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Mixed Monotone Iterative Technique for Singular Hadamard Fractional Integro-Differential Equations in Banach Spaces

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Abstract

This paper deals with fractional integro-differential equations involving Hadamard fractional derivatives and nonlinear boundary conditions in an ordered Banach space. The nonlinearity is allowed to be singular with respect to time variable. Under some monotonicity conditions and noncompactness measure conditions, we use the method of coupled lower and upper L-quasisolutions associated with the mixed monotone iterative technique to investigate the existence of extremal L-quasisolutions. A unique solution between coupled lower and upper L-quasisolutions is also obtained. An example is given to illustrate our theoretical results. The results got in this paper are new and enrich the existing related work.

Keywords

Hadamard Fractional Derivative, Nonlinear Boundary Condition, Monotone Iterative Technique, Noncompactness Measure

1. Introduction

In this work, we consider the following boundary value problem (BVP for short) in a Banach space E

$$\begin{cases} {}^{H}\mathscr{D}_{a^+}^{\alpha} x(t) = f\left(t, x(t), x(t), Gx(t)\right), t \in (a,b], 0 < a < b < \infty, \\ g\left(\tilde{x}(a), \tilde{x}(b)\right) = \theta, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, ${}^{H}\mathscr{D}_{a^+}^{\alpha}$ denotes left-sided Hadamard fractional derivative of order α with the low limit *a*. The nonlinear term $f(t,x,y,z)$ is an *E*-value continuous function on $(a,b]\times E\times E\times E$ and may be singular at $t=a$. The operator G is given by $Gx(t)=\int_a^t k(t,s)x(s)\mathrm{d}s$ and $k(t,s)\in C(D,\mathbb{R}^+)$, $D=\{(t,s)\in\mathbb{R}\times\mathbb{R}\mid a\le s\le t\le b\}$, $\mathbb{R}^+=[0,\infty)$. The function $g\in C(E\times E,E)$. $\tilde{x}(a)=\lim_{t\to a^+}\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}x(t)$ and $\tilde{x}(b)=\lim_{t\to b^-}\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}x(t)$, $\log(\cdot)=\log_{\mathrm{e}}(\cdot)$. θ denotes the zero element in *E*. Throughout the article, the integrals of the functions with values in E are taken in Bochner's sense.

Fractional calculus and fractional differential equations have been studied extensively during the last decades. An effective technique for discussing the existence of solutions for initial and boundary value problems of differential equations is the monotone iterative technique combined with the lower and upper solutions method. This method is widely used to investigate Riemann-Liouville and Caputo type fractional differential equations, see, for example, [1]-[11] and the references therein. In [12], the Hadamard fractional calculus was introduced. In the definition of Hadamard fractional derivative, the kernel of the integral contains a logarithmic function of arbitrary exponent which is different from the fractional derivatives of Riemann-Liouville and Caputo type. Some recent contributions to the existence of solutions for Hadamard fractional differential equations via various fixed point theorems can be found in [13]-[18]. For details as regards the application of the iterative method in Hadamard fractional differential equations, see [19] [20] [21] [22] and the references therein.

In [20], Pei *et al.* discussed the existence of positive solutions for Hadamard fractional integro-differential equation

$$\begin{cases} {}^H\mathscr{D}^{\alpha}u(t)+f\left(t,u(t),{}^H\mathscr{I}^{r}u(t),{}^H\mathscr{D}^{\alpha-1}u(t)\right)=0, 1<\alpha<2, t\in(1,\infty), \\ u(1)=0, {}^H\mathscr{D}^{\alpha-1}u(\infty)=\sum_{i=1}^{m}\lambda_i\,{}^H\mathscr{I}^{\beta_i}u(\eta), \end{cases}$$

where $\eta\in(1,\infty)$, $r,\beta_i,\lambda_i>0$ $(i=1,2,\cdots,m)$ are given constants and satisfy $\Gamma(\alpha)>\sum_{i=1}^{m}\frac{\lambda_i\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log\eta)^{\alpha+\beta_i-1}$. The authors not only established the existence of positive solutions but also sought the positive minimal and maximal solutions and got two explicit monotone iterative sequences which converge to the extremal solutions.

In [22], by employing the monotone iterative method, the authors investigated the iterative positive solutions of nonlocal Hadamard fractional boundary value problem with nonlocal Hadamard integral and discrete boundary conditions

$$\begin{cases} {}^H\mathscr{D}^{q}x(t)+\sigma(t)f(t,x(t))=0, 2<q\le 3, t\in(1,\infty), \\ x(1)=x'(1)=0, {}^H\mathscr{D}^{q-1}x(\infty)=a\,{}^H\mathcal{I}^{\beta}x(\xi)+b\sum_{i=1}^{m-2}\alpha_i x(\eta_i), \end{cases}$$

where $\beta>0$, $1<\xi<\eta_1<\eta_2<\cdots<\eta_{m-2}<\infty$, a and b are real constants, and α_i is positive real constants. Some explicit monotone iterative sequences were established for approximating the extreme positive solutions and the unique positive solution.

It is worth pointing out that the iteration sequences in [19] [20] [21] [22] are constructed from the appropriate initial functions rather than from the lower and upper solutions. The literature on the monotone iterative technique and the method of lower and upper solutions for Hadamard fractional differential equations is scarce. In [23], using the method of lower and upper solutions and its associated monotone iterative technique, the author investigated the existence of extremal solutions of the following system of nonlinear Hadamard fractional differential equations with Cauchy initial value conditions

$$\begin{cases} {}^{H}\mathscr{D}_{a^+}^{\alpha} x(t) = f\left(t, x(t), y(t)\right), \mathscr{J}_{a^+}^{1-\alpha} x\left(a^+\right) = x_0^*, t \in (a,b], \\ {}^{H}\mathscr{D}_{a^+}^{\alpha} y(t) = g\left(t, x(t), y(t)\right), \mathscr{J}_{a^+}^{1-\alpha} y\left(a^+\right) = y_0^*, t \in (a,b], \end{cases}$$

where $0 < \alpha \le 1$, *f* and *g* are continuous on $[a,b] \times \mathbb{R} \times \mathbb{R}$. $x_0^*, y_0^* \in \mathbb{R}$, $x_0^* \le y_0^*$. ${}^{H}\mathscr{D}_{a^+}^{\alpha}$ and $\mathscr{J}_{a^+}^{\alpha}$ are the left-sided Hadamard fractional derivative and integral of order α, respectively.

To the best of our knowledge, the existence of solutions for fractional BVP (1.1) in ordered Banach spaces has not been considered up to now. In this work, combining the theory of noncompactness measure with mixed monotone iterative technique and coupled lower and upper *L*-quasisolutions, we prove the existence of extremal *L*-quasisolutions of BVP (1.1). Also, we establish the uniqueness result of solutions between coupled lower and upper *L*-quasisolutions. It is allowable in our main result that the nonlinear term $f(t, x, x, Gx)$ is non-decreasing with respect to one variable *x* and non-increasing with respect to another variable *x*. Moreover, due to the weighted boundary value condition in (1.1), we establish the explicit solution to the weighted Cauchy type problem of linear Hadamard fractional differential equations, which is different from Lemma 2.1 in [23] and Theorem 4.5 in [24]. Consequently, the results got in this paper will enrich the existing related work and also can serve as an interesting complement to the work in [23] [24].

2. Preliminaries

In this section, we introduce some notations and preliminary facts which are used throughout this paper. Let $J = [a,b]$ and E be an ordered Banach space with the norm $\|\cdot\|$ and the partial order "$\le$", whose positive cone $P = \{x \in E, x \ge \theta\}$ is normal with normal constant *N*. Let $C(J,E)$ denotes the ordered Banach space of all continuous *E*-value functions on the interval *J* with the norm $\|x\|_c = \max_{t\in J} \|x(t)\|$ and the partial order "$\le$" deduced by the positive cone $P_c = \{x : x \in C(J,E), x(t) \ge \theta\}$. P_c is also normal with the same normal constant *N*. Let

$$C_{r,\log}(J,E) = \left\{ x : x \in C\left((a,b],E\right), \left(\log\left(\frac{t}{a}\right)\right)^r x(t) \in C(J,E) \right\} \quad \text{for} \quad 0 < r < 1.$$

Evidently, $C_{r,\log}(J,E)$ also is an ordered Banach space with the norm

$\|x\|_{C_{r,\log}} = \left\| \left(\log\left(\frac{t}{a}\right) \right)^r x(t) \right\|_C$ and the partial order "$\le$" deduced by the positive cone $P_{C_{r,\log}} = \left\{ x : x \in C_{r,\log}(J,E), \left(\log\left(\frac{t}{a}\right) \right)^r x(t) \ge \theta \right\}$. The normal constant of $P_{C_{r,\log}}$ also is *N*. It is easy to verify that $C_{r,\log}(J,E) \subset L(J,E)$, where $L(J,E)$ denotes the Banach space of all *E*-value Bochner integrable functions defined on *J* with the norm $\|u\|_L = \int_J \|u(t)\| \mathrm{d}t$.

A function $x \in C_{1-\alpha,\log}(J,E)$ is called a solution of BVP (1.1) if it satisfies the equation and the boundary value condition in (1.1).

Now let us recall some fundamental facts of the notion of Kuratowski non-compactness measure.

Definition 1. ([25]) Let *E* be a Banach space and let Ω_E be the family of bounded subsets of *E*. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0,\infty]$ defined by

$$\mu(B) = \inf\left\{ \varepsilon > 0 : B \subseteq \bigcup_{i=1}^n B_i, diam(B_i) \le \varepsilon \right\}, \text{here } B \in \Omega_E.$$

Property 1. ([25]) The Kuratowski measure of noncompactness satisfies some useful properties.

1) $\mu(B) = 0 \Leftrightarrow \bar{B}$ is compact (*B* is relatively compact), where $\bar{B}$ denotes the closure of *B*.

2) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.

3) $\mu(A + B) \le \mu(A) + \mu(B)$.

4) $\mu(cB) = |c| \mu(B), c \in \mathbb{R}$.

Denote the Kuratowski noncompactness measures of bounded sets in $C_{r,\log}(J,E)$ by $\mu_{C_{r,\log}}$. Similar to the proof of Lemma 2.1 in [26], we can obtain the following useful result.

Lemma 1. Let $H \subset C_{r,\log}(J,E)$ be bounded and equicontinuous. Then

$$\mu_{C_{r,\log}}(H) = \max_{t \in J} \left\{ \mu\left(\left(\log\left(\frac{t}{a}\right) \right)^r H(t) \right) \right\},$$

where $H(t) = \{ x(t) \mid x \in H \}, t \in J$.

The following lemma is necessary in the proof of our main results.

Lemma 2. ([27] [28] [29]) Let *E* be a Banach space, $H = \{x_n\} \subset L(J,E)$ be a countable set with $\|x_n(t)\| \le \rho(t)$ for a.a. $t \in J$ and every $x_n \in H$, where $\rho(t) \in L(J)$. Then $\mu(H(t))$ is Lebesgue integrable on *J*, and $\mu\left(\left\{ \int_J x_n(t) \mathrm{d}t \right\} \right) \le 2 \int_J \mu(H(t)) \mathrm{d}t$.

Next, we review definitions and some useful properties of Hadamard fractional integrals and derivatives which are used in the following sequels.

Definition 2. ([12] [24] [30]) Let $(a,b)(0 < a < b < \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$ and $\alpha > 0$. The left-sided Hadamard fractional integral and fractional derivative of order α are defined respectively by

$$\mathscr{I}_{a^+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}x(s)\frac{\mathrm{d}s}{s}$$

and

$$^{H}\mathscr{D}_{a^+}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{n-\alpha-1}x(s)\frac{\mathrm{d}s}{s},$$

provided that the right-hand sides are pointwise defined on (a,b), where $n-1<\alpha<n$.

Property 2. ([24]) If $\alpha,\beta>0$, then

$$\mathscr{I}_{a^+}^{\alpha}\left(\log\left(\frac{t}{a}\right)\right)^{\beta-1}(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\left(\frac{x}{a}\right)\right)^{\beta+\alpha-1}.$$

Property 3. ([24]) Let $n-1<\alpha<n$, then

1) $^{H}\mathscr{D}_{a^+}^{\alpha}\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-i}=0$, $i=1,2,\cdots,n$.

2) $^{H}\mathscr{D}_{a^+}^{\alpha}\mathscr{I}_{a^+}^{\alpha}x(t)=x(t)$ for $x\in L^p(a,b), 1\le p\le\infty, 0<a<b<\infty$.

3) $\mathscr{I}_{a^+}^{\alpha\,H}\mathscr{D}_{a^+}^{\alpha}x(t)=x(t)+\sum_{i=1}^{n}c_i\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-i}$ for

$^{H}\mathscr{D}_{a^+}^{\alpha}x(t)\in C(a,b)\cap L(a,b)$, where $c_i\in\mathbb{R}, i=1,2,\cdots,n$.

3. Main Results

Definition 3. Let $L\ge 0$ and $v,w\in C_{1-\alpha,\log}(J,E)$. We call v,w coupled lower and upper L-quasisolutions of BVP (1.1) if v and w satisfy

$$\begin{cases}^{H}\mathscr{D}_{a^+}^{\alpha}v(t)\le f\left(t,v(t),w(t),Gv(t)\right)+L\left(v(t)-w(t)\right), t\in(a,b],\\ g\left(\tilde{v}(a),\tilde{v}(b)\right)\le\theta,\end{cases}\tag{3.1}$$

and

$$\begin{cases}^{H}\mathscr{D}_{a^+}^{\alpha}w(t)\ge f\left(t,w(t),v(t),Gw(t)\right)+L\left(w(t)-v(t)\right), t\in(a,b],\\ g\left(\tilde{w}(a),\tilde{w}(b)\right)\ge\theta.\end{cases}\tag{3.2}$$

Remark 1. Only choose "=" in (3.1) and (3.2), we call v and w coupled L-quasisolutions of BVP (1.1). In particular, v and w are coupled quasisolutions of BVP (1.1) for $L=0$. Furthermore, if $v=w:=u$, then u is a solution of BVP (1.1).

In what follows, we assume that v and w are coupled lower and upper L-quasisolutions of BVP (1.1), respectively, and $v\le w$. Define the ordered interval in space $C_{1-\alpha,\log}(J,E)$

$$[v,w]=\left\{x(t)\in C_{1-\alpha,\log}(J,E): v(t)\le x(t)\le w(t), t\in(a,b], \tilde{v}(a)\le\tilde{x}(a)\le\tilde{w}(a)\right\}.$$

Let $\lambda\in\mathbb{R}$ be a constant and $h\in C_{1-\alpha,\log}(J,E)$. Consider the weighted linear initial value problem in E

$$\begin{cases}^{H}\mathscr{D}_{a^+}^{\alpha}x(t)-\lambda x(t)=h(t), t\in(a,b],\\ \tilde{x}(a)=x_a.\end{cases}\tag{3.3}$$

Lemma 3. *For any* $h \in C_{1-\alpha,\log}(J,E)$, $x_a \in E$, *the unique solution of* (3.3) *in the space* $C_{1-\alpha,\log}(J,E)$ *has the following form*

$$\begin{aligned} x(t) = {} & x_a \Gamma(\alpha)\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\right) \\ & + \int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right) h(s)\frac{\mathrm{d}s}{s}, \end{aligned} \tag{3.4}$$

where $E_{\alpha,\alpha}(\cdot)$ is the Mittag-Leffler function defined by

$E_{\alpha,\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+\alpha)}$.

Proof. According to Theorem 3.32 in [24], we assert that (3.3) has a unique solution in the space $C_{1-\alpha,\log}(J,E)$. Applying Theorem 4.5 and similar relations to Lemma 3.2 on Hadamard fractional integrals in [24], we can prove this lemma. In what follows we use the method of successive approximations to solve (3.3). First of all, by Property 3, we derive that (3.3) is equivalent to the following integral equation

$$\begin{aligned} x(t) = {} & x_a\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1} + \frac{\lambda}{\Gamma(\alpha)}\int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} x(s)\frac{\mathrm{d}s}{s} \\ & + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} h(s)\frac{\mathrm{d}s}{s}. \end{aligned} \tag{3.5}$$

Now we set

$$x_0(t) = x_a\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1} \tag{3.6}$$

and

$$\begin{aligned} x_m(t) = {} & x_0(t) + \frac{\lambda}{\Gamma(\alpha)}\int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} x_{m-1}(s)\frac{\mathrm{d}s}{s} \\ & + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} h(s)\frac{\mathrm{d}s}{s}. \end{aligned} \tag{3.7}$$

Using (3.6), (3.7) and taking Property 2 into account we arrive at

$$\begin{aligned} x_1(t) = {} & x_a\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1} + x_a\lambda\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{2\alpha-1} \\ & + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} h(s)\frac{\mathrm{d}s}{s}. \end{aligned} \tag{3.8}$$

Similarly, using (3.6), (3.7), (3.8) and Property 2 we obtain

$$\begin{aligned} x_2(t) = {} & x_a\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1} + x_a\lambda\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{2\alpha-1} + x_a\lambda^2\frac{\Gamma(\alpha)}{\Gamma(3\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{3\alpha-1} \\ & + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} h(s)\frac{\mathrm{d}s}{s} + \frac{\lambda}{\Gamma(2\alpha)}\int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{2\alpha-1} h(s)\frac{\mathrm{d}s}{s}. \end{aligned}$$

Continuing this process, we derive the following relation for $x_m(t)(m\in\mathbb{N})$:

$$x_m(t)=x_a\Gamma(\alpha)\sum_{k=1}^{m+1}\frac{\lambda^{k-1}}{\Gamma(k\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{k\alpha-1}+\int_a^t\left(\sum_{k=1}^{m}\frac{\lambda^{k-1}}{\Gamma(k\alpha)}\left(\log\left(\frac{t}{s}\right)\right)^{k\alpha-1}\right)h(s)\frac{\mathrm{d}s}{s}.$$

Taking the limit as $m\to\infty$, we obtain the explicit solution $x(t)$ to the integral Equation (3.5):

$$x(t)=x_a\Gamma(\alpha)\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{\Gamma(k\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{k\alpha-1}+\int_a^t\left(\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{\Gamma(k\alpha)}\left(\log\left(\frac{t}{s}\right)\right)^{k\alpha-1}\right)h(s)\frac{\mathrm{d}s}{s},$$

replacing the index of summation k by $k-1$ we have

$$x(t)=x_a\Gamma(\alpha)\sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(k\alpha+\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{k\alpha+\alpha-1}$$
$$+\int_a^t\left(\sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(k\alpha+\alpha)}\left(\log\left(\frac{t}{s}\right)\right)^{k\alpha+\alpha-1}\right)h(s)\frac{\mathrm{d}s}{s},$$

and thus, using the expression of the Mittag-Leffler function, we get the explicit solution (3.4) to the problem (3.3).

Remark 2. ([31] [32] [33]) For $\alpha,\beta>0$, the well-known two-parameter Mittag-Leffler function $E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+\beta)}$, $x\in\mathbb{R}$ is continuous on $\mathbb{R}$. Moreover, Mittag-Leffler function has the following useful properties:

1) For $0<\alpha\le 1$, $E_{\alpha,\alpha}(x)>0$, $x\in\mathbb{R}$ and $E_{\alpha,\alpha}(x)\le\frac{1}{\Gamma(\alpha)}$, $x\le 0$.

2) For $0<\alpha\le 1$ and $\lambda\in\mathbb{R},\lambda\ne 0$,

$$E_{\alpha,2\alpha}(\lambda t^\alpha)=\lambda^{-1}t^{-\alpha}\left(E_{\alpha,\alpha}(\lambda t^\alpha)-\frac{1}{\Gamma(\alpha)}\right),t\ne 0.$$

3) For all $t>0$, $E_{\alpha,\alpha}(\lambda t^\alpha)$ is decreasing in t for $\lambda<0$ and increasing in t for $\lambda>0$.

4) For $0<\alpha\le 1$, $\beta,\gamma>0$ and $\lambda\in\mathbb{R}$,

$$\int_a^t(t-s)^{\gamma-1}(s-a)^{\beta-1}E_{\alpha,\beta}\left(\lambda(s-a)^\alpha\right)\mathrm{d}s$$
$$=\Gamma(\gamma)(t-a)^{\beta+\gamma-1}E_{\alpha,\beta+\gamma}\left(\lambda(t-a)^\alpha\right),$$

Equivalently,

$$\int_a^t(t-s)^{\beta-1}(s-a)^{\gamma-1}E_{\alpha,\beta}\left(\lambda(t-s)^\alpha\right)\mathrm{d}s$$
$$=\Gamma(\gamma)(t-a)^{\beta+\gamma-1}E_{\alpha,\beta+\gamma}\left(\lambda(t-a)^\alpha\right).$$

Remark 3. By Remark 2 and (3.4), if $x_a\ge\theta$, $h(t)\ge\theta$, the solution of (3.3) $x(t)\ge\theta$. This comparison result will play a very important role in this paper.

Further, for any $\eta_1,\eta_2\in[v,w]$, let $\sigma_{(\eta_1,\eta_2)}(t)=f(t,\eta_1(t),\eta_2(t),G\eta_1(t))+M\eta_1(t)+L(\eta_1(t))-\eta_2(t)$ and consider the weighted linear initial value problem

$$\begin{cases} {}^{H}\mathscr{D}_{a^+}^{\alpha} x(t) + Mx(t) = \sigma_{(\eta_1,\eta_2)}(t),\ t \in (a,b], \\ \tilde{x}(a) = r_{\eta_1} := \tilde{\eta}_1(a) - \dfrac{g(\tilde{\eta}_1(a), \tilde{\eta}_1(b))}{L_1}. \end{cases} \tag{3.9}$$

Lemma 3 indicates that (3.9) has exactly one solution $x(t)$ given by

$$\begin{aligned} x(t) &= r_{\eta_1}\Gamma(\alpha)\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\right) \\ &\quad + \int_a^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)\sigma_{(\eta_1,\eta_2)}(s)\frac{\mathrm{d}s}{s}. \end{aligned} \tag{3.10}$$

For any $\eta_1, \eta_2 \in [v, w]$, define the operator T as

$$T(\eta_1, \eta_2)(t) = x(t).$$

Then, obviously, the coupled fixed points of operator T are exactly the coupled L-quasisolutions of (1.1) and the fixed points of T are the solutions of (1.1).

We work with the following conditions on the functions f and g in (1.1).

(H0) $f(t,x,y,z) \in C_{1-\alpha,\log}(J,E)$ for any $(x,y,z) \in E \times E \times E$.

(H1) There exist constants $M > 0, L \ge 0$ such that

$$f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1) \ge -M(x_2 - x_1) - L(y_1 - y_2),\ t \in (a,b],$$

where $v \le x_1 \le x_2 \le w$, $v \le y_2 \le y_1 \le w$ and $Gv \le z_1 \le z_2 \le Gw$.

(H2)There exist constants $L_1 > 0, L_2 \ge 0$ such that

$$g(x_2, y_2) - g(x_1, y_1) \le L_1(x_2 - x_1) - L_2(y_2 - y_1),$$

where $\tilde{v}(a) \le x_1 \le x_2 \le \tilde{w}(a)$, $\tilde{v}(b) \le y_1 \le y_2 \le \tilde{w}(b)$.

(H3)There exist constants $N_1 \ge -L_1, N_2 \ge L_2 \ge 0$ such that

$$g(x_2, y_2) - g(x_1, y_1) \ge -N_1(x_2 - x_1) - N_2(y_2 - y_1),$$

where $\tilde{v}(a) \le x_1 \le x_2 \le \tilde{w}(a)$, $\tilde{v}(b) \le y_1 \le y_2 \le \tilde{w}(b)$.

(H4)There exists a constant $K \ge 0$ such that

$$\begin{aligned} &\mu\left(\left\{f(t,x_n,y_n,Gx_n) + Mx_n(t) + L(x_n(t) - y_n(t))\right\}\right) \\ &\le K\left(\mu(\{x_n(t)\}) + \mu(\{y_n(t)\}) + \mu(\{Gx_n(t)\})\right),\ t \in (a,b], \end{aligned}$$

for any mixed monotone sequence $\{(x_n, y_n)\} \subset [v,w] \times [v,w]$. Moreover,

$$N\frac{L_1 + N_1 + N_2}{L_1} + \frac{4KA\Gamma(\alpha)}{\Gamma(2\alpha)}\left(1 + \frac{\bar{K}b}{2\alpha}\right) < 1,$$

where $\bar{K} = \max_{(t,s)\in D}\{k(t,s)\}$, $A = \max\left\{\left(\log\dfrac{b}{a}\right)^{\alpha}, \left(\log\dfrac{b}{a}\right)^{\alpha+1}\right\}$.

(H5)There exist constants $M_1, M_2, M_3 \ge 0$ such that for $t \in (a,b]$,

$$f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1) \le M_1(x_2 - x_1) + M_2(y_1 - y_2) + M_3(z_2 - z_1),$$

where $v \le x_1 \le x_2 \le w$, $v \le y_2 \le y_1 \le w$ and $Gv \le z_1 \le z_2 \le Gw$. Moreover,

$$N\left[\frac{L_1+N_1+N_2}{L_1}+\frac{A\Gamma(\alpha)}{\Gamma(2\alpha)}\left(M_1+M_2+M+2L+\frac{M_3\bar{K}b}{2\alpha}\right)\right]<1.$$

Now we are in the position to state our main results.

Theorem 1. Let E be an ordered Banach space, whose positive cone P is normal. Assume that $f:(a,b]\times E\times E\times E\to E$ is continuous, $g\in C(E\times E,E)$, v,w are coupled lower and upper L-quasisolutions of BVP (1.1), respectively. The conditions (H0) - (H4) are valid. Then BVP (1.1) has coupled minimal and maximal L-quasisolutions $v^*,w^*\in[v,w]$ with $v^*\le w^*$. Moreover, there exist monotone iterative sequences $\{v_n\},\{w_n\}\subset C_{1-\alpha,\log}(J,E)$ starting from v and w which converge to the coupled minimal and maximal L-quasisolutions v^* and w^* respectively.

Proof. For clarity, we divide the proof into the following several steps.

Step 1: First of all, we need to show that the operator $T:[v,w]\times[v,w]\to C_{1-\alpha,\log}(J,E)$ is well defined. Indeed, for any $\eta_1,\eta_2\in[v,w]$, by the condition (H1), we have for $t\in(a,b]$

$$f(t,v,w,Gv)+Mv+L(v-w)\le\sigma_{(\eta_1,\eta_2)}(t)\le f(t,w,v,Gw)+Mw+L(w-v).$$

By the normality of cone P and (H0), there exists $\bar{L}>0$ such that $\left\|\sigma_{(\eta_1,\eta_2)}(t)\right\|_{C_{1-\alpha,\log}}\le\bar{L}$, that is,

$$\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\left\|\sigma_{(\eta_1,\eta_2)}(t)\right\|\le\bar{L},\ t\in J. \tag{3.11}$$

Combining (3.10), (3.11), Property 2 and Remark 2, for any $t\in J$ we have

$$\begin{aligned}&\left\|\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}T(\eta_1,\eta_2)(t)\right\|\\&\le\left\|r_{\eta_1}\right\|+\frac{\bar{L}}{\Gamma(\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1}\frac{\mathrm{d}s}{s}\\&=\left\|r_{\eta_1}\right\|+\frac{\bar{L}}{\Gamma(\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{2\alpha-1}\\&=\left\|r_{\eta_1}\right\|+\bar{L}\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{\alpha},\end{aligned}$$

this implies that the integral in (3.10) exists and belongs to $C_{1-\alpha,\log}(J,E)$.

Step 2: We show that the operator T is equicontinuous. Let $\eta_1,\eta_2\in[v,w]$ and $t_1,t_2\in[a,b]$ with $t_1<t_2$. Evidently,

$$\left\|\left(\log\frac{t_2}{a}\right)^{1-\alpha}T(\eta_1,\eta_2)(t_2)-\left(\log\frac{t_1}{a}\right)^{1-\alpha}T(\eta_1,\eta_2)(t_1)\right\|\to0$$

if $t_1=a$ and $|t_1-t_2|\to0$. In the following we set $t_1>a$. In view of the condition (H2), for any $\eta\in[v,w]$, one has

$$\tilde{\eta}(a)-\frac{g(\tilde{\eta}(a),\tilde{\eta}(b))}{L_1}\le \tilde{w}(a)-\frac{g(\tilde{w}(a),\tilde{w}(b))}{L_1}-\frac{L_2}{L_1}(\tilde{w}(b)-\tilde{\eta}(b))$$
$$\le \tilde{w}(a)-\frac{g(\tilde{w}(a),\tilde{w}(b))}{L_1}$$

and

$$\tilde{\eta}(a)-\frac{g(\tilde{\eta}(a),\tilde{\eta}(b))}{L_1}\ge \tilde{v}(a)-\frac{g(\tilde{v}(a),\tilde{v}(b))}{L_1}+\frac{L_2}{L_1}(\tilde{\eta}(b)-\tilde{v}(b))$$
$$\ge \tilde{v}(a)-\frac{g(\tilde{v}(a),\tilde{v}(b))}{L_1}.$$

By the normality of the cone *P*, there exists a constant $\bar{M}>0$ such that

$$\|r_\eta\|=\left\|\tilde{\eta}(a)-\frac{g(\tilde{\eta}(a),\tilde{\eta}(b))}{L_1}\right\|\le \bar{M}.$$

Thus we get

$$\begin{aligned}
&\left\|\left(\log\frac{t_2}{a}\right)^{1-\alpha}T(\eta_1,\eta_2)(t_2)-\left(\log\frac{t_1}{a}\right)^{1-\alpha}T(\eta_1,\eta_2)(t_1)\right\|\\
&\le \bar{M}\Gamma(\alpha)\left|E_{\alpha,\alpha}\left(-M\left(\log\frac{t_2}{a}\right)^{\alpha}\right)-E_{\alpha,\alpha}\left(-M\left(\log\frac{t_1}{a}\right)^{\alpha}\right)\right|\\
&\quad+\left\|\int_a^{t_1}\left[\left(\log\frac{t_2}{a}\right)^{1-\alpha}\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha}\right)\right.\right.\\
&\quad\left.\left.-\left(\log\frac{t_1}{a}\right)^{1-\alpha}\left(\log\left(\frac{t_1}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t_1}{s}\right)\right)^{\alpha}\right)\sigma_{(\eta_1,\eta_2)}(s)\frac{\mathrm{d}s}{s}\right\|\right.\\
&\quad+\left\|\int_{t_1}^{t_2}\left(\log\frac{t_2}{a}\right)^{1-\alpha}\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha}\right)\sigma_{(\eta_1,\eta_2)}(s)\frac{\mathrm{d}s}{s}\right\|\\
&:=I_1+I_2+I_3.
\end{aligned}\tag{3.12}$$

Note that $E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\right)$ is continuous. The expression I_1 has limit zero as $|t_2-t_1|\to 0$. Next we estimate the integral I_3. By (3.11) and Remark 2, we obtain

$$\begin{aligned}
I_3&\le\frac{\bar{L}}{\Gamma(\alpha)}\left(\log\frac{t_2}{a}\right)^{1-\alpha}\int_{t_1}^{t_2}\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1}\frac{\mathrm{d}s}{s}\\
&\le\frac{\bar{L}}{\Gamma(\alpha)}\left(\log\frac{t_2}{a}\right)^{1-\alpha}\left(\log\frac{t_1}{a}\right)^{\alpha-1}\int_{t_1}^{t_2}\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha-1}\frac{\mathrm{d}s}{s}\\
&=\frac{\bar{L}}{\Gamma(\alpha+1)}\left(\log\frac{t_2}{a}\right)^{1-\alpha}\left(\log\frac{t_1}{a}\right)^{\alpha-1}(\log t_2-\log t_1)^{\alpha}.
\end{aligned}\tag{3.13}$$

Hence, the expression I_3 has limit zero as $|t_2-t_1|\to 0$. Observing Remark 2 and nonincreasing property of the function $l(t)=\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}$

for $a \le s < t \le b$, for the rest term I_2, we can deduce

$$\begin{aligned}
I_2 \le \int_a^{t_1} \Bigg[& \left(\log\frac{t_1}{a}\right)^{1-\alpha}\left(\log\left(\frac{t_1}{s}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t_1}{s}\right)\right)^{\alpha}\right) \\
& -\left(\log\frac{t_2}{a}\right)^{1-\alpha}\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha}\right)\Bigg]\left\|\sigma_{(\eta_1,\eta_2)}(s)\right\|\frac{\mathrm{d}s}{s} \\
\le & \bar{L}\int_a^{t_1}\left(\log\frac{t_1}{a}\right)^{1-\alpha}\left(\log\left(\frac{t_1}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t_1}{s}\right)\right)^{\alpha}\right)\frac{\mathrm{d}s}{s} \\
& -\bar{L}\int_a^{t_2}\left(\log\frac{t_2}{a}\right)^{1-\alpha}\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha}\right)\frac{\mathrm{d}s}{s} \\
& +\bar{L}\int_{t_1}^{t_2}\left(\log\frac{t_2}{a}\right)^{1-\alpha}\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t_2}{s}\right)\right)^{\alpha}\right)\frac{\mathrm{d}s}{s} \\
:= & I_{21}+I_{22}+I_{23}.
\end{aligned} \tag{3.14}$$

The relation (3.13) ensures that I_{23} has limit zero as $|t_2-t_1|\to 0$. Using (2) and (4) in Remark 2 and by simple calculations, we know that

$$\begin{aligned}
& \int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)\frac{\mathrm{d}s}{s} \\
& =\Gamma(\alpha)\left(\log\left(\frac{t}{a}\right)\right)^{2\alpha-1} E_{\alpha,2\alpha}\left(-M\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\right) \\
& =\frac{-\Gamma(\alpha)}{M}\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1}\left[E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\right)-\frac{1}{\Gamma(\alpha)}\right],
\end{aligned}$$

so we have

$$\begin{aligned}
I_{21}+I_{22} & =\frac{-\Gamma(\alpha)\bar{L}}{M}\left[E_{\alpha,\alpha}\left(-M\left(\log\frac{t_1}{a}\right)^{\alpha}\right)-\frac{1}{\Gamma(\alpha)}\right] \\
& \quad -\frac{-\Gamma(\alpha)\bar{L}}{M}\left[E_{\alpha,\alpha}\left(-M\left(\log\frac{t_2}{a}\right)^{\alpha}\right)-\frac{1}{\Gamma(\alpha)}\right] \\
& =\frac{-\Gamma(\alpha)\bar{L}}{M}\left[E_{\alpha,\alpha}\left(-M\left(\log\frac{t_1}{a}\right)^{\alpha}\right)-E_{\alpha,\alpha}\left(-M\left(\log\frac{t_2}{a}\right)^{\alpha}\right)\right].
\end{aligned} \tag{3.15}$$

Consequently, the relations (3.12), (3.13), (3.14) and (3.15) guarantee that the operator T is equicontinuous.

Step 3: In this part we show that $T:[v,w]\times[v,w]\to[v,w]$ is a continuous mixed monotone operator.

Firstly, from (3.10), (3.11), the continuity of f and g together with the Lebesgue dominated convergence theorem, it is easy to know T is continuous.

Secondly, we prove T is a mixed monotone operator, that is, $T(\eta_1,\eta_2)\le T(\xi_1,\xi_2)$ for $\eta_1,\eta_2,\xi_1,\xi_2\in[v,w]$ with $\eta_1\le\xi_1,\xi_2\le\eta_2$. Since by

(H2)

$$\left(\tilde{\xi}_1(a)-\frac{g\left(\tilde{\xi}_1(a),\tilde{\xi}_1(b)\right)}{L_1}\right)-\left(\tilde{\eta}_1(a)-\frac{g\left(\tilde{\eta}_1(a),\tilde{\eta}_1(b)\right)}{L_1}\right)$$
$$\geq\frac{L_2}{L_1}\left(\tilde{\xi}_1(b)-\tilde{\eta}_1(b)\right)\geq\theta,$$

we have

$$r_{\eta_1}\leq r_{\xi_1}. \tag{3.16}$$

Moreover, by (H1) we obtain

$$\left[f\left(t,\xi_1,\xi_2,G\xi_1\right)+M\xi_1+L\left(\xi_1-\xi_2\right)\right]-\left[f\left(t,\eta_1,\eta_2,G\eta_1\right)+M\eta_1+L\left(\eta_1-\eta_2\right)\right]$$
$$\geq L\left(\xi_1-\eta_1\right)\geq\theta,$$

that is,

$$\sigma_{(\eta_1,\eta_2)}(t)\leq\sigma_{(\xi_1,\xi_2)}(t). \tag{3.17}$$

As a result, (3.10), (3.16) and (3.17) ensure $T\left(\eta_1,\eta_2\right)\leq T\left(\xi_1,\xi_2\right)$.

Finally, we prove $T\left(\eta_1,\eta_2\right)\in[v,w]$ for any $\eta_1,\eta_2\in[v,w]$. Since T is a mixed monotone operator, $T(v,w)\leq T\left(\eta_1,\eta_2\right)\leq T(w,v)$ for any $\eta_1,\eta_2\in[v,w]$, it is sufficient to prove $v\leq T(v,w)$ and $T(w,v)\leq w$. Let $p(t)={}^{H}\mathscr{D}_{a^+}^{\alpha}v(t)+Mv(t)$, then by (3.1)

$$p(t)\leq f(t,v,w,Gv)+Mv+L(v-w)=\sigma_{(v,w)}(t),$$

and by Lemma 3,

$$v(t)=\tilde{v}(a)\Gamma(\alpha)\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\right)$$
$$+\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)p(s)\frac{\mathrm{d}s}{s}.$$

Thus, for any $t\in J$, one gets

$$\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}v(t)$$
$$\leq\left(\tilde{v}(a)-\frac{g\left(\tilde{v}(a),\tilde{v}(b)\right)}{L_1}\right)\Gamma(\alpha)E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\right)$$
$$+\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)\sigma_{(v,w)}(s)\frac{\mathrm{d}s}{s}$$
$$=\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}T(v,w)(t),$$

that is $v\leq T(v,w)$. Similarly, we can derive $T(w,v)\leq w$.

Step 4: Now, we can define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v,w]$ by the iterative scheme $v_n=T\left(v_{n-1},w_{n-1}\right)$ and $w_n=T\left(w_{n-1},v_{n-1}\right)$, where $v_0=v$ and $w_0=w$. Then from the mixed monotonicity of T, it follows that

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{3.18}$$

Next we verify $\{v_n\}$ and $\{w_n\}$ are convergent in *J*. For convenience, let $V = \{v_n : n = 1, 2, \cdots\}$ and $W = \{w_n : n = 1, 2, \cdots\}$. Note that $\mu_{c_{1-\alpha,\log}}(V) = \mu_{c_{1-\alpha,\log}}(V \cup \{v_0\})$ and $\mu_{c_{1-\alpha,\log}}(W) = \mu_{c_{1-\alpha,\log}}(W \cup \{w_0\})$ by Property 1. Observing that for any $\eta_1, \eta_2 \in [v, w]$ with $\eta_1 \le \eta_2$, by (H3),

$$\begin{aligned}&\left(\tilde{\eta}_2(a) - \frac{g(\tilde{\eta}_2(a), \tilde{\eta}_2(b))}{L_1}\right) - \left(\tilde{\eta}_1(a) - \frac{g(\tilde{\eta}_1(a), \tilde{\eta}_1(b))}{L_1}\right)\\ &\le \left(1 + \frac{N_1}{L_1}\right)(\tilde{\eta}_2(a) - \tilde{\eta}_1(a)) + \frac{N_2}{L_1}(\tilde{\eta}_2(b) - \tilde{\eta}_1(b)),\end{aligned}$$

together with (3.16) and the normality of the cone *P*, we can find

$$\begin{aligned}&\left\|\left(\tilde{\eta}_2(a) - \frac{g(\tilde{\eta}_2(a), \tilde{\eta}_2(b))}{L_1}\right) - \left(\tilde{\eta}_1(a) - \frac{g(\tilde{\eta}_1(a), \tilde{\eta}_1(b))}{L_1}\right)\right\|\\ &\le N\left(1 + \frac{N_1}{L_1}\right)\|\tilde{\eta}_2(a) - \tilde{\eta}_1(a)\| + N\frac{N_2}{L_1}\|\tilde{\eta}_2(b) - \tilde{\eta}_1(b)\|,\end{aligned}$$

hence, for any equicontinuous sequence $\{\eta_n\} \subset [v, w]$, we arrive at by Lemma 1

$$\begin{aligned}\mu\left(\left\{\tilde{\eta}_n(a) - \frac{g(\tilde{\eta}_n(a), \tilde{\eta}_n(b))}{L_1}\right\}\right) &\le N\left(1 + \frac{N_1}{L_1}\right)\mu(\{\tilde{\eta}_n(a)\}) + N\frac{N_2}{L_1}\mu(\{\tilde{\eta}_n(b)\})\\ &\le N\frac{L_1 + N_1 + N_2}{L_1}\mu_{c_{1-\alpha,\log}}(\{\eta_n\}),\end{aligned}$$

thus, in view of Lemma 1, Lemma 2, Property 1, Property 2, condition (H4) and (3.10), it follows that

$$\begin{aligned}&\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\mu(V(t))\\ &\le \mu\left(\left\{\tilde{v}_{n-1}(a) - \frac{g(\tilde{v}_{n-1}(a), \tilde{v}_{n-1}(b))}{L_1}\right\}\right)\\ &\quad + 2K\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)\\ &\quad \cdot\left[\mu(\{v_{n-1}(s)\}) + \mu(\{w_{n-1}(s)\}) + \mu(\{Gv_{n-1}(s)\})\right]\frac{\mathrm{d}s}{s}\\ &\le N\frac{L_1 + N_1 + N_2}{L_1}\mu_{c_{1-\alpha,\log}}(\{v_{n-1}\})\\ &\quad + \frac{2K}{\Gamma(\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1}\frac{\mathrm{d}s}{s}\cdot\mu_{c_{1-\alpha,\log}}(V)\\ &\quad + \frac{2K}{\Gamma(\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1}\frac{\mathrm{d}s}{s}\cdot\mu_{c_{1-\alpha,\log}}(W)\\ &\quad + \frac{4K\bar{K}}{\Gamma(\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}\left[\int_a^s\left(\log\left(\frac{\tau}{a}\right)\right)^{\alpha-1}\mathrm{d}\tau\right]\frac{\mathrm{d}s}{s}\cdot\mu_{c_{1-\alpha,\log}}(V)\end{aligned}$$

$$
\begin{aligned}
&\le N\frac{L_1+N_1+N_2}{L_1}\mu_{c_{1-\alpha,\log}}(V)+\frac{2K\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\mu_{c_{1-\alpha,\log}}(V)\\
&\quad+\frac{2K\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\mu_{c_{1-\alpha,\log}}(W)\\
&\quad+\frac{4K\bar{K}b}{\Gamma(\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}\left[\int_a^s\left(\log\left(\frac{\tau}{a}\right)\right)^{\alpha-1}\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s}\cdot\mu_{c_{1-\alpha,\log}}(V)\\
&= N\frac{L_1+N_1+N_2}{L_1}\mu_{c_{1-\alpha,\log}}(V)+\frac{2K\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\log\left(\frac{t}{a}\right)\right)^{\alpha}\\
&\quad\cdot\left[\mu_{c_{1-\alpha,\log}}(V)+\mu_{c_{1-\alpha,\log}}(W)\right]+\frac{4K\bar{K}b\Gamma(\alpha)}{\Gamma(2\alpha+1)}\left(\log\left(\frac{t}{a}\right)\right)^{\alpha+1}\mu_{c_{1-\alpha,\log}}(V)\\
&\le\left[N\frac{L_1+N_1+N_2}{L_1}+\frac{4KA\Gamma(\alpha)}{\Gamma(2\alpha)}\left(1+\frac{\bar{K}b}{2\alpha}\right)\right]\cdot\max\left\{\mu_{c_{1-\alpha,\log}}(V),\mu_{c_{1-\alpha,\log}}(W)\right\}.
\end{aligned}
\tag{3.19}
$$

Similarly,

$$
\begin{aligned}
&\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\mu\left(W(t)\right)\\
&\le\left[N\frac{L_1+N_1+N_2}{L_1}+\frac{4KA\Gamma(\alpha)}{\Gamma(2\alpha)}\left(1+\frac{\bar{K}b}{2\alpha}\right)\right]\cdot\max\left\{\mu_{c_{1-\alpha,\log}}(V),\mu_{c_{1-\alpha,\log}}(W)\right\}.
\end{aligned}
\tag{3.20}
$$

Combining (H4), (3.19), (3.20) and Lemma 1 we have $\mu_{c_{1-\alpha,\log}}(V)=\mu_{c_{1-\alpha,\log}}(W)=0$. Then Property 1 guarantees that $\{v_n\}$ and $\{w_n\}$ are relatively compact sets of $C_{1-\alpha,\log}(J,E)$, and thus there exist subsequences converging to $v^*,w^*\in C_{1-\alpha,\log}(J,E)$. Furthermore, from monotonicity (3.18) we easily obtain that $\{v_n\}$ and $\{w_n\}$ are convergent in $C_{1-\alpha,\log}(J,E)$, and the limits v^*,w^* satisfy

$$v=v_0\le v_1\le\cdots\le v_n\le v^*\le w^*\le w_n\le\cdots\le w_1\le w_0=w.$$

Moreover, $v^*=T(v^*,w^*), w^*=T(w^*,v^*)$. By the monotonicity of T, it is easy to deduce that v^* and w^* are the minimal and maximal coupled fixed points of T in $[v,w]$. Therefore, v^* and w^* are the minimal and maximal coupled L-quasisolutions of the problem (1.1) in $[v,w]$, respectively. We complete the proof of this theorem.

If E is weakly sequentially complete, Theorem 2.2 in [34] asserts that any monotonic and order-bounded sequence in E is precompact. In this situation, the conditions (H3) and (H4) which ensure convergence of the monotonic and order-bounded sequences $\{v_n\}$ and $\{w_n\}$ in Theorem 1 are superfluous. As a result, we obtain the following corollary.

Corollary 1. Let E be an ordered and weakly sequentially complete Banach space, whose positive cone P is normal. Assume that $f:(a,b]\times E\times E\times E\to E$ is continuous, $g\in C(E\times E,E)$, v,w are coupled lower and upper L-quasi-solutions of BVP (1.1), respectively. The conditions (H0), (H1) and (H2) are valid. Then BVP (1.1) has coupled minimal and maximal L-quasisolutions $v^*,w^*\in[v,w]$ with $v^*\le w^*$. Moreover, there exist monotone iterative sequences $\{v_n\},\{w_n\}\subset C_{1-\alpha,\log}(J,E)$ starting from v and w which converge to the coupled minimal and maximal L-quasisolutions v^* and w^* respectively.

Obviously, if the cone P is regular, then monotonic and order-bounded sequences $\{v_n\}$ and $\{w_n\}$ obtained in Theorem 1 are convergent. Consequently, we have the following corollary from Theorem 1.

Corollary 2. Let E be an ordered Banach space, whose positive cone P is regular. Assume that $f:(a,b]\times E\times E\times E\to E$ is continuous, $g\in C(E\times E,E)$, v,w are coupled lower and upper L-quasisolutions of BVP (1.1), respectively. The conditions (H0), (H1) and (H2) are valid. Then BVP (1.1) has coupled minimal and maximal L-quasisolutions $v^*,w^*\in[v,w]$ with $v^*\le w^*$. Moreover, there exist monotone iterative sequences $\{v_n\},\{w_n\}\subset C_{1-\alpha,\log}(J,E)$ starting from v and w which converge to the coupled minimal and maximal L-quasisolutions v^* and w^* respectively.

Now, we discuss the existence of solutions to the problem (1.1) in $[v,w]$.

Theorem 2. Let E be an ordered Banach space, whose positive cone P is normal. Assume that $f:(a,b]\times E\times E\times E\to E$ is continuous, $g\in C(E\times E,E)$, v,w are coupled lower and upper L-quasisolutions of BVP (1.1), respectively. The conditions (H0) - (H5) are valid. Then BVP (1.1) has a unique solution u in $[v,w]$, which can be obtained from monotone iterative sequences $\{v_n\},\{w_n\}\subset C_{1-\alpha,\log}(J,E)$ starting from v and w, respectively.

Proof. From the proof of Theorem 1, we know that the iterative sequences $\{v_n\}$ and $\{w_n\}$ starting from v and w are convergent and satisfy (3.18). Next, we verify that there exists a unique $u\in[v,w]$ such that $u=T(u,u)$. For any $t\in(a,b]$, by (H3) and (H5), we obtain

$$\begin{aligned}
&\theta\le w_n(t)-v_n(t)=T(w_{n-1},v_{n-1})(t)-T(v_{n-1},w_{n-1})(t)\\
&\le\left[\frac{L_1+N_1}{L_1}\left(\tilde{w}_{n-1}(a)-\tilde{v}_{n-1}(a)\right)+\frac{N_2}{L_1}\left(\tilde{w}_{n-1}(b)-\tilde{v}_{n-1}(b)\right)\right]\left(\log\left(\frac{t}{a}\right)\right)^{\alpha-1}\\
&\quad+\left(M_1+M_2+M+2L\right)\\
&\quad\cdot\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)\left(w_{n-1}(s)-v_{n-1}(s)\right)\frac{\mathrm{d}s}{s}\\
&\quad+M_3\bar{K}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)\left[\int_a^s\left(w_{n-1}(\tau)-v_{n-1}(\tau)\right)\mathrm{d}\tau\right]\frac{\mathrm{d}s}{s}.
\end{aligned}$$

From the normality of the cone P, it follows that

$$\begin{aligned}
&\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}\left\|w_n(t)-v_n(t)\right\|\\
&\le N\frac{L_1+N_1+N_2}{L_1}\left\|w_{n-1}-v_{n-1}\right\|_{C_{1-\alpha,\log}}+\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}N\left(M_1+M_2+M+2L\right)\\
&\quad\cdot\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}\left(\log\left(\frac{s}{a}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)\frac{\mathrm{d}s}{s}\cdot\left\|w_{n-1}-v_{n-1}\right\|_{C_{1-\alpha,\log}}\\
&\quad+\left(\log\left(\frac{t}{a}\right)\right)^{1-\alpha}NM_3\bar{K}\int_a^t\left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1}E_{\alpha,\alpha}\left(-M\left(\log\left(\frac{t}{s}\right)\right)^{\alpha}\right)\\
&\quad\cdot\left[\int_a^s\left(\log\left(\frac{\tau}{a}\right)\right)^{\alpha-1}\mathrm{d}\tau\right]\frac{\mathrm{d}s}{s}\left\|w_{n-1}-v_{n-1}\right\|_{C_{1-\alpha,\log}}
\end{aligned}$$

$$\leq N\left[\frac{L_1+N_1+N_2}{L_1}+\frac{A\Gamma(\alpha)}{\Gamma(2\alpha)}(M_1+M_2+M+2L)+\frac{A\Gamma(\alpha)}{\Gamma(2\alpha+1)}M_3b\bar{K}\right]$$
$$\cdot\|w_{n-1}-v_{n-1}\|_{C_{1-\alpha,\log}}$$
$$=N\left[\frac{L_1+N_1+N_2}{L_1}+\frac{A\Gamma(\alpha)}{\Gamma(2\alpha)}\left(M_1+M_2+M+2L+\frac{M_3\bar{K}b}{2\alpha}\right)\right]\cdot\|w_{n-1}-v_{n-1}\|_{C_{1-\alpha,\log}}.$$

Therefore,

$$\|w_n-v_n\|_{C_{1-\alpha,\log}}\leq N\left[\frac{L_1+N_1+N_2}{L_1}+\frac{A\Gamma(\alpha)}{\Gamma(2\alpha)}\left(M_1+M_2+M+2L+\frac{M_3\bar{K}b}{2\alpha}\right)\right]$$
$$\cdot\|w_{n-1}-v_{n-1}\|_{C_{1-\alpha,\log}}.$$

Again using the above inequality, we get

$$\|w_n-v_n\|_{C_{1-\alpha,\log}}$$
$$\leq\left\{N\left[\frac{L_1+N_1+N_2}{L_1}+\frac{A\Gamma(\alpha)}{\Gamma(2\alpha)}\left(M_1+M_2+M+2L+\frac{M_3\bar{K}b}{2\alpha}\right)\right]\right\}^n\|w-v\|_{C_{1-\alpha,\log}}.$$

which implies $\|w_n-v_n\|_{C_{1-\alpha,\log}}\to 0$ as $n\to\infty$. Then there exists $u\in[v,w]$ such that $\lim_{n\to\infty}w_n=\lim_{n\to\infty}v_n=u$. So let $n\to\infty$ in $w_n=T(w_{n-1},v_{n-1})$, we have $u=T(u,u)$, which means that u is a unique solution of problem (1.1) in $[v,w]$. This completes the proof of Theorem 2.

If E is weakly sequentially complete, then the condition (H4) is unnecessary in Theorem 2.

Corollary 3. Let E be an ordered and weakly sequentially complete Banach space, whose positive cone P is normal. Assume that $f:(a,b]\times E\times E\times E\to E$ is continuous, $g\in C(E\times E,E)$, v,w are coupled lower and upper *L*-quasisolutions of BVP (1.1), respectively. The conditions (H0) - (H3) and (H5) are valid. Then BVP (1.1) has a unique solution u in $[v,w]$, which can be obtained from monotone iterative sequences $\{v_n\},\{w_n\}\subset C_{1-\alpha,\log}(J,E)$ starting from v and w, respectively.

4. Example

Let $E=l^2=\left\{x:x=(x_1,x_2,\cdots,x_n,\cdots)\mid\sum_{n=1}^{\infty}|x_n|^2<\infty\right\}$ with the norm

$$\|x\|=\left(\sum_{n=1}^{\infty}|x_n|^2\right)^{\frac{1}{2}}$$

and $P=\{x\in E:x_n\geq 0,n=1,2,3,\cdots\}$. Then E is a weakly sequentially complete Banach space and P is a normal cone in E. Consider the BVP of infinite system in E

$$\begin{cases}{}^{H}\mathscr{D}_{1^+}^{\frac{1}{2}}x_n(t)=\dfrac{\arctan(x_{n+2}(t))}{50n(\log(t))^{\frac{1}{2}}}+\dfrac{t}{100}x_n(t)+\dfrac{1}{50ne^{|x_n(t)|}}\\ \qquad+\dfrac{1}{40}\displaystyle\int_1^t\frac{t}{s}x_n(s)\mathrm{d}s,\ t\in(1,\mathrm{e}],\\ \tilde{x}_n(1)-\dfrac{1}{2}\tilde{x}_n(1)\tilde{x}_n(\mathrm{e})=0.\end{cases}\tag{4.1}$$

Evidently, (4.1) can be regarded as a BVP of the form (1.1) in E. In this situation, $x=(x_1,x_2,\cdots,x_n,\cdots)$, $y=(y_1,y_2,\cdots,y_n,\cdots)$, $f=(f_1,f_2,\cdots,f_n,\cdots)$ and $g=(g_1,g_2,\cdots,g_n,\cdots)$, in which

$$f_n(t,x,y,Gx)=\frac{\arctan(x_{n+2})}{50n(\log(t))^{\frac{1}{2}}}+\frac{t}{100}x_n+\frac{1}{50ne^{|y_n|}}+\frac{1}{40}\int_1^t k(t,s)x_n(s)\mathrm{d}s,$$

where $k(t,s)=\frac{t}{s}$, and

$$g_n(x,y)=x_n-\frac{1}{2}x_ny_n.$$

It is clear $f\in C((1,\mathrm{e}]\times E\times E\times E,E)$ and $f(t,x,y,z)\in C_{\frac{1}{2},\log}([1,\mathrm{e}],E)$ for any $x,y,z\in E$, $g\in C(E\times E,E)$. Let

$$w=\left(1+(\log(t))^{-\frac{1}{2}},\frac{1+(\log(t))^{-\frac{1}{2}}}{2},\cdots,\frac{1+(\log(t))^{-\frac{1}{2}}}{n},\cdots\right),v=(0,0,\cdots,0,\cdots),$$

then $v,w\in C_{\frac{1}{2},\log}([1,\mathrm{e}],E)$, $v(t)\le w(t),t\in(1,\mathrm{e}]$, $\tilde{v}(1)\le\tilde{w}(1)$ and

$$g_n(\tilde{v}(1),\tilde{v}(\mathrm{e}))=0,\ g_n(\tilde{w}(1),\tilde{w}(\mathrm{e}))=\frac{1}{n}-\frac{1}{n^2}\ge 0.$$

Moreover, set $L=0$, one has

$$f_n(t,v(t),w(t),Gv(t))+L(v(t)-w(t))=\frac{1}{50ne^{|w_n(t)|}}\ge 0,$$

and

$$\begin{aligned}
&f_n(t,w(t),v(t),Gw(t))+L(w(t)-v(t))\\
&\le\frac{\pi}{100n(\log(t))^{\frac{1}{2}}}+\frac{\mathrm{e}}{100}\frac{1+(\log(t))^{-\frac{1}{2}}}{n}+\frac{1}{50n}+\frac{t}{40n}\int_1^t\frac{1+(\log(s))^{-\frac{1}{2}}}{s}\mathrm{d}s\\
&\le\frac{\pi}{100n(\log(t))^{\frac{1}{2}}}+\frac{\mathrm{e}}{100n}+\frac{\mathrm{e}}{100n(\log(t))^{\frac{1}{2}}}+\frac{1}{50n}+\frac{\mathrm{e}\log(t)}{40n}+\frac{\mathrm{e}(\log(t))^{\frac{1}{2}}}{20n}\\
&\le\frac{1}{n(\log(t))^{\frac{1}{2}}}\frac{2+\pi+9.5\mathrm{e}}{100}\le\frac{1}{n(\log(t))^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)}={}^{H}\mathscr{D}_{1^+}^{\frac{1}{2}}w_n(t).
\end{aligned}$$

Hence, v,w are coupled lower and upper quasisolutions of BVP (4.1).

The condition (H1) is satisfied. In fact, let $x^{(i)}=\left(x_1^{(i)},x_2^{(i)},\cdots,x_n^{(i)},\cdots\right)$, $y^{(i)}=\left(y_1^{(i)},y_2^{(i)},\cdots,y_n^{(i)},\cdots\right)$ and $z^{(i)}=\left(z_1^{(i)},z_2^{(i)},\cdots,z_n^{(i)},\cdots\right)$, $i=1,2$ be such that $v\le x^{(1)}\le x^{(2)}\le w$, $v\le y^{(2)}\le y^{(1)}\le w$ and $Gv\le z^{(1)}\le z^{(2)}\le Gw$. It is easy to get

$$\begin{aligned}
&f_n\left(t,x^{(2)},y^{(2)},z^{(2)}\right)-f_n\left(t,x^{(1)},y^{(1)},z^{(1)}\right)\\
&\ge\frac{1}{100}\left(x_n^{(2)}(t)-x_n^{(1)}(t)\right)\ge -M\left(x_n^{(2)}(t)-x_n^{(1)}(t)\right)
\end{aligned}$$

for any $M > 0$.

Furthermore, for $\tilde{v}(1) \le x^{(1)} \le x^{(2)} \le \tilde{w}(1)$, $\tilde{v}(\mathrm{e}) \le y^{(1)} \le y^{(2)} \le \tilde{w}(\mathrm{e})$, we can obtain

$$g_n\left(x^{(2)}, y^{(2)}\right) - g_n\left(x^{(1)}, y^{(1)}\right) \le x_n^{(2)} - x_n^{(1)}.$$

Thus, (H2) is valid with $L_1 = 1, L_2 = 0$. Therefore, Corollary 1 ensures BVP (4.1) has coupled minimal and maximal quasisolutions $v^*, w^* \in [v, w]$ with $v^* \le w^*$.

5. Conclusion

This paper explores a nonlinear boundary value problem involving Hadamard fractional derivatives and singularity in Banach spaces. Under suitable monotonicity conditions and noncompactness measure conditions, the existence of extremal L quasisolutions and uniqueness of solutions between coupled lower and upper L quasisolutions are derived from mixed monotone iterative technique and coupled lower and upper L quasisolutions method. Similarly, applying lower and upper solutions method and monotone iterative technique, we can investigate the existence of extremal solutions and uniqueness of solutions between lower and upper solutions to the problem (1.1) with the nonlinearity $f(t, x, Gx)$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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