Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2022, 10, 3629-3634
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

On Some Properties of the Norm of the Spectral Geometric Mean

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How to cite this paper: Kong, X.R. (2022) On Some Properties of the Norm of the Spectral Geometric Mean. *Journal of Applied Mathematics and Physics*, **10**, 3629-3634.
https://doi.org/10.4236/jamp.2022.1012241

Received: November 10, 2022
Accepted: December 18, 2022
Published: December 21, 2022

Abstract

In this paper, we consider the norms related to spectral geometric means and geometric means. When A and B are positive and invertible, we have $\|A^{-1}\# B\| \le \|A^{-1}\sigma_s B\|$. Let H be a Hilbert space and $B(H)$ be the set of all bounded linear operators on H. Let $A \in B(H)$. If $\|A\# X\| = \|A\sigma_s X\|, \forall X \in B(H)^{++}$, then A is a scalar. When $\mathscr{A}$ is a C*-algebra and for any $A, B \in \mathscr{A}^{++}$, we have that $\|\log A\# B\| = \|\log A\sigma_s B\|$, then $\mathscr{A}$ is commutative.

Keywords

Kubo-Ando Means, Spectral Geometric Mean, Geometric Mean, C*-Algebra

1. Introduction

The concept of means plays an important role in mathematics in general. In matrix theory and operator theory, the study of means represents a very active research field with wide spearing applications in various areas of pure and applied mathematics ([1]-[6]). There are many different approaches to matrix or operator means and the Kubo-Ando theory [7] is the mean we want to consider in this paper ([8]-[14]). Means are originally rather algebraic objects and they have close connection with the geometric features of the underlying structures. For example, the weighted arithmetic means $(1-t)A + tB$, $0 \le t \le 1$ of two elements A and B in a Euclidean space form the unique geodesic between A and B. Also the weighted geometric means

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}, \quad 0 \le t \le 1$$

of two given positive definite matrices A, B form the unique geodesic in a

Riemannian structure on the positive definite cone of matrices which has many applications (see, e.g., Chapter 6 in [1]).

Throughout this paper, we always assume that $\mathscr{A}$ is an unital C*-algebra with unit *I*. Let

$$\mathscr{A}^+ = \{x \in \mathscr{A} : x \text{ selfadjoint}, \sigma(x) \subset [0,\infty)\}$$

and

$$\mathscr{A}^{++} = \{x \in \mathscr{A} : x \text{ selfadjoint}, \sigma(x) \subset (0,\infty)\}.$$

We say A^+ and $\mathscr{A}^{++}$ are positive semidefinite cone and positive definite cone of C*-algebra $\mathscr{A}$ respectively. For basic of C*-algebras and von Neumann algebras, we refer to [15]. For a complex Hilbert space $\mathscr{H}$, we let $B(\mathscr{H})$ be the set of all bounded linear operators on $\mathscr{H}$ and $B(\mathscr{H})^+$ be the positive semidefinite cone of $B(\mathscr{H})$. There are several kinds of means defined on the positive definite cone $\mathscr{A}^{++}$ of a C*-algebra $\mathscr{A}$. The arithmetic mean, the harmonic mean and the geometric means are defined by $\frac{A+B}{2}$, $2\left(A^{-1}+B^{-1}\right)^{-1}$ and $A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$ respectively. These three means are special case of the Kubo-Ando means [7].

Definition 1.1. A binary operation σ on $B(\mathscr{H})^+$ is a Kubo-Ando mean if

1) $I\sigma I = I$;

2) If $A \le C, B \le D$, then $A\sigma C \le B\sigma D$;

3) $C(A\sigma B)C \le (CAC)\sigma(CBC)$;

4) If $A_n \downarrow A, B_n \downarrow B$ in strong operator topology, then $A_n\sigma B_n \downarrow A\sigma B$ in strong operator topology (here $\downarrow$ means monotone decreasing convergent in usual order on $B(\mathscr{H})$ and all operators appeared are assumed in $B(\mathscr{H})^+$).

Suppose $\mathscr{A}$ is a C*-algebra. Let $\mathscr{A}^{++}$ be the set of all positive invertible elements in $\mathscr{A}$. We use $A > 0$ to denote that $A \in \mathscr{A}^{++}$. The *spectral geometric mean* is the operation defined by

$$A\sigma_s B = \left(A^{-1}\# B\right)^{1/2} A \left(A^{-1}\# B\right)^{1/2}, \forall A, B \in \mathscr{A}^{++},$$

where $A\# B = \sqrt{A}\left(\frac{1}{\sqrt{A}}B\frac{1}{\sqrt{A}}\right)^{\frac{1}{2}}\sqrt{A}$ is the geometric mean of *A* and *B*.

In [8], the authors studied the maps preserving the spectral geometric mean and many interesting results are obtained. There are many interesting and important results related to the norms of means (see [3] [9] [11] [13] and references therein).

In this paper, we give some results on norms related to spectral geometric means and geometric means. We first give a norm inequality related to the spectral geometric mean and the geometric mean. We give a condition for an operator to be a scalar using norm equality between the spectral geometric mean and the geometric mean. We also show that a C*-algebra is commutative under certain conditions.

2. Main Results

Suppose $\mathscr{A}$ is a C*-algebra. For $A, B \in \mathscr{A}$, $A, B > 0$ (*i.e.*, A and B are positive and invertible), AB^2A is unitary equivalent to BA^2B. Note that since $A\sigma_s B$ is unitary equivalent to $\sqrt{A}\left(A^{-1}\# B\right)\sqrt{A} = \sqrt{\sqrt{A}B\sqrt{A}}$, then we have that

$$\|A\sigma_s X\| = \sqrt{\left\|\sqrt{A}X\sqrt{A}\right\|}.$$

Proposition 2.1. *Suppose* $\mathscr{A}$ *is a C*-algebra. For* $A, B \in \mathscr{A}$, $A, B > 0$, *we have* $\left\|A^{-1}\# B\right\| \le \left\|A^{-1}\sigma_s B\right\|$.

Proof. If $\left\|A^{-1}\sigma_s B\right\| \le t$, that is, $\sqrt{\left\|\frac{1}{\sqrt{A}}B\frac{1}{\sqrt{A}}\right\|} \le t$ and then $\left\|\frac{1}{\sqrt{A}}B\frac{1}{\sqrt{A}}\right\| \le t^2$, this shows that $\frac{1}{\sqrt{A}}B\frac{1}{\sqrt{A}} \le t^2 I$. Then $B \le t^2 A$, that is, $\sqrt{A}B\sqrt{A} \le t^2 A^2$.

This implies that $\sqrt{\sqrt{A}B\sqrt{A}} \le tA$, and this is equivalent to

$$\sqrt{A}\left(A^{-1}\# B\right)\sqrt{A} \le tA,$$

that is, $A^{-1}\# B \le tI$. Therefore, $\left\|A^{-1}\# B\right\| \le t$. □

Proposition 2.2. *Let H be a Hilbert space and* $B(H)$ *be the set of all bounded linear operators on H. Let* $A \in B(H)$. *If*

$$\|A\# X\| = \|A\sigma_s X\|, \quad \forall X \in B(H)^{++},$$

then A is a scalar.

Proof. For any projection P, since $P + \frac{1}{n}I \to P$, we have that

$$\|A\# P\| = \sqrt{\left\|\sqrt{A}P\sqrt{A}\right\|} = \sqrt{\|PAP\|}.$$

In particular, if P is a rank-one projection (written as $P = x \otimes x$), we have that

$$A\# P = \sqrt{\lambda(A, P)}P,$$

where $\lambda(A, P)$ is the strength of A along P. This implies that

$$\lambda(A, P) = \|PAP\| = \langle Ax, x\rangle.$$

Above equation is true for all $x \in H$. Note that

$$\lambda(A, P) = \left\|A^{-1/2}x\right\|^{-2} = \left\langle A^{-1}x, x\right\rangle^{-1}$$

for all $x \in H$ with $\|x\| = 1$. Hence $\langle Ax, x\rangle\left\langle A^{-1}x, x\right\rangle = 1$ for every unit vector $x \in H$. Then one can derive that

$$\|x\|^2 = \sqrt{\langle Ax, x\rangle\left\langle A^{-1}x, x\right\rangle}, \quad \forall x \in H.$$

Put $\sqrt{A}x$ in the above equation, we can see that

$$\left\|\sqrt{A}x\right\|^2 = \sqrt{\left\langle A\sqrt{A}x, \sqrt{A}x\right\rangle\left\langle A^{-1/2}x, A^{1/2}x\right\rangle},$$

that is,

$$\langle Ax, x\rangle = \sqrt{\langle A^2x, x\rangle}\|x\| = \|Ax\|\|x\|$$

for all $x \in H$. Then $Ax = \lambda_x x$ for some $\lambda_x \in \mathbb{C}$. For any $x, y \in H$ with $\|x\| = \|y\| = 1$, if $x = \alpha y$ for some $\alpha \in \mathbb{T}$, then we have that $A(\alpha x) = \lambda_{\alpha x}(\alpha x)$ and hence $\lambda_{\alpha x} = \lambda_x$. If $x \neq \alpha y$ for any $\alpha \in \mathbb{T}$, it follows from

$$A\left(\frac{x+y}{2}\right) = \lambda_{x+y/2}\left(\frac{x+y}{2}\right)$$

that

$$\lambda_x = \lambda_y = \lambda_{(x+y)/2}.$$

Therefore, we have that $A = \lambda I$ for some $\lambda \in \mathbb{C}$. □

Proposition 2.3. *Let* $\mathscr{A}$ *be a C*-algebra. Suppose for any* $A, B \in \mathscr{A}^{++}$, *we have that*

$$\|\log A \# B\| = \|\log A\sigma_s B\|.$$

Then $\mathscr{A}$ is commutative.

Proof. Let $X = \frac{1}{\sqrt{A}} B \frac{1}{\sqrt{A}}$ and $Y = A$. It follows that

$$\|\log A \# B\| = \left\|\log \sqrt{Y}\sqrt{X}\sqrt{Y}\right\| = \left\|\log \sqrt{\sqrt{Y}\sqrt{Y}X\sqrt{Y}\sqrt{Y}}\right\|$$
$$= \left\|\log \sqrt{YXY}\right\| = \frac{1}{2}\|\log YXY\|.$$

Put $Y = A^{-1}, X = B^2$, this shows that

$$\left\|\log \frac{1}{\sqrt{A}} B \frac{1}{\sqrt{A}}\right\| = \frac{1}{2}\left\|\log \frac{1}{\sqrt{A^2}} B^2 \frac{1}{\sqrt{A^2}}\right\|,$$

that is,

$$d_T(A, B) = \frac{1}{2} d_T(A^2, B^2),$$

where d_T is the Thompson metric. Then $A \mapsto A^2$ is a non-isometric dilation, this forces $\mathscr{A}$ is commutative (see [14], Theorem 18).

3. Conclusion

Mean is an important concept in mathematics. There are many interesting results from studying operator means. In this paper, we give some results on norms related to spectral geometric means and geometric means. We first give a norm inequality related to the spectral geometric mean and the geometric mean. We give a condition for an operator to be a scalar using norm equality between the spectral geometric mean and the geometric mean. We also show that a C*-algebra is commutative under certain conditions.

Acknowledgements

The author would like to thank the anonymous referee for constructive criticisms and valuable comments.

Funding

Partially supported by NFS of China (11871303, 11971463) and NSF of Shandong Province (ZR2019MA039 and ZR2020MA008).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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