi.org/10.1016/S0096-3003(01)00020-0

[5] Liao, S.J. (1992) The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems. PhD Thesis, Shanghai Jiao Tong University, Shanghai.

[6] Liao, S.J. (2004) On the Homotopy Analysis Method for Nonlinear Problems. *Applied Mathematics and Computation*, **147**, 499-513. https://doi.org/10.1016/S0096-3003(02)00790-7

[7] Liao, S.J. (2014) Advances in the Homotopy Analysis Method. World Scientific, Singapore. https://doi.org/10.1142/8939

[8] Bataineh, A.S., Noorani, M.S.M. and Hashim, I. (2009) Modified Homotopy Analysis Method for Solving Systems of Second-Order BVPs. *Communications in Nonlinear Science and Numerical Simulation*, **14**, 430-442. https://doi.org/10.1016/j.cnsns.2007.09.012

[9] Huseen, S.N. (2020) A Comparative Study of q-Homotopy Analysis Method and Liao's Optimal Homotopy Analysis Method. *Advances in Computer and Communication*, **1**, 36-45. https://doi.org/10.26855/acc.2020.12.004

[10] Eshkuvatov, Z.K. (2022) New Development of Homotopy Analysis Method for Non-Linear Integro-Differential Equations with Initial Value Problems. *Mathematical Modeling and Computing*, **9**, 842-859. https://doi.org/10.23939/mmc2022.04.842

[11] He, J. (1999) Homotopy Perturbation Technique. *Computer Methods in Applied Mechanics and Engineering*, **178**, 257-262. https://doi.org/10.1016/S0045-7825(99)00018-3

[12] He, J. (2000) A Coupling Method of a Homotopy Technique and a Perturbation Technique for Non-Linear Problems. *International Journal of Non-Linear Mechanics*, **35**, 37-43. https://doi.org/10.1016/S0020-7462(98)00085-7

[13] Yildirim, A. (2010) He's Homotopy Perturbation Method for Nonlinear Differential-Difference Equations. *International Journal of Computer Mathematics*, **87**, 992-996. https://doi.org/10.1080/00207160802247646

[14] Golbabai, A. and Javidi, M. (2007) Application of He's Homotopy Perturbation Method for nth-Order Integro-Differential Equations. *Applied Mathematics and Computation*, **190**, 1409-1416. https://doi.org/10.1016/j.amc.2007.02.018

[15] Al-Saif, N.S.M. and Ameen, A.S. (2020) Numerical Solution of Mixed Volter-ra-Fredholm Integral Equation Using the Collocation Method. *Baghdad Science Journal*, **17**, 849-853. https://doi.org/10.21123/bsj.2020.17.3.0849

[16] Hamoud, A.A., Hussain, K.H., Mohammed, N.M. and Ghadle, K.P. (2019) Solving Fredholm Integro-Differential Equations by Using Numerical Techniques. *Nonlinear Functional Analysis and Applications*, **24**, 533-542. http://nfaa.kyungnam.ac.kr/journal-nfaa/index.php/NFAA/article/view/1202

[17] Dhari, I.A., Mustafa, N.R. and Mustafa, M.M. (2020) Solving Linear Volter-ra-Fredholm Integral Equation of the Second Type Using Linear Programming Method. *Baghdad Science Journal*, **17**, 342-347. https://doi.org/10.21123/bsj.2020.17.1(Suppl.).0342

[18] Khan, M., Gondal, M.A. and Suleymani, F. (2011) A Novel Two Step Laplasw Decomposition Algorithm for Fredholm Integro-Differential Equation. *World Applied Sciences Journal*, **13**, 2258-2262. https://doi.org/10.1016/j.apm.2011.11.083

[19] Eshkuvatov, Z.K., Kammuji, M. and Muminov, Z. (2018) Approximate Solution for Linear Integro-Differential Equations of Order One by Legendre Polynomials. *Indian Journal of Mathematics*, *Dharma Prakash Gupta Memorial*, **60**, 415-438. https://doi.org/10.1063/1.5041639

[20] Avudainayagam, A. and Van, C. (2000) Wavelet Galerkin Method for Integro-Differential Equations. *Applied Numerical Mathematics*, **32**, 247-254. https://doi.org/10.1016/S0168-9274(99)00026-4

[21] El-Sayed, S.M. and Abdel Aziz, M.R. (2003) A Comparison of Adomian's De-Composition Method and Wavelet Galerkin Method for Solving Integro-Differential Equations. *Applied Mathematics and Computation*, **136**, 151-159. https://doi.org/10.1016/S0096-3003(02)00024-3

[22] Manafianheris, J. (2012) Solving the Integro-Differential Equations Using the Modified Laplace Adomian Decomposition Method. *Journal of Mathematical Extension*, **6**, 1-15.