Journal of Applied Mathematics and Physics, 2022, 10, 3586-3600
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Towards a Surreal Spin Theory: Surreal Superstrings?

Juan Antonio Nieto, Jesús Adrián Félix-Algandar

Facultad de Ciencias Fsico-Matemáticas de la Universidad Autónoma de Sinaloa, Culiacán Sinaloa, México
Email: niet@uas.edu.mx, janieto1@asu.edu, adrianfelix.fcfm@uas.edu.mx

How to cite this paper: Nieto, J.A. and Félix-Algandar, J.A. (2022) Towards a Surreal Spin Theory: Surreal Superstrings? *Journal of Applied Mathematics and Physics*, **10**, 3586-3600.
https://doi.org/10.4236/jamp.2022.1012238

Received: October 5, 2022
Accepted: December 16, 2022
Published: December 19, 2022

Abstract

We propose a surreal spin theory. Our basic mathematical tools are the dyadic rational number which is one of the key mathematical notions in surreal number theory. We argue that from the perspective of such surreal numbers, bosons and fermions, and therefore supersymmetry, are mere particular cases of a very much larger dyadic spin structure which include $\frac{1}{4}$-spin (semionics), $\frac{1}{8}$-spin and in general $\frac{m}{2^n}$-spin, with m odd integer and n a positive natural number. Finally, we conjecture that this development could imply the existence of a surreal superstring theory.

Keywords

Higher Spin Theory, Surreal Numbers, Dyadic Rationals

1. Introduction

In mathematics, from number theory history [1], one learns that, roughly speaking, the starting point was the natural number N and after centuries of thought evolution one ends up with the real number R from which one constructs the differential and integral calculus. In physics, ironically, one starts with classical mechanics which is based on continuous function over the real R and ends up with quantum mechanics whose structure leads to change in our physical continuous observable quantities for discrete values based on the natural numbers N. The main lesson learned from these histories is that the underlying mathematical structure is, indeed, number theory.

The orbital angular momentum L provides us the clearer example of the importance of the underlying number theory. In fact, in classical mechanics, L is considered in terms of the position and the momentum, both of which are assumed to be continuous functions of time over the real *R*, while in quantum mechanics, in the Bohr model of the hydrogen atom, one has $L = n\hbar$ with $\hbar$ the Planck fundamental constant and $n \in N$. (In order to give emphasis to the numerical aspect let us just set units such that $\hbar = 1$.) So, let's say, one uses *L* over *R* in planetary systems, but *L* over *N* in atomic scale systems. In any case, one observes that number theory is the basic mathematical notion, somehow hidden in this analysis.

When one makes a reflection on the above comments at the level of fundamental particles one discovers that one needs to add the spin of particles which is in some cases, as the photon, a number in *N*, while in other cases, such as the electron ($\frac{1}{2}$) or the gravitino ($\frac{3}{2}$), is a number of the type $\frac{m}{2}$ with *m* odd. It turns out that particles with integer spin are called bosons, while particles with half integer spin are called fermions. It is worth mentioning that supersymmetry combines both types of particles: bosons and fermions. The interesting thing is that such numbers, integers and half integers are subsets of the rationals *Q* which in turn are contained in *R*. Thus, at the level of fundamental particles one seems to go backward from *N* to *Q* and perhaps later on to *R*, as we shall see below. So, one may ask the question: is it possible to have a mathematical structure that describes this numerical evolution of the spin concept at the level of fundamental particles? Here, we would like to propose the answer: surreal numbers theory.

2. A Short Comment about Surreal Number Theory and Dyadic Rational Spin

Let us briefly describe what a surreal number means. For this purpose consider the set

$$x = \{ X_L \mid X_R \} \tag{1}$$

and call X_L and X_R the left and right sets of *x*, respectively. In 1973, Conway [2] (see also Ref. [3]) developed the surreal numbers structure $\mathcal{S}$ from two axioms:

Axiom 1. Every surreal number corresponds to two sets X_L and X_R of previously created numbers, such that no member of the left set $x_L \in X_L$ is greater or equal to any member x_R of the right set X_R.

Let us denote by the symbol $\not\geq$ the notion of no greater or equal to. So the axiom establishes that if *x* is a surreal number then for each $x_L \in X_L$ and $x_R \in X_R$ one has $x_L \not\geq x_R$. This is denoted by $X_L \not\geq X_R$.

Axiom 2. One number $x = \{ X_L \mid X_R \}$ is less than or equal to another number $y = \{ Y_L \mid Y_R \}$ if and only if the two conditions $X_L \not\geq y$ and $x \not\geq Y_R$ are satisfied.

This can be simplified by saying that $x \leq y$ if and only if $X_L \not\geq y$ and $x \not\geq Y_R$.

Observe that Conway definition relies in an inductive method; before a surreal number x is introduced one needs to know the two sets X_L and X_R of surreal numbers. Using Conway algorithm one finds that at the l_2-day one obtains $2^{l_2+1}-1$ numbers, all of which are of form

$$x = \frac{m}{2^n}, \tag{2}$$

where m is an integer and n is a natural number, $n > 0$. Of course, the numbers (2) are dyadic rationals which are dense not only in the rationals Q but also in the real R. It is also possible to show that the real numbers R are contained in the surreals $\mathcal{S}$ (see Ref. [2] [3] for details). Of course, in some sense, the proof relies on the fact that the dyadic numbers (2) are dense in the real R.

In 1986, Gonshor [4] introduced a different but equivalent definition of surreal numbers:

Definition 1. A surreal number is a function f from initial segment of the ordinals into the set $\{+,-\}$.

For instance, if f is the function so that $f(1)=+$, $f(2)=+$, $f(3)=-$, $f(4)=+$ then f is the surreal number $(++-+)$. In the Gonshor approach, one obtains the sequence: 1-day

$$-1=(-), \quad (+)=+1, \tag{3}$$

in the 2-day

$$-2=(--), \quad -\frac{1}{2}=(-+), \quad (+-)=+\frac{1}{2}, \quad (++)=+2, \tag{4}$$

and 3-day

$$\begin{gathered} -3=(---), \ -\frac{3}{2}=(--+), \ -\frac{3}{4}=(-+-), \ -\frac{1}{4}=(-++) \\ (+--)=+\frac{1}{4}, \ (+-+)=+\frac{3}{4}, \ (++-)=+\frac{3}{2}, \ (+++)=+3, \end{gathered} \tag{5}$$

respectively. Moreover, in Gonshor approach one finds the different numbers through the formula [4]

$$\mathcal{J} = l_1\varepsilon_0 + \frac{\varepsilon_1}{2} + \sum_{k=1}^{p} \frac{\varepsilon_{k+1}}{2^{k+1}}, \tag{6}$$

where $l_1 \in \{1,2,\cdots\}$ and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p \in \{+,-\}$ and $\varepsilon_0 \neq \varepsilon_1$. As in the case of Conway definition, through (6) one gets the dyadic rationals. Just for clarity, let us consider the example:

$$(++-+-+) = 2 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \frac{27}{16}. \tag{7}$$

By defining the order $x<y$ if $x(\alpha)<y(\alpha)$, where α is the first place where x and y differ and the convention $-<0<+$, it is possible to show that the Conway and Gonshor definitions of surreal numbers are equivalent (see Ref. [4] for details).

Let us focus in (6) with $\varepsilon_0=+$ (and therefore with $\varepsilon_1=-1$). Thus, (6) becomes

$$\mathcal{J}_{(+)} = l_1 - \frac{1}{2} + \sum_{k=1}^{p} \frac{\varepsilon_{k+1}}{2^{k+1}}. \tag{8}$$

It remains to clarify what index p, in the sum symbol, means. In order to clarify this aspect instead of (8) one may write the more complete expression [5]:

$$\mathcal{J}_{(+)}(l_1,l_2)=\begin{cases}(\mathrm{I})\, l_1, & \text{if } l_2-l_1=0\\ (\mathrm{II})\, l_1-\dfrac{1}{2}, & \text{if } l_2-l_1=1\\ (\mathrm{III})\, l_1-\dfrac{1}{2}+\displaystyle\sum_{k=1}^{l_2-(l_1+1)}\frac{\varepsilon_{k+1}}{2^{k+1}}, & \text{if } l_2-l_1>1\end{cases} \tag{9}$$

Notice that according to this expression one always has $l_2-l_1\geq 0$.

Similar analysis for $\varepsilon_0=-$ and $\varepsilon_1=+$ in (6) lead us to

$$\mathcal{J}_{(-)}(l_1,l_2)=\begin{cases}(\mathrm{I})\, -l_1, & \text{if } l_2-l_1=0\\ (\mathrm{II})\, -l_1+\dfrac{1}{2}, & \text{if } l_2-l_1=1\\ (\mathrm{III})\, -l_1+\dfrac{1}{2}-\displaystyle\sum_{k=1}^{l_2-(l_1+1)}\frac{\varepsilon_{k+1}}{2^{k+1}}, & \text{if } l_2-l_1>1\end{cases} \tag{10}$$

Observe that $\mathcal{J}_{(-)}(l_1,l_2)=-\mathcal{J}_{(+)}(l_1,l_2)$. The first thing that one notes is that (I) in (9) and (10) gives the integer numbers $Z_{(+)}$ and $Z_{(-)}$. By completeness one set $\mathcal{J}_{(\pm)}(0,0)=0$. While (II) provides with the dyadic rationals $\frac{m}{2}$ where m is an odd element in the rationals $Q_{(+)}$ and $Q_{(-)}$. It turns out that from the point of view of the spin concept, (I) gives integer spin numbers Z corresponding to bosons, while (II) establishes the half integer spin numbers corresponding to fermions. So, one concludes that (I) and (II) in (9) and (10) determine the spin numerical values for bosons and fermions, just the kind of particles considered in supersymmetry. So our contribution to this spin framework must come from (III) in (9) and (10). In order to have a better understanding of what exactly one means by this, we shall consider several examples.

Assume $l_1=1$. In this case (9) becomes

$$\mathcal{J}_{(+)}(1,l_2)=\begin{cases}(\mathrm{I})\, 1, & \text{if } l_2=1\\ (\mathrm{II})\, \dfrac{1}{2}, & \text{if } l_2=2\\ (\mathrm{III})\, \dfrac{1}{2}+\displaystyle\sum_{k=1}^{l_2-2}\frac{\varepsilon_{k+1}}{2^{k+1}}, & \text{if } l_2>2\end{cases} \tag{11}$$

This implies that from (I) and (II) one gets $\mathcal{J}_{(+)}(1,1)=1$, $\mathcal{J}_{(+)}(1,2)=\frac{1}{2}$ and from (III) one obtains $\mathcal{J}_{(+)}(1,3)=\left\{\frac{1}{4},\frac{3}{4}\right\}$, $\mathcal{J}_{(+)}(1,4)=\left\{\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\right\}$ and so on. Since $\mathcal{J}_{(-)}(l_1,l_2)=-\mathcal{J}_{(+)}(l_1,l_2)$ one also has $\mathcal{J}_{(-)}(1,1)=-1$, $\mathcal{J}_{(-)}(1,2)=-\frac{1}{2}$ and $\mathcal{J}_{(-)}(1,3)=\left\{-\frac{1}{4},-\frac{3}{4}\right\}$, $\mathcal{J}_{(-)}(1,4)=\left\{-\frac{1}{8},-\frac{3}{8},-\frac{5}{8},-\frac{7}{8}\right\}$ and so on. Hence, according to these results in addition to particles with 1-spin and $\frac{1}{2}$-spin obtained from (I) and (II) our prediction is that there must exist particles with $\left\{\frac{1}{4},\frac{3}{4}\right\}$-spin and $\left\{\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\right\}$-spin and so on. It is interesting that particles with $\frac{1}{4}$-spin have already considered in the literature [6] [7] and [8]. But our prediction here is that $\left\{\frac{1}{4}\right\}$-spin gauge fields must be just part of a larger list of new dyadic $\frac{m}{2^n}$-spin particles, with m odd.

An interesting combination between (11) and (9) (and (10)) is

$$\mathcal{J}_{(+)}(l_1,l_2)=\mathcal{J}_{(+)}\left(1,l_2^{(0)}\right)+(l_1-1), \tag{12}$$

with $l_2^{(0)}=l_2-(l_1-1)$ [5]. This means that the tree $\mathcal{J}_{(+)}\left(1,l_2^{(0)}\right)$ (and $\mathcal{J}_{(-)}\left(1,l_2^{(0)}\right)$) plays the role of a main building block; any other tree $\mathcal{J}_{(+)}(l_1,l_2)$ with $l_1>1$ can be obtained from (12). Surprisingly, $\mathcal{J}_{(+)}\left(1,l_2^{(0)}\right)$ has been studied in the context of Zeno algorithm [9] and Minkowskis question mark function [10]. In some sense if $\mathcal{J}_{(-)}\left(1,l_2^{(0)}\right)$ were added to $\mathcal{J}_{(+)}\left(1,l_2^{(0)}\right)$ and (12) was used the surreal numbers could be discovered for another routes, different than game theory.

Another interesting aspect of the tree structure $\mathcal{J}_{(+)}(l_1,l_2)$, $\mathcal{J}_{(-)}(l_1,l_2)$ and $\mathcal{J}_{(\pm)}(0,0)$ is that one can derive in an alternative way how many numbers are created in the l_2-day. It is worth to mention that this notion of "day" is used by the mathematicians in spire of their development of surreal numbers theory is considered only in the mathematical context. First, let us use Gonshor formalism to answer this question. In the 0-day one starts with the number 0 and in the 1-day the numbers −1 and +1 are created, namely (−) and (+). While in the 2-day 4 numbers are created, namely $(++)=2$, $(+-)=\frac{1}{2}$, $(--)=-2$, $(-+)=-\frac{1}{2}$, and so on. So one has that the series $t=1+2\left(1+2+4+8+\cdots+2^{l_2-1}\right)$ determines the total numbers of surreal numbers that at the l_2-day are created. But using the identity $2+2+4+8+16+\cdots+2^{l_2}=2^{l_2+1}$ one discovers that $t=2^{l_2+1}-1$. Now, $\mathcal{J}_{(\pm)}(l_1,l_2)$ is two parameter function l_1 and l_2. If one sets l_1 and change l_2 one moves vertically producing the corresponding tree, as $\mathcal{J}_{(\pm)}(1,l_2)$. While if one sets l_2 and change l_1 one is moving horizontally. In this sense l_2 determines the day parameter used by the mathematician. As an example set $l_2=3$. From (9) one obtains $\mathcal{J}_{(+)}(3,3)=3$, $\mathcal{J}_{(+)}(2,3)=\frac{3}{2}$, $\mathcal{J}_{(+)}(1,3)=\left\{\frac{1}{4},\frac{3}{4}\right\}$, and the corresponding negatives. So, in the 3-day we have 8 numbers and so one discovers the series $t=1+2+4+8+\cdots+2^{l_2}$ which is what one obtains with Gonshor approach.

3. Surreal Numbers and Rational Dyadic Spin-Field

Until now we have established a link between surreal number theory and the physical concept of spin. A natural question is; how such a numerical scenario can be associated with gauge fields? Consider a plane wave gauge field $\psi\left(x^{\alpha}\right)$. One says that ψ is a gauge field of spin s if satisfies the transformation rule

$$\psi' = \mathrm{e}^{is\theta}\psi. \tag{13}$$

In the case of the electromagnetism it is shown that for a wave propagating in the z-direction the 4-vector potential $A_{\mu}\left(x^{\alpha}\right)$, in the Lorentz gauge, leads to the transformation

$$A_{\pm}' = \mathrm{e}^{i\theta}A_{\pm}, \tag{14}$$

with $A_{\pm} = A_1 \mp iA_2$. While in the case of gravitational gauge field $h_{\mu\nu} = h_{\nu\mu}\left(x^{\alpha}\right)$, again propagating in the z-direction and in the Lorentz gauge, transforms as

$$h_{\pm}' = \mathrm{e}^{i2\theta}h_{\pm}, \tag{15}$$

with $h_{\pm} = h_{11} \mp ih_{12}$. Thus, (14) and (15) are used to say that electromagnetism and gravitation are gauge fields of 1-spin and 2-spin, respectively. From this rough analysis, one sees a patter; if there is just one spacetime index in the gauge field A_{μ} the corresponding spin is 1, while if there are two indices in the symmetric gauge field $A_{\mu\nu}$ the spin is 2, whereby convenience we define $A_{\mu\nu} \equiv h_{\mu\nu}$. Therefore, if one considers a completely symmetric gauge field $A_{\mu_1\mu_2\cdots\mu_{l_1}}$ it is expected that the associated spin must be l_1. Just the integer numbers associated with $Z_{(+)}, Z_{(-)}$ and the neutral element $Z_{(0)}$ predicted in the surreal numbers (I) given in (9) and (10).

We would like now to consider the case of $\frac{1}{2}$-spin gauge spinor field. In this case one has a complex spinor $\psi^{a}\left(x^{\alpha}\right)$ which of course satisfies de Dirac equation [11]

$$\left[\gamma^{\mu}_{ab}\hat{p}_{\mu} + m_0\delta_{ab}\right]\psi^{b} = 0. \tag{16}$$

Here, as it is known, the γ-matrices satisfy the Clifford algebra

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = -2\eta^{\mu\nu}, \tag{17}$$

with $\eta^{\mu\nu} = diag\left(-1,1,1,1\right)$ and $\hat{p}_{\mu} = -i\partial_{\mu}$.

Comparing ψ^{a} with A_{μ} one notes two important differences; ψ^{a} is complex and the index a is spinorial, while A_{μ} is real and contains the target space-time index μ. The complexity of ψ^{a} can be solved by imposing a Majorana condition $C^{a}_{b}\psi^{b} = \mathrm{e}^{i\theta}\psi^{a}$, where C^{a}_{b} is the charge conjugation matrix. Similar observation applies to a field of the form $A_{\mu_1\mu_2\cdots\mu_{l_1}}$ which only contains the spacetime index $\mu_1\mu_2\cdots\mu_{l_1}$. So according to the previous discussion although both fields ψ^{a} and $A_{\mu_1\mu_2\cdots\mu_{l_1}}$ are continuous function over the real their associated spin structure is discrete. What if one now combines indices in the form A^{a}_{μ}? For A^{a}_{μ} one imposes, of course, the Majorana condition. The question emerges what is it the spin associated with the gauge field A^{a}_{μ}? It is not difficult to guess that A^{a}_{μ} is a field of $\frac{3}{2}$-spin (gravitino field). If we now consider a Majorana gauge field of the form $A^{a}_{\mu_1\mu_2\cdots\mu_{l_1}}$ one must find that this corresponds to a $\frac{2l_1-1}{2}$-spin field. But one has $\frac{2l_1-1}{2}=l_1-\frac{1}{2}$, just as the surreal numbers in (II) in (9) and (10). Summarizing the l_1-spin (integer spin) structure of $A_{\mu_1\mu_2\cdots\mu_{l_1}}$ corresponds to (I) in (9) and (10), and $A^a_{\mu_1\mu_2\cdots\mu_{l_1}}$ with $\left(l_1-\frac{1}{2}\right)$-spin (half integer spin) structure corresponds to (II) in (9) and (10). So, one can say that (I) and (II) correspond what in the literature is called "higher spin theories" (see Ref. [12] and references therein). Again the question emerges what could be the gauge field associated with (III) in (9) and (10)?

In order to answer the above question let us again observe the indices in the gauge fields $A_{\mu_1\mu_2\cdots\mu_{l_1}}$ and $A^a_{\mu_1\mu_2\cdots\mu_{l_1}}$. One notes that, only because one introduces an appropriate spinor index *a*, the numerical nature of the spin changes drastically from integer l_1 to half integer $\left(l_1-\frac{1}{2}\right)$. Observe also that the number $\left(l_1-\frac{1}{2}\right)$ enter (in the two first numbers) also in (III). So one should expect that in order to include lets say terms like $\pm\frac{1}{4}$-spin one must introduce a new type of index in the gauge field $A^a_{\mu_1\mu_2\cdots\mu_{l_1}}$. In fact, one finds that a gauge field of the form $A^{aA^{\pm}}_{\mu_1\mu_2\cdots\mu_{l_1}}$ makes the job. But immediately the question arises: What could be the field equation that $A^{aA^{\pm}}_{\mu_1\mu_2\cdots\mu_{l_1}}$ must satisfy? One can explore first $A^{aA^{+}}$ which we simply write as A^{aA}. In the case of A^a the field equation is, of course, the Dirac Equation (16). It turns out that this equation is so important in quantum mechanics and supersymmetry that it seems difficult to modify in order to make some room for the field A^{aA}. Fortunately, we have a possible proposal and this goes with the name of Elko field equation [13] [14].

First, recall that Dirac equation (16) can be considered as the "square root" of the Klein-Gordon equation. Surprisingly, in 2005 Ahluwalia and Grumiller [13] [14] proved that such a "square root" is not unique. In fact, assuming more general helicity eigenvalues they proved that an alternative and different field equation

$$\left[\gamma^{\mu}_{ab}\hat{p}_{\mu}\delta^B_A+im_0\delta_{ab}\varepsilon^B_A\right]\psi^b_B=0 \tag{18}$$

emerges which can also be reduced to the Klein-Gordon equation. They call their formalism Elko theory [13] [14]. Here, the indices μ,ν, etc. run from 0 to 3, the indices A,B run from 1 to 2 and the spinorial indices a,b run from 1 to 4, with ε^{AB} is the totally antisymmetric ε-symbol, with $\varepsilon^{12}=1=-\varepsilon^{21}$. One of the interesting aspects of the Elko fields is that they provide one of the most interesting candidates for dark matter [15].

In contrast to the usual Dirac equation, (18) requires 8-component complex spinor field ψ^a_A rather than 4-component ψ^a which is the case in equation (3). This is solved by imposing in ψ^a_A the kind of Majorana condition. Furthermore, the quantities δ^B_A and ε^B_A in (1) establish that ψ^a_A is not eigen-spinor of the $\gamma^{\mu}\hat{p}_{\mu}$ operator as ψ^a in the Dirac equation (16).

It turns out that (18) can be generalized to the form [16]

$$\left[\frac{1}{l!}\gamma^{\mu}_{ab}\hat{p}_{\mu}\delta^{B_1\cdots B_l}_{A_1\cdots A_l}+\frac{i}{l!}m_0\delta_{ab}\varepsilon^{B_1\cdots B_l}_{A_1\cdots A_l}\right]\psi^{a}_{B_1\cdots B_l}=0, \tag{19}$$

where $\delta^{B_1\cdots B_l}_{A_1\cdots A_l}$ is the generalized Kronecker delta which is related to $\varepsilon^{B_1\cdots B_l}_{A_1\cdots A_l}$ by

$$\delta^{B_1\cdots B_l}_{A_1\cdots A_l}=\frac{1}{l!}\varepsilon^{B_1\cdots B_l}_{C_1\cdots C_l}\varepsilon^{C_1\cdots C_l}_{A_1\cdots A_l}. \tag{20}$$

It seems that l-rank totally antisymmetric tensor spinor fields of the form $\psi_{aA_1^{\pm}\cdots A_l^{\pm}}\left(x^{\mu}\right)$, is what one is looking for. An interesting feature of our formalism arises when one requires that the field $\psi_{aA_1\cdots A_l}$ satisfies the Grassmann-Plücker relations

$$\psi^{a}_{A_1\cdots[A_l}\psi^{a}_{B_1\cdots B_l]}=0, \tag{21}$$

for each value of the index a in $\psi^{a}_{A_1\cdots A_l}$. Here, the bracket $\left[A_l B_1\cdots B_l\right]$ means totally antisymmetric. Thus, one finds that due to (21) the field $\psi^{a}_{A_1\cdots A_l}$ is decomposable and therefore can be written as

$$\psi^{a}_{A_1\cdots A_l}=\Omega^{a}_{a_1\cdots a_l}\varepsilon_{i_1\cdots i_l}\psi^{a_1 i_1}_{A_1}\cdots\psi^{a_l i_l}_{A_l}, \tag{22}$$

where the only non-vanishing terms of $\Omega^{a}_{a_1\cdots a_l}$ are $\Omega^{1}_{1\cdots 1}=\Omega^{2}_{2\cdots 2}=\Omega^{3}_{3\cdots 3}=\Omega^{4}_{4\cdots 4}=1$. The Majorana condition for $\psi^{a}_{A_1\cdots A_l}$ looks like

$$C^{a}_{b}\psi^{b}_{A_1\cdots A_l}=\mathrm{e}^{i\theta}\psi^{a}_{A_1\cdots A_l}. \tag{23}$$

One recognizes in (22) a kind of generalization of the Plücker coordinates for ψ^{ai}_{A}. Hence, (22) determines one-to-one correspondence between the fields $\psi^{a}_{A_1\cdots A_l}$ and ψ^{ai}_{A}. Using (23) one can make identification $\psi^{ai}_{A}=A^{ai}_{A}$. Thus, it seems that A^{ai}_{A} is the gauge field that one is looking for in order to make sense of $\frac{m}{2^n}$-spin gauge field, but further work is required.

Just for completeness consider a complex state $|\psi\rangle\in C^{2^N}$ expressed as (see Ref. [17] and references therein)

$$|\psi\rangle=\sum_{\hat{A}_1,\hat{A}_2,\cdots,\hat{A}_N=0}^{1}Q_{\hat{A}_1\hat{A}_2\cdots\hat{A}_N}\left|\hat{A}_1\hat{A}_2\cdots\hat{A}_N\right\rangle. \tag{24}$$

This is a *N*-qubit sate, where $\left|\hat{A}_1\hat{A}_2\cdots\hat{A}_N\right\rangle=\left|\hat{A}_1\right\rangle\otimes\left|\hat{A}_2\right\rangle\otimes\cdots\otimes\left|\hat{A}_N\right\rangle$ correspond to a standard basis. In a particular subclass of *N*-qubit entanglement, the Hilbert space can be broken into the form $C^{2^N}=C^{L}\otimes C^{l}$, with $L=2^{N-n}$ and $l=2^n$. Such a partition allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L,l)$ of l plains in C^{L} *via* the Plücker embedding. The idea is to associate the first $N-n$ and the last n indices of $Q_{\hat{A}_1\hat{A}_2\cdots\hat{A}_N}$ with a $L\times l$ matrix $\boldsymbol{v}^{i}_{A}$. This can be interpreted as the coordinates of the Grassmannian $Gr(L,l)$ of l-plains in C^{L}. Using the matrix $\boldsymbol{v}^{i}_{A}$ one can define the Plücker coordinates

$$Q_{A_1\cdots A_l}=\varepsilon_{i_1\cdots i_l}\boldsymbol{v}^{i_1}_{A_1}\cdots\boldsymbol{v}^{i_l}_{A_l}, \tag{25}$$

which one can associate with (25) by making the identification $\boldsymbol{v}^{i}_{A}\to\psi^{ai}_{A}$. Just as in qubit theory the transformation $v \to Sv$, with $S \in GL(l,C)$, the Plücker coordinates transform as $\mathcal{Q} \to Det(S)\mathcal{Q}$ (see Ref. [17] for details) one discovers that under $\psi_A^{ai} \to S_j^i \psi_A^{aj}$ the field $\psi_{A_1 \cdots A_l}^a$ transform as $\psi_{A_1 \cdots A_l}^a \to Det(S)\psi_{A_1 \cdots A_l}^a$. This shows that according to our previous discussion that the Plücker coordinates may be the key concept for surreal numbers framework and therefore in $\frac{m}{2^n}$-spin gauge field theory. It turns out that the Plücker coordinates are also a key concept in one of the possible definitions of realizable oriented matroids [18] (see also Refs. [19]-[26] and references therein).

Surprisingly, the rebits (the real part of the complex qubits) can be connected surreal numbers and therefore one can also associate the Plücker coordinates of $\psi_{A_1 \cdots A_l}^a$ with the surreal numbers. As we mentioned using Conway algorithm one finds that at the " j_2-day" one obtains $2^{j_2+1}-1$ numbers all of which are of form $x = \frac{m}{2^n}$, where m and n are integers numbers. Of course, these numbers are dyadic rationals which we already know that are dense in the real *R*. The mathematicians call ω-day when the surreal numbers lead to the real *R*. It is not difficult to prove that in the ω-day the tree $\mathcal{J}_{(\pm)}(1,l_2)$ leads to the interval $-1 \le \mathcal{J}_{(\pm)}(1,l_2) \le 1$ over the real *R*. In this context, one finds that in the ω-day one must associate a continuous spin over the real *R*. These types of continuous spins are of course very well known; particles with such spin are called anyons. The surprisingly thing is that the creation of surreal numbers do not finish in the ω-day, but can continue let's say to the $(\omega+1)$-day, $(\omega+2)$-day, etc. What this means in the framework of physical spin is at present mysterious for us.

4. Surreal *sl*(2,*r*) Correspondence

It is worth mentioning that in Ref. [22] a connection between oriented matroid theory and super *p*-branes was established. Thus, according to the present development, one may expect that eventually a link between super *p*-branes and surreal numbers may appear. Therefore one is tempting to believe that our proposed surreal spin theory may emerge as a key structure in the quest for quantum gravity. A strongly evidence for this conjecture may come from the observation that $\frac{1}{4}$-spin and $\frac{3}{4}$-spin particles (semionic) have already been considered in a kind supergravity theories [7]. The relevant symmetry, in this case, is the $SL(2,R)$ group which has already been connected with superstrings [27] [28]. Let us express from the point of view of the surreal number structure the relevance of this group in superstrings.

Let us first introduce the generators J_i of the group $SU(1,1)$ which is isomorphic to $SL(2,R)$, $SO(1,2)$ or $Sp(2,R)$

$$J^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{26}$$

which are of course related to the Pauli matrices in the form $J^1 = \frac{i\sigma_1}{2}, J^2 = \frac{i\sigma_2}{2}$ and $J^0 = \frac{\sigma_3}{2}$. Thus, we find that the J^i satisfies the algebra

$$\left[J^1, J^2\right] = iJ^3, \quad \left[J^3, J^1\right] = iJ^2, \quad \left[J^3, J^2\right] = -iJ^1 \tag{27}$$

and also

$$J^i J^j + J^j J^i = \frac{1}{2}\eta^{ij},$$

with $\eta^{ij} = diag(-1,+1,+1)$. As usual, in order to discuss any representation of $SL(2,R)$ one introduces the raising and lowering operators

$$\begin{aligned} J^+ &= J^1 + iJ^2, \\ J^- &= J^1 - iJ^2. \end{aligned} \tag{28}$$

Thus, one finds that the Casimir operator

$$(J)^2 = -(J^3)^2 + (J^1)^2 + (J^2)^2, \tag{29}$$

becomes

$$(J)^2 = -(J^3)^2 + \frac{1}{2}(J^- J^+ + J^+ J^-). \tag{30}$$

With these algebraic tools, the unitary irreducible representation of $SL(2,R)$ can be obtained. Typically one has discrete and continuos representations, depending of the eigenvalues k of J^3 and the eigenvalues q of the Casimir operator $(J)^2$. Here, however we are more interested in considering the exceptional representation

$$J^+ = \frac{1}{2}a^\dagger a^\dagger, \quad J^- = \frac{1}{2}aa, \quad J^3 = \frac{1}{4}(a^\dagger a + aa^\dagger), \tag{31}$$

where one assumes that $a^\dagger$ and a satisfies the typical expressions of the harmonic oscillator, namely

$$\begin{aligned} a^\dagger |n\rangle &= \sqrt{n+1}|n+1\rangle, \\ a|n\rangle &= \sqrt{n}|n-1\rangle, \end{aligned} \tag{32}$$

with $n = 0,1,2,\cdots$. Observe that

$$J^- = J^{+\dagger}. \tag{33}$$

It is not difficult to see that (31) is consistent with (27). From (30), (31) and (32) one obtains

$$\frac{1}{2}(J^- J^+ + J^+ J^-)|n\rangle = \frac{1}{4}(n^2 + n + 1)|n\rangle \tag{34}$$

and

$$J^3|n\rangle = \frac{1}{4}(2n+1)|n\rangle, \tag{35}$$

and therefore we discover that

$$(J)^2|n\rangle = \frac{3}{16}. \tag{36}$$

This means that the combination

$$J^- J^+ = J^{+\dagger} J^+ = (J^1)^2 + (J^2)^2 + i[J^1, J^2] = (J)^2 + (J^3)^2 - (J^3)^2 \tag{37}$$

leads to the formula

$$\frac{3}{16} + j^2 - j \geq 0, \tag{38}$$

with

$$J^3|n\rangle = j|n\rangle. \tag{39}$$

In the minimum case, the solution of (38) corresponds to $j = \frac{1}{4}$ and $j = \frac{3}{4}$. Since

$$J^-|n\rangle = \frac{1}{2}n(n-1)|n\rangle \tag{40}$$

annihilate the vacuum for $n = 0$ and $n = 1$, that is $J^-|0\rangle = 0$ and $J^-|1\rangle = 0$ the whole solution for $|n\rangle$ can be separated in the even part

$$J^3|n\rangle = \frac{1}{2}\left(2n + \frac{1}{2}\right)|n\rangle, \tag{41}$$

corresponding to $j = \frac{1}{4}$ and odd part

$$J^3|n\rangle = \frac{1}{2}\left(2n + 1 + \frac{1}{2}\right)|n\rangle. \tag{42}$$

for the case $j = \frac{3}{4}$. One can make a contact with surreal numbers if one now observes that $\mathcal{J}_{(+)}(1,3) = \left\{\frac{1}{4}, \frac{3}{4}\right\}$, $\mathcal{J}_{(-)}(1,3) = \left\{-\frac{1}{4}, -\frac{3}{4}\right\}$, obtained from (III) in (9) and (10). But from (12) one gets

$$\mathcal{J}_{(+)}(l_1,3) = \mathcal{J}_{(+)}(1,3) + (l_1 - 1). \tag{43}$$

This means that

$$\mathcal{J}_{(\pm)}(l_1,3) = l_1 - 1 + 1 - \frac{1}{2} \pm \frac{1}{4}. \tag{44}$$

So, by taking

$$n = l_1 - 1, \tag{45}$$

one obtains the two eigenvalues (41) and (42) corresponding to $j = \frac{1}{4}$ and $j = \frac{3}{4}$.

5. Surreal Superstrings?

Now we would like to pursue the idea of using the previous surreal $sl(2,r)$

correspondence in order to develop a kind of surreal superstring theory. Fortunately, there are a number of works, including "Cosets as gauge slices in $SU(1,1)$ strings" by Hwang [27] and "Strings in AdS3 and the SL(2, R) WZW Model; 1, 2, 3" by J. Maldacena and H. Ooguri [28] among others, that establishes a bridge between the group $SL(2,R)$ and strings. The starting point in these developments is to Fourier expand the generators of $SL(2,R)$ in the form

$$J_L = \sum_{-\infty}^{\infty} J_n^a \mathrm{e}^{-inx^-}, \quad J_R = \sum_{-\infty}^{\infty} \bar{J}_n^a \mathrm{e}^{-inx^+}, \tag{46}$$

with $a = 0, -, +$ and

$$x^{\pm} = \tau \pm \sigma. \tag{47}$$

We require that the generators J_n^a satisfies the algebra

$$\left[J_n^3, J_m^3\right] = -\frac{kn}{2}\delta_{n,-m}, \tag{48}$$

$$\left[J_n^3, J_m^{\pm}\right] = \pm J_{n+m}^{\pm}, \tag{49}$$

$$\left[J_n^-, J_m^+\right] = -2J_{n+m}^3 + kn\delta_{n,-m} \tag{50}$$

and the same for $\bar{J}_n^a$. Of course, this algebra corresponds to the Hilbert space of the WZW model. It is not difficult to see that the origin of this algebra can be traced back to the algebra of the generators of $SL(2,R)$, namely the expressions (27). For this reason this algebra we shall called $SL(2,\tilde{R})$.

The Virasoro generators are defined by

$$L_0 = \frac{1}{k-2}\left[J_0^+ J_0^- + J_0^- J_0^+ - \left(J^3\right)^2 + \sum_{m=1}^{\infty} J_{-m}^+ J_m^- + J_{-m}^- J_m^+ - 2J_{-m}^3 J_m^3\right] \tag{51}$$

and

$$L_{n\neq 0} = \frac{1}{k-2}\left[\sum_{m=1}^{\infty} J_{n-m}^+ J_m^- + J_{n-m}^- J_m^+ - 2J_{n-m}^3 J_m^3\right], \tag{52}$$

which presumably satisfies the Virasoro algebra

$$\left[L_n, L_m\right] = (n-m)L_{n-m} + \frac{c}{12}\left(n^3 - n\right)\delta_{n,-m}, \tag{53}$$

where

$$c = \frac{3k}{k-2} \tag{54}$$

is the central charge.

6. Final Remarks

From the above development, it becomes evident that the Fourier series of the generators (31) leads to straightforward contact with string theory. However, the eigenvalues $j = \frac{1}{4}$ and $j = \frac{3}{4}$ which are predicted by (31) become hidden spin structures. This is part due that our discussion has been based on a bosonic perspective but in fact from the original discussion one knows that semionic particles with spin $j=\frac{1}{4}$ and $j=\frac{3}{4}$ behaves in very peculiar way. In particular, Zee [6] has studied the idea that the quasi-particles in the superconductors may be semions. Similarly, Sorokin and Volkov [8], using twistor-like formulation, obtained supersymmetric equations for semions with spin $j=\frac{1}{4}$ and $j=\frac{3}{4}$. Finally, Mezincescu and Townsend [7] develop the theory of semionic supersymmetric solitons. These works show, among other things, that the particles with spin $j=\frac{1}{4}$ and $j=\frac{3}{4}$ may be relevant at the supersymmetric level. As far as we know, besides the article [26], until now there seems not to be works considering $\left\{\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\right\}$-spin or in general dyadic $\frac{m}{2^n}$-spin and therefore it turns out that surreal supersymmetry remains as an open problem. But at least from the above comments it becomes clear that surreal supersymmetry and surreal *p*-branes must be the relevant route for further work.

Moreover, the present work may be of special relevant interest to be connected with other related subjects in theoretical physics scenario such as the ones described in the Refs [29] and [30] in which it investigated the generation of 1-spin field and the Pythagoreans figurative numbers, respectively.

Acknowledgments

JA Nieto would like to thank the Mathematical, Computational & Modeling Sciences Center of the Arizona State University where part of this work was developed. This work was partially supported by PROFAPI-UAS.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Ore, O. (1948) Number Theory and Its History. McGraw Hill Book Company, London.

[2] Conway, J.H. (1976) On Number and Games. London Mathematical Society Monographs, Academic Press, Cambridge.

[3] Knuth, D.E. (1974) Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness: A Mathematical Novelette. Addison-Wesley Publishing Co., Boston.

[4] Gonshor, H. (1986) An Introduction to the Theory of Surreal Numbers. London Mathematical Society Lectures Notes Series, Vol. 110, Cambridge Univ. Press, Cambridge. https://doi.org/10.1017/CBO9780511629143

[5] Avalos-Ramos, C., Félix-Algandar, J.A. and Nieto, J.A. (2020) Dyadic Rationals and Surreal Number Theory. *IOSR Journal of Mathematics* (*IOSR-JM*), **16**, 35-43.

[6] Zee, A. (1990) Semionics: A Theory of High Temperature Superconductivity. In:

Bedell, K., Coffey, D., Meltzer, D.E., Pines, D. and Schrieffer, J.R., Eds., *High Temperature Superconductivity*, Addison-Wesley, Boston, 206-212.

[7] Mezincescu, L. and Townsend, P.K. (2010) Anyons from Strings. *Physical Review Letters*, **105**, Article ID: 191601. https://doi.org/10.1103/PhysRevLett.105.191601

[8] Soroking, D. (2002) The Heisenberg Algebra and Spin, Fortsch. *Physics*, **50**, 724-728. https://doi.org/10.1002/1521-3978(200205)50:5/7<724::AID-PROP724>3.0.CO;2-J

[9] Hayes, B. (2008) Wagering with Zeno. *American Scientist*, **96**, 194-199. https://doi.org/10.1511/2008.71.3620

[10] Vepstas, L. (2004) The Minkowski Question Mark, GL(2,Z) and the Modular Group. https://www.academia.edu/3221306

[11] Dirac, P.A.M. (1928) The Quantum Theory of the Electron. *Proceedings of the Royal Society A*, **117**, 610-624. https://doi.org/10.1098/rspa.1928.0023

[12] Danehkar, A. (2019) Electric-Magnetic Duality in Gravity and Higher-Spin Fields. *Frontiers in Physics*, **6**, Article No. 146. https://doi.org/10.3389/fphy.2018.00146

[13] Ahluwalia, D.V. and Grumiller, D. (2005) Spin-Half Fermions with Mass Dimension One: Theory, Phenomenology, and Dark Matter. *Journal of Cosmology and Astroparticle Physics*, **7**, 12-72. https://doi.org/10.1088/1475-7516/2005/07/012

[14] Ahluwalia, D.V. and Grumiller, D. (2005) Dark Matter: A Spin One-Half Fermion Field with Mass Dimension One? *Physical Review D*, **72**, 67701-67705. https://doi.org/10.1103/PhysRevD.72.067701

[15] Bertone, G. (2010) Particle Dark Matter: Observations, Models and Searches. Cambridge University Press, Cambridge.

[16] Nieto, J.A. (2019) Generalized Elko Theory. *Modern Physics Letters A*, **34**, Article ID: 1950211. https://doi.org/10.1142/S0217732319502110

[17] Levay, P. (2005) On the Geometry of a Class of N-qubit Entanglement Monotones. *Journal Physics A*: *Mathematical General*, **38**, 9075-9085. https://doi.org/10.1088/0305-4470/38/41/016

[18] Björner, A., Las Vergnas, M., Sturmfels, B., White, N. and Ziegler, G.M. (1993) Oriented Matroids. Cambridge University Press, Cambridge.

[19] Nieto, J.A. (2004) Matroids and P-Branes. *Advances in Theoretical and Mathematical Physics*, **8**, 177-188. https://doi.org/10.4310/ATMP.2004.v8.n1.a4

[20] Nieto, J.A. (2006) Oriented Matroid Theory as a Mathematical Framework for M Theory. *Advances in Theoretical and Mathematical Physics*, **10**, 747-757. https://doi.org/10.4310/ATMP.2006.v10.n5.a5

[21] Nieto, J.A. (2004) Searching for a Connection between Matroid Theory and String Theory. *Journal of Mathematical Physics*, **45**, 285-301. https://doi.org/10.1063/1.1625416

[22] Nieto, J.A. (2014) Phirotopes, Super p-Branes and Qubit Theory. *Nuclear Physics B*, **883**, 350-372. https://doi.org/10.1016/j.nuclphysb.2014.04.001

[23] Nieto, J.A. and Marin, M.C. (2000) Matroid Theory and Chern-Simons. *Journal of Mathematical Physics*, **41**, 7997. https://doi.org/10.1063/1.1319518

[24] Nieto, J.A. (2013) Qubits and Oriented Matroids in Four Time and Four Space Dimensions. *Physics Letters B*, **718**, 1543-1547. https://doi.org/10.1016/j.physletb.2012.12.034

[25] Nieto, J.A. (2010) Qubits and Chirotopes. *Physics Letters B*, **692**, 43-46. https://doi.org/10.1016/j.physletb.2010.07.010

[26] Nieto, J.A. (2018) Cosmological Duality in Four Time and Four Space Dimensions.

Frontiers in Physics, **6**, Article No. 106.

[27] Hwang, S. (1991) No-Ghost Theorem for SU(1,1) String Theories. *Nuclear Physics B*, **354**, 100-112. https://doi.org/10.1016/0550-3213(91)90177-Y

[28] Maldacena, J. and Ooguri, H. (2001) Strings in AdS3 and the SL(2,R) WZW Model. I: The Spectrum. *Journal Mathematical Physics*, **42**, 2929-2959. https://doi.org/10.1063/1.1377273

[29] Srour, M.Kh., Sahmoud, A.N., Al-Mokayed, M.T., Sbaih, M.A.A. and Sumour, M.A. (2020) Generation of Spin 1 System. *Journal of Applied Mathematics and Physics*, **8**, 1706-1714. https://doi.org/10.4236/jamp.2020.89128

[30] Agarwal, R. (2021) Pythagoreans Figurative Numbers: The Beginning of Number Theory and Summation of Series. *Journal of Applied Mathematics and Physics*, **9**, 2038-2113. https://doi.org/10.4236/jamp.2021.98132