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Symmetrically Harmonic Kaluza-Klein Metrics on Tangent Bundles

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Abstract

Let (M,g) be a Riemannian manifold and G be a Kaluza-Klein metric on its tangent bundle TM. A metric H on TM is said to be symmetrically harmonic to G if the metrics G and H are harmonic w.r.t. each other; that is the identity maps $\mathrm{id}:(TM,G)\to(TM,H)$ and $\mathrm{id}:(TM,H)\to(TM,G)$ are both harmonic maps. In this work we study Kaluza-Klein metrics H on TM which are symmetrically harmonic to G. In particular, we characterize and determine horizontally and vertically conformal Kaluza-Klein metrics H on TM, which are symmetrically harmonic to G.

Keywords

Harmonic Maps, Kaluza-Klein Metrics, Conformal Metrics

1. Introduction

The geometry of the tangent bundle TM of a Riemannian manifold (M,g) started with the work of S. Sasaki in 1958. The Sasaki metric is naturally defined, it has been shown in many papers that a lot of geometric properties (like local symmetricity, having constant scalar curvature, being Einstein manifold, etc.) of tangent bundle with the Sasaki metric cannot be ensured unless the base manifold is local flat. Recall that when the base manifold is local flat, the tangent bundle with the Sasaki metric is local flat too. This rigidity leads mathematicians to search for other metrics like Cheeger-Gromoll metric ([1] [2] [3] [4] [5]). Later, Kaluza-Klein metrics are introduced on tangent bundles which generalize both Sasaki and Cheeger-Gromoll metrics (see [6] [7] [8] [9] for more information).

Furthermore, harmonic metrics arised from an interesting application of harmonic maps and have been introduced by the authors in [10]. Let $\phi:(M,g)\to(N,h)$ be an immersion between two Riemannian manifolds (M,g) and (N,h). If ϕ is a harmonic map, then ϕ^*h is a Riemannian metric on *M* such that the identity map $Id_M:(M,g)\to(M,\phi^*h)$ is a harmonic map [11]. Thus for a given Riemannian manifold (M,g), it became natural and interesting to seek for pseudo-Riemannian metrics $\overline{g}$ on *M* for which the identity map $Id_M:(M,g)\to(M,\overline{g})$ is a harmonic map. Such metrics are said to be harmonic with respect to the given metric *g*. The authors in [12], who introduced formally the notion, obtained an intrinsic characterization of harmonic metrics and used it to extend the definition of harmonicity to symmetric $(0,2)$-tensors.

Let G be a Kaluza-Klein metric on the tangent bundle TM of a Riemannian manifold (M,g). We study Kaluza-Klein metrics H on the tangent bundle TM such that G and H are harmonic with respect to each other. We then say that the metrics G and H are symmetrically harmonic. In the next section, Section 2, we give some basics and known results on the Kaluza-Klein metrics, and on harmonic metrics. In Section 3, we characterize symmetrically harmonic Kaluza-Klein metrics on (TM,G) and we study the special cases where such metrics are moreover horizontally and vertically conformal to *G*. We determine all these metrics for tangent bundles (TM,G) on Riemannian surfaces (M,g). In the case of *n*-dimensional Riemannian manifold with $n\geq 3$, we determine these metrics for a subclass of Kaluza-Klein metrics *G*.

2. Preliminaries

2.1. Kaluza-Klein Metrics on Tangent Bunbles

Let (M,g) be a Riemannian manifold and ∇ the Levi-Civita connection of *g*. The tangent space of TM at a point $(x,u)\in TM$ splits into the horizontal and vertical subspaces with respect to ∇:

$$T_{(x,u)}TM = H_{(x,u)}M \oplus V_{(x,u)}M.$$

A system of local coordinates $(U;x_i,\, i=1,\cdots,m)$ in M induces on TM a system of local coordinates $(\pi^{-1}(U);x_i,u^i,\, i=1,\cdots,m)$.

Let $X=\sum_{i=1}^m X^i\frac{\partial}{\partial x_i}$ be the local expression in U of a vector field X on *M*.

Then, the horizontal lift X^h and the vertical lift X^v of X are given.

With respect to the induced coordinates, by:

$$X^h=\sum_i X^i\frac{\partial}{\partial x_i}-\sum_{i,j,k}\Gamma^i_{jk}u^jX^k\frac{\partial}{\partial u^i} \tag{1}$$

and

$$X^v=\sum_i X^i\frac{\partial}{\partial u^i}, \tag{2}$$

where the Γ^i_{jk} is Christoffel's symbols defined by *g*.

Next, we introduce some notations which will be used to describe vectors obtained from lifted vectors by basic operations on *TM*. Let *T* be a tensor field of type $(1,s)$ on *M*. If $X_1, X_2, \cdots, X_{s-1} \in T_x M$, then $h\{T(X_1,\cdots,u,\cdots,X_{s-1})\}$ and $v\{T(X_1,\cdots,u,\cdots,X_{s-1})\}$) are horizontal and vertical vectors respectively at the point (x,u) which are defined by:

$$h\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum u^\lambda \left(T\left(X_1,\cdots,\frac{\partial}{\partial x_{\lambda\,|x}},\cdots,X_{s-1}\right)\right)^h$$

$$v\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum u^\lambda \left(T\left(X_1,\cdots,\frac{\partial}{\partial x_{\lambda\,|x}},\cdots,X_{s-1}\right)\right)^v.$$

In particular, if *T* is the identity tensor of type $(1,1)$, then we obtain the geodesic flow vector field at (x,u), $\xi_{(x,u)} = \sum_\lambda u^\lambda \left(\frac{\partial}{\partial x_\lambda}\right)^h_{(x,u)}$, and the canonical vertical vector at (x,u), $\mathcal{U}_{(x,u)} = \sum_\lambda u^\lambda \left(\frac{\partial}{\partial x_\lambda}\right)^v_{(x,u)}$.

Also $h\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ and $v\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ are defined by similar way.

Let us introduce the notations

$$h\{T(X_1,\cdots,X_s)\} =: T(X_1,\cdots,X_s)^h \tag{3}$$

and

$$v\{T(X_1,\cdots,X_s)\} =: T(X_1,\cdots,X_s)^v. \tag{4}$$

Thus $h\{X\} = X^h$ and $v\{X\} = X^v$, for each vector field *X* on *M*.

From the preceding quantities, one can define vector fields on *TU* in the following way: If $u = \sum_i u^i \left(\frac{\partial}{\partial x_i}\right)_x$ is a given point in *TU* and $X_1,\cdots,X_{s-1}$ are vector fields on *U*, then we denote by

$$h\{T(X_1,\cdots,u,\cdots,X_{s-1})\} \quad \left(\text{respectively } v\{T(X_1,\cdots,u,\cdots,X_{s-1})\}\right)$$

the horizontal (respectively vertical) vector field on *TU* defined by

$$h\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum_\lambda u^\lambda T\left(X_1,\cdots,\frac{\partial}{\partial x_\lambda},\cdots,X_{s-1}\right)^h$$

$$\left(\text{resp. } v\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum_\lambda u^\lambda T\left(X_1,\cdots,\frac{\partial}{\partial x_\lambda},\cdots,X_{s-1}\right)^v\right).$$

Moreover, for vector fields $X_1,\cdots,X_{s-t}$ on *U*, where $s,t \in \mathbb{N}^*(s>t)$, the vector fields $h\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ and $v\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ on *TU*, are defined by similar way.

Definition 2.1 *Let* (M,g) *be a Riemannian manifold and* $\dim M \ge 2$. *A metric G on TM is called a Kaluza-Klein metric induced by the metric g, if there exists three functions* α_0, α_1 *and* $\beta: \mathbb{R}_+ \to \mathbb{R}$, *such that for any* $x \in M$ *and* *all vectors* $u, X, Y \in T_x M$ *we have*:

$$G\left(X^h, Y^h\right) = \left(\alpha_1 + \alpha_0\right)(t) g(X, Y), \tag{5}$$

$$G\left(X^h, Y^v\right) = 0,$$

$$G\left(X^v, Y^v\right) = \alpha_1(t) g(X, Y) + \beta(t) g(X, u) g(Y, u);$$

where $t = g_x(u, u)$, X^h and X^v are respectively the horizontal lift and the vertical lift of the vector $X \in T_x M$ at the point $(x, u) \in TM$.

For $\dim M = 1$, the same holds with $\beta = 0$.

Notations 2.2

- $\phi_0(t) = \alpha_0(t) - t\beta(t), \forall t \in \mathbb{R}_+$;
- $\phi_1(t) = \alpha_1(t) + t\beta(t), \forall t \in \mathbb{R}_+$;
- $\alpha = \alpha_1(\alpha_1 + \alpha_0)$;
- $\phi = \phi_1(\phi_1 + \phi_0)$.

Proposition 2.3 *Let* (M, g) *be a Riemannian manifold. A Kaluza-Klein metric on the tangent bundle TM of* (M, g) *and defined by the functions* α_0, α_1 *and* β *of Defintion 2.1, is Riemannian if and only if*:

$$\alpha_1(t) > 0, \phi_1(t) > 0 \text{ and } \left(\alpha_1 + \alpha_0\right)(t) > 0, \forall t \in \mathbb{R}_+. \tag{6}$$

Next, we are going to present the inverse of a Riemannian Kaluza-Klein metric.

Let $(U, x_i, i = 1, \cdots, m)$ be a local coordinates system of a Riemannian manifold (M, g) and $\left(\pi^{-1}(U); x_i, u^i, i = 1, \cdots, m\right)$ its induced coordinates system on *TM*.

Let G be a Riemannian Kaluza-Klein metric on *TM* defined by the functions α_0, α_1 and β of the Definition 2.1. Let us consider the matrix-valued functions on $\pi^{-1}(U)$ defined by

$$M_0 = \left(\alpha_0 g_{ij} - \beta u_i u_j\right), 1 \le i, j \le m, \tag{7}$$

$$M_1 = \left(\alpha_1 g_{ij} + \beta u_i u_j\right), 1 \le i, j \le m; \tag{8}$$

where g_{ij} and u_i are the functions on $\pi^{-1}(U)$ given by $g_{ij} = g \circ \pi\left(\partial_{x_i}, \partial_{x_j}\right)$, $u_i = g_{ik} u^k$ and $\partial_{x_i} = \dfrac{\partial}{\partial_{x_i}}; i, j = 1, \cdots, m$.

So $\begin{pmatrix} M_1 + M_0 & 0 \\ 0 & M_1 \end{pmatrix}$ is the matrix-valued functions of $G_{|_{\pi^{-1}(U)}}$ with respect to the local frame $\left(\partial_{x_i}^h, \partial_{x_i}^v\right)_{i=1,\cdots,m}$ on $\pi^{-1}(U)$. We shall denote

$$G \equiv \begin{pmatrix} M_1 + M_0 & 0 \\ 0 & M_1 \end{pmatrix}. \tag{9}$$

If G is a Riemannian Kaluza-Klein metric, its inverse G^{-1} has the form

$$G^{-1} \equiv \begin{pmatrix} \Lambda & 0 \\ 0 & \Omega \end{pmatrix}, \tag{10}$$

where $\Lambda = \left(\lambda^{ij}\right)_{1\le i,j\le m}$ and $\Omega = \left(\omega^{ij}\right)_{1\le i,j\le m}$ are square matrix-valued functions of order *m*, defined on $\pi^{-1}(U)$.

Proposition 2.4 [13] *If G is a Riemannian Kaluza-Klein metric, the elements of the matrix-valued functions in* (*10*) *are given on* $\pi^{-1}(U)$ *by*

$$\lambda^{ij} = \frac{\alpha_1(t)}{\alpha(t)} g^{ij}, \tag{11}$$

$$\omega^{ij} = \frac{(\alpha_1+\alpha_0)(t)}{\alpha(t)} g^{ij} - \frac{\beta(t)}{\alpha_1(t)\phi_1(t)} u^i u^j, \tag{12}$$

where $\left(g^{ij}\right)_{1\le i,j\le m}$ denotes the inverse of $g \equiv \left(g_{ij}\right)_{1\le i,j\le m}$, $t = g(u,u)$ for any $p \in U$ and for any $u = u^k \partial_{x_k} \in T_p M$. Then the blocks of the matrix-valued functions in (10) satisfy:

$$\Lambda(p,u) \equiv \left(\lambda^{ij}(p,u)\right)_{1\le i,j\le m}, \tag{13}$$

$$\Omega(p,u) \equiv \left(\omega^{ij}(p,u)\right)_{1\le i,j\le m}; \tag{14}$$

for all $p \in U$, $u = \sum_{i=1}^m u^i \partial_{x_i}|_p \in T_p M$. Furthermore, the Levi-Civita connection $\tilde{\nabla}$ of a Riemannian Kaluza-Klein metric is defined by the following Proposition. Let $(x,u) \in TM$ and $X, Y \in \mathfrak{X}(M)$, we have

$$\left(\tilde{\nabla}_{X^h} Y^h\right)_{(x,u)} = \left(\nabla_X Y\right)^h_{(x,u)} + v\left\{B\left(u; X_x, Y_x\right)\right\}, \tag{15}$$

$$\left(\tilde{\nabla}_{X^h} Y^v\right)_{(x,u)} = \left(\nabla_X Y\right)^v_{(x,u)} + h\left\{C\left(u; X_x, Y_x\right)\right\}, \tag{16}$$

$$\left(\tilde{\nabla}_{X^v} Y^h\right)_{(x,u)} = h\left\{C\left(u; Y_x, X_x\right)\right\}, \tag{17}$$

$$\left(\tilde{\nabla}_{X^v} Y^v\right)_{(x,u)} = v\left\{F\left(u; Y_x, X_x\right)\right\}; \tag{18}$$

where

$$B(u;X,Y) = -\frac{1}{2} R(X,u)Y + \frac{1}{2} R(Y,u)X - \frac{(\alpha_1+\alpha_0)'}{\phi_1} g(X,Y)u, \tag{19}$$

$$C(u;X,Y) = -\frac{\alpha_1}{2(\alpha_1+\alpha_0)} R(Y,u)X + \frac{(\alpha_1+\alpha_0)'}{\alpha_1+\alpha_0} g(Y,u)X, \tag{20}$$

$$\begin{aligned} F(u;X,Y) = {} & \frac{\alpha_1'}{\alpha_1} g(X,u)Y + \frac{\alpha_1'}{\alpha_1} g(Y,u)X + \left(\frac{\beta-\alpha_1'}{\phi_1}\right) g(X,Y)u \\ & + \frac{1}{\phi_1}\left(\beta' - 2\frac{\alpha_1'\beta}{\alpha_1}\right) g(X,u) g(Y,u) u; \end{aligned} \tag{21}$$

for all $x \in M$ and $u, X, Y \in T_x M$, ∇ is the Levi-Civita connection of *g*, and *R* is the Riemannian curvature of *g*.

2.2. Harmonic Maps, Harmonic Metrics

Definition 2.5 *Let* $\varphi:(M,g) \to (N,h)$ *be a* C^2 *map between two Riemannian manifolds* (M,g) *and* (N,h) *with compact support. The energy density of* φ, *denoted by* $e(\varphi)$ *is defined by:*

$$e(\varphi) = \frac{|d\varphi|^2}{2},$$

where $|d\varphi|$ is the Hilbert-Schmidt norm of $d\phi$ induced by the metrics g and h on $T^*(M) \otimes \varphi^{-1}TN$ that is defined by:

$$|d\varphi|^2 = tr_g(\varphi^* h).$$

In local coordinates, $e(\varphi) = \frac{g^{ij}}{2} h_{\alpha\beta} \partial_i \varphi^\alpha \partial_j \varphi^\beta$. The Dirichlet energy of φ, over M is defined by

$$E(\varphi) = \int_M e(\varphi) dv_g,$$

where dv_g is the volume measure induced by g.

The map φ is said to be harmonic, if it is a critical point of the energy functionnal E.

In the case where the map φ has a noncompact support, the map φ is said to be harmonic if its restriction to any compact subset of M is harmonic.

The Euler-Lagrange equations with respect to the energy functional E obtained by the first variation formula give rise to the following characterization: the map $\varphi : (M, g) \to (N, h)$ is harmonic if and only if its tension field $\tau(\phi)$ vanishes identically, where $\tau(\phi)$ is the contraction w.r.t. g of the second fundamental form $\nabla^{\varphi^{-1}TN} d\varphi$ of φ defined by

$$\nabla^{\varphi^{-1}TN} d\varphi(X, Y) = \left(\nabla_X^{\varphi^{-1}TN} d\varphi\right)(Y)$$
$$= \nabla^N_{d\varphi(X)} d\varphi(Y) - d\varphi\left(\nabla_X^M Y\right), \forall X, Y \in \Gamma(TM),$$

with ∇^M and ∇^N the Levi-Civita connections on (M, g) and (N, h) respectively.

In local coordinates $(x_i)_{i=1}^m$ at $x \in M$ and $(u^\alpha)_{\alpha=1}^n$ at $f(x) \in N$, the Euler-Lagrange equations are given by the system:

$$-\Delta_g f^\alpha + g^{ij} \overline{\Gamma}^\alpha_{\gamma\beta} \frac{\partial f^\gamma}{\partial x_j} \frac{\partial f^\beta}{\partial x_i} = 0 \text{ for all } \alpha = 1, \cdots, n,$$

where Δ_g is the Laplace-operator on (M, g) and $\overline{\Gamma}^\alpha_{\gamma\beta}$ the Christoffel symbols of (N, h).

Let (M, g) be an m-dimensional Riemannian manifold. It is easy to check that the identity map $Id_M : (M, g) \to (M, g)$ is harmonic. However if we consider another metric h on M, then the identity map $Id_M : (M, g) \to (M, h)$ is not any more automatically harmonic. A metric h on M is said to be harmonic w.r.t. g if the identity map $Id : (M, g) \to (M, h)$ is harmonic.

In a local coordinate system $(x_i)_{i=1}^m$ on M, the metric h is harmonic w.r.t. g if and only if:

$$g^{ij}\left(\overline{\Gamma}^k_{ij} - \Gamma^k_{ij}\right) = 0, k = 1, \cdots, m,$$

where Γ^k_{ij} and $\overline{\Gamma}^k_{ij}$ are the Christoffel symbols w.r.t. g and h respectively.

Furthermore

$$\tau = g^{ij}\left(\overline{\Gamma}^k_{ij} - \Gamma^k_{ij}\right)\partial_{x_k}, \tag{22}$$

is called the tension field of the identity map $Id_M : (M, g) \to (M, h)$.

Equivalently the metric h is harmonic w.r.t. g if and only if

$$d(tr_g h) + 2\delta h = 0,$$

where d and δ are the differential and the codifferential operators defined on (M, g) respectively. From this characterization, a symmetric $(0,2)$-tensor T on (M, g) is said to be harmonic with respect to g if it satisfies equation

$$d(tr_g T) + 2\delta T = 0.$$

Some interesting results have been obtained on harmonic symmetric $(0,2)$-tensors by some authors like in [14].

Let us notice that the relation "be harmonic to" between metrics is not a symmetric relation; the fact that the metric h is harmonic w.r.t. g does not imply that g is harmonic w.r.t. h.

3. Symmetrically Harmonic Kaluza-Klein Metrics

Definition 3.1 *Let (M, g) be a Riemannian manifold, G, $\bar{G}$ be two Riemannian Kaluza-Klein metrics on its tangent bundle TM, respectively defined by the functions $(\alpha_0, \alpha_1, \beta)$ and $(\bar{\alpha}_0, \bar{\alpha}_1, \bar{\beta})$ as in Definition* 2.1. *The metrics G and $\bar{G}$ are said to be symmetrically harmonic metrics if the metric $\bar{G}$ is harmonic with respect to G, and the metric G is harmonic with respect to $\bar{G}$.*

Let (M, g) be a Riemannian manifold and G, $\bar{G}$ be two Riemannian Kaluza-Klein metrics on its tangent bundle TM, respectively defined by the functions $(\alpha_0, \alpha_1, \beta)$ and $(\bar{\alpha}_0, \bar{\alpha}_1, \bar{\beta})$ as in Definition 2.1 with $\phi_1 = \alpha_1 + t\beta$, $\bar{\phi}_1 = \bar{\alpha}_1 + t\bar{\beta}$, $\forall t \in \mathbb{R}_+$. Then by direct computations, we obtain the tension field τ of the identity map $Id_{TM} : (TM, G) \to (TM, \bar{G})$ as follows:

$$\begin{aligned}\tau = \Bigg\{ & \left(-\frac{(\bar{\alpha}_1 + \bar{\alpha}_0)'}{\bar{\phi}_1} + \frac{(\alpha_1 + \alpha_0)'}{\phi_1} \right) \frac{n}{\alpha_1 + \alpha_0} + 2\left(\frac{\bar{\alpha}_1'}{\bar{\alpha}_1} - \frac{\alpha_1'}{\alpha_1} \right)\left(\frac{1}{\alpha_1} - g(u,u)\frac{\beta}{\alpha_1\phi_1} \right) \\ & + \left(\frac{\bar{\beta} - \bar{\alpha}_1'}{\bar{\phi}_1} - \frac{\beta - \alpha_1'}{\phi_1} \right)\left(\frac{n}{\alpha_1} - g(u,u)\frac{\beta}{\alpha_1\phi_1} \right) \\ & + \left[\left(\frac{\bar{\beta}'}{\bar{\phi}_1} - 2\frac{\bar{\alpha}_1'\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1} \right) - \left(\frac{\beta'}{\phi_1} - 2\frac{\alpha_1'\beta}{\alpha_1\phi_1} \right) \right]\left(\frac{1}{\alpha_1} g(u,u) - g(u,u)^2 \frac{\beta}{\alpha_1\phi_1} \right) \Bigg\} u^v_{(x,u)},\end{aligned} \tag{23}$$

for all $x \in M$ and for all $u \in TM_x$; where $n = \dim M$, $n \geq 2$.

Then we obtain the tension field $\bar{\tau}$ of the identity map $Id_{TM} : (TM, \bar{G}) \to (TM, G)$ as follows:

$$\begin{aligned}\bar{\tau} = \Bigg\{ & \left(-\frac{(\alpha_1 + \alpha_0)'}{\phi_1} + \frac{(\bar{\alpha}_1 + \bar{\alpha}_0)'}{\bar{\phi}_1} \right) \frac{n}{\bar{\alpha}_1 + \bar{\alpha}_0} + 2\left(\frac{\alpha_1'}{\alpha_1} - \frac{\bar{\alpha}_1'}{\bar{\alpha}_1} \right)\left(\frac{1}{\bar{\alpha}_1} - g(u,u)\frac{\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1} \right) \\ & + \left(\frac{\beta - \alpha_1'}{\phi_1} - \frac{\bar{\beta} - \bar{\alpha}_1'}{\bar{\phi}_1} \right)\left(\frac{n}{\bar{\alpha}_1} - g(u,u)\frac{\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1} \right) \\ & + \left[\left(\frac{\beta'}{\phi_1} - 2\frac{\alpha_1'\beta}{\alpha_1\phi_1} \right) - \left(\frac{\bar{\beta}'}{\bar{\phi}_1} - 2\frac{\bar{\alpha}_1'\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1} \right) \right]\left(\frac{1}{\bar{\alpha}_1} g(u,u) - g(u,u)^2 \frac{\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1} \right) \Bigg\} u^v_{(x,u)},\end{aligned} \tag{24}$$

for all $x \in M$ and for all $u \in T_x M$; where $n = \dim M$, $n \geq 2$.

It follows that:

Proposition 3.2 *Let* (M, g) *be a n-dimensional Riemannian manifold with* $n \geq 2$, *G and* $\bar{G}$ *be two Riemannian Kaluza-Klein metrics TM, respectively defined by the functions* $(\alpha_0, \alpha_1, \beta)$ *and* $(\bar{\alpha}_0, \bar{\alpha}_1, \bar{\beta})$ *as in Definition* 2.1 *with* $\dim M = n \geq 2$ *and* $\phi_1(t) = \alpha_1(t) + t\beta(t)$, $\bar{\phi}(t) = \bar{\alpha}_1(t) + t\bar{\beta}(t)$ *for all* $t \in \mathbb{R}_+$.

Then the Kaluza-Klein metrics G and $\bar{G}$ are symmetrically harmonic if and only if

$$\begin{aligned}&\left(-\frac{(\bar{\alpha}_1+\bar{\alpha}_0)'}{\bar{\phi}_1}+\frac{(\alpha_1+\alpha_0)'}{\phi_1}\right)\frac{n}{\alpha_1+\alpha_0}+2\left(\frac{\bar{\alpha}_1'}{\bar{\alpha}_1}-\frac{\alpha_1'}{\alpha_1}\right)\left(\frac{1}{\alpha_1}-t\frac{\beta}{\alpha_1\phi_1}\right)\\&+\left(\frac{\bar{\beta}-\bar{\alpha}_1'}{\bar{\phi}_1}-\frac{\beta-\alpha_1'}{\phi_1}\right)\left(\frac{n}{\alpha_1}-t\frac{\beta}{\alpha_1\phi_1}\right)\\&+\left[\left(\frac{\bar{\beta}'}{\bar{\phi}_1}-2\frac{\bar{\alpha}_1'\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1}\right)-\left(\frac{\beta'}{\phi_1}-2\frac{\alpha_1'\beta}{\alpha_1\phi_1}\right)\right]\left(\frac{1}{\alpha_1}t-t^2\frac{\beta}{\alpha_1\phi_1}\right)=0\end{aligned}\tag{25}$$

and

$$\begin{aligned}&\left(-\frac{(\alpha_1+\alpha_0)'}{\phi_1}+\frac{(\bar{\alpha}_1+\bar{\alpha}_0)'}{\bar{\phi}_1}\right)\frac{n}{\bar{\alpha}_1+\bar{\alpha}_0}+2\left(\frac{\alpha_1'}{\alpha_1}-\frac{\bar{\alpha}_1'}{\bar{\alpha}_1}\right)\left(\frac{1}{\bar{\alpha}_1}-t\frac{\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1}\right)\\&+\left(\frac{\beta-\alpha_1'}{\phi_1}-\frac{\bar{\beta}-\bar{\alpha}_1'}{\bar{\phi}_1}\right)\left(\frac{n}{\bar{\alpha}_1}-t\frac{\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1}\right)\\&+\left[\left(\frac{\beta'}{\phi_1}-2\frac{\alpha_1'\beta}{\alpha_1\phi_1}\right)-\left(\frac{\bar{\beta}'}{\bar{\phi}_1}-2\frac{\bar{\alpha}_1'\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1}\right)\right]\left(\frac{1}{\bar{\alpha}_1}t-t^2\frac{\bar{\beta}}{\bar{\alpha}_1\bar{\phi}_1}\right)=0\end{aligned}\tag{26}$$

for all $t \in \mathbb{R}_+$, where the functions $\alpha_0, \alpha_1, \beta, \phi_1, \bar{\alpha}_0, \bar{\alpha}_1, \bar{\beta}, \bar{\phi}_1$ and their derivatives are evaluated at *t*.

Conformally and Symmetrically Harmonic Kaluza-Klein Metrics

Definition 3.3 *Let* (M, g) *be a Riemannian manifold, G and* $\bar{G}$ *be two Kaluza-Klein metrics on the tangent bundle TM.*

The metrics G and $\bar{G}$ on *TM* are said to be respectively horizontally and vertically conformal if there exist two positive functions l_0 and l_1 defined on $\mathbb{R}_+$ such that

$$\bar{G}(X^h, Y^h) = l_0(t) G(X^h, Y^h), \tag{27}$$

$$\bar{G}(X^v, Y^v) = l_1(t) G(X^v, Y^v); \tag{28}$$

for all $x \in M$ and for all $u, X, Y \in T_x M$, where $t = g(u, u)$, X^h and X^v are respectively the horizontal lift and the vertical lift of the vector $X \in T_x M$ at the point $(x, u) \in TM$.

By direct computations using Proposition 3.2 and Definition 3.3 we have:

Proposition 3.4 *Let* (M, g) *be an n-dimensional Riemannian manifold with* $n \geq 2$, *G and* $\bar{G}$ *be two Kaluza-Klein metrics on the tangent bundle TM such* *as G and $\bar{G}$ are horizontally and vertically conformal with conformal factors l_0 and l_1 as in Definition* 3.3.

Then the Kaluza-Klein metrics G and $\bar{G}$ are symmetrically harmonic if and only if

$$nl_0' + (n-2)l_1' + n\frac{(\alpha_1+\alpha_0)'}{\alpha_1+\alpha_0}(l_0 - l_1) = 0 \tag{29}$$

and

$$nl_0'l_1 + (n-2)l_1'l_0 + n\frac{(\alpha_1+\alpha_0)'}{\alpha_1+\alpha_0}l_1(l_0 - l_1) = 0, \tag{30}$$

where $(\alpha_1+\alpha_0)$ is the positive function such as

$$G(X^h, Y^h) = (\alpha_1+\alpha_0)(t)g(X,Y),$$

for all $x \in M$ and for all $u, X, Y \in T_xM$.

In the cases of tangent bundles on Riemannian surfaces (M,g), we obtain the following result:

Theorem 3.5 *Let (M,g) be a Riemannian surface, G and $\bar{G}$ be two horizontally and vertically conformal Kaluza-Klein metrics TM with conformal factors l_0 and l_1 as in Definition* 3.3.

Then the metrics G and $\bar{G}$ are symmetrically harmonic if and only if $l_0 > 0$, $l_1 > 0$ and there exists a constant real number k_0 such as

$$l_0 = \frac{k_0 + \int_0^t (\alpha_1+\alpha_0)'(s)l_1(s)\mathrm{d}s}{(\alpha_1+\alpha_0)(t)}; \tag{31}$$

where $\alpha_1+\alpha_0$ is the positive function such as $G(X^h, Y^h) = (\alpha_1+\alpha_0)(t)g(X,Y)$ for all $x \in M$ and for all $u, X, Y \in T_xM$, with $t = g(u,u)$ and X^h is the horizontal lift of X at (x,u).

Proof.

Let (M,g) be a Riemannian manifold such as $\dim M = n = 2$. Let G and $\bar{G}$ be two Riemannian Kaluza-Klein metrics induced by g on the tangent bundle TM such as

$$\bar{G}(X^h, Y^h) = l_0(t)G(X^h, Y^h),$$
$$\bar{G}(X^v, Y^v) = l_1(t)G(X^v, Y^v);$$

for all $x \in M$ and for all $u, X, Y \in T_xM$, where $t = g(u,u)$, X^h and X^v are respectively the horizontal lift and the vertical lift of the vector $X \in T_xM$ at the point $(x,u) \in TM$.

Then G and $\bar{G}$ are symmetrically harmonic if and only if Equations (29) and (30) hold. Now these equations are equivalent to the linear differential equation of first degree with second member

$$l_{0'} + \frac{(\alpha_1+\alpha_0)'}{\alpha_1+\alpha_0}l_0 = \frac{(\alpha_1+\alpha_0)'}{\alpha_1+\alpha_0}l_1. \tag{32}$$

So by solving this differential equation by the method of variation of constants by example, we obtain the solutions in (31). This completes the proof.

□

Remark 3.6

1) If $l_1 = 1$ and $k_0 = (\alpha_1 + \alpha_0)(0) > 0$ in (31) then $l_0 = 1$, and we obtain then the trivial case $\bar{G} = G$.

2) In the cases where G is Sasaki metric or Cheeger-Gromoll metric induced by *g*, we have $(\alpha_1 + \alpha_0)' = 0$ in (31), so for any positive function l_1 on $\mathbb{R}_+$ and any positive constant real number l_0, the pair (l_0, l_1) induces a horizontally and vertically conformal Kaluza-Klein metric $\bar{G}$ which is symmetrically harmonic to *G*.

For a special class of Kaluza-Klein metrics we have the following result:

Theorem 3.7 *Let* (M, g) *be a Riemannian manifold with* $\dim M \geq 3$, *G be a Kaluza-Klein metric induced on TM such that the sum* $(\alpha_1 + \alpha_0)$ *of Definition* 2.1 *is constant positive on* $\mathbb{R}_+$ *with* $G(X^h, Y^h) = (\alpha_1 + \alpha_0) g(X, Y)$, *for all* $x \in M$ *and for all* $u, X, Y \in T_x M$, *with* X^h *is the horizontal lift of X at* (x, u).

Let $\bar{G}$ be a Riemannian Kaluza-Klein on *TM* and which is horizontally and vertically conformal to G with conformal factors l_0 and l_1 as in Definition 3.3.

Then the metrics G and $\bar{G}$ are symmetrically harmonic if and only if there exist two positive constant real numbers k_0 and k_1 such that:

1)

$$l_0 = \frac{n-2}{2(n-1)}\left(k_0 - \sqrt{k_0^2 - 4k_1\left(\frac{n-1}{n-2}\right)}\right) \tag{33}$$

and

$$l_1 = -\frac{n}{n-2} l_0 + k_0, \tag{34}$$

with $0 < k_1 \leq \left(\frac{n-2}{4(n-1)}\right) k_0^2$;

Or

2)

$$l_0 = \frac{n-2}{2(n-1)}\left(k_0 + \sqrt{k_0^2 - 4k_1\left(\frac{n-1}{n-2}\right)}\right) \tag{35}$$

and

$$l_1 = -\frac{n}{n-2} l_0 + k_0, \tag{36}$$

with $\left(\frac{n-2}{n^2}\right) k_0^2 < k_1 \leq \frac{n-2}{4(n-1)} k_0^2$.

Proof.

Let (M, g) be a Riemannian manifold such as $\dim M = n \geq 3$. Let G be a

Riemannian Kaluza-Klein metric induced by g on TM such as it exits a constant positive function $(\alpha_1+\alpha_0)$ on $\mathbb{R}_+$ and

$$G\left(X^h,Y^h\right)=(\alpha_1+\alpha_0)g(X,Y), \tag{37}$$

for all $x\in M$ and for all $u,X,Y\in T_xM$, with X^h is the horizontal lift of X at (x,u). Let $\bar{G}$ be a Riemannian Kaluza-Klein metric on the tangent bundle TM induced by g such as

$$\bar{G}\left(X^h,Y^h\right)=l_0(t)G\left(X^h,Y^h\right),$$
$$\bar{G}\left(X^v,Y^v\right)=l_1(t)G\left(X^v,Y^v\right);$$

for all $x\in M$ and for all $u,X,Y\in T_xM$, where $t=g(u,u)$, X^h and X^v are respectively the horizontal lift and the vertical lift of the vector $X\in T_xM$ at the point $(x,u)\in TM$, and l_0,l_1 are positive functions defined on $\mathbb{R}_+$.

Then G and $\bar{G}$ are symmetrically harmonic if and only if Equations (29) and (30) hold. Now $(\alpha_1+\alpha_0)'=0$ so these equations are equivalent

$$nl_0'+(n-2)l_1'=0 \tag{38}$$

$$nl_0'l_1+(n-2)l_0l_1'=0, \tag{39}$$

$$l_0>0 \text{ and } l_1>0. \tag{40}$$

Then since $n\geq 3$, Equation (38) gives

$$l_1'=-\frac{n}{n-2}l_0'. \tag{41}$$

So there exists a constant real number k_0 such that

$$l_1=-\frac{n}{n-2}l_0+k_0, \tag{42}$$

and necessarily k_0 is positive since $\frac{n}{n-2}$, l_0 and l_1 are positive. Furthermore, by inserting the Formulas (41) and (42) in the differential Equation (39), we obtain

$$-2\frac{n-1}{n-2}l_0'l_0+k_0l_0'=0. \tag{43}$$

So there exists a constant real number k_1 such

$$-\left(\frac{n-1}{n-2}\right)l_0^2+k_0l_0=k_1. \tag{44}$$

That means, l_0 is a solution of a polynomial equation of degree two

$$\left(\frac{n-1}{n-2}\right)X^2-k_0X+k_1=0, \tag{45}$$

with $k_0>0$ and $k_1\in\mathbb{R}$.

Then to determine l_0 and l_1 with respect to Equations (40), (42) and (44) we have the following cases:

1) If $k_1\leq 0$, then $k_0^2-4k_1\left(\frac{n-1}{n-2}\right)\geq 0$, and since $\frac{n-1}{n-2}>0$, Equation (44)

has an unique positive solution

$$l_0 = \frac{n-2}{2(n-1)}\left(k_0 + \sqrt{k_0^2 - 4k_1\left(\frac{n-1}{n-2}\right)}\right). \tag{46}$$

So by Equation (42), we obtain

$$l_1 = \frac{1}{2(n-1)}\left(\frac{-4(n-1)k_0^2 + 4k_1 n^2\left(\frac{n-1}{n-2}\right)}{(n-2)k_0 + n\sqrt{k_0^2 - 4k_1\left(\frac{n-1}{n-2}\right)}}\right) < 0. \tag{47}$$

This is absurd.

So there is no (l_0, l_1) which satisfies Equations (40), (42) and (44).

2) If $k_1 > \frac{n-2}{4(n-1)}k_0^2$ then the polynomial Equation (45) has no solution. So there is no (l_0, l_1) which satisfies Equations (40), (42) and (44).

3) If $0 < k_1 \le \frac{n-2}{n^2}k_0^2$ then $k_1 \le \frac{n-2}{n^2}k_0^2 < \frac{n-2}{4(n-1)}k_0^2$ (since $k_0 > 0$ and $n \ge 3$) and the polynomial Equation (45) has the positive solutions (since k_0, k_1 and $\frac{n-1}{n-2}$ are positive):

$$a = \frac{n-2}{2(n-1)}\left(k_0 - \sqrt{k_0^2 - 4k_1\left(\frac{n-1}{n-2}\right)}\right) \tag{48}$$

and

$$a^* = \frac{n-2}{2(n-1)}\left(k_0 + \sqrt{k_0^2 - 4k_1\left(\frac{n-1}{n-2}\right)}\right). \tag{49}$$

So if $l_0 = a$ then by (42),

$$l_1 = \frac{1}{2(n-1)}\left((n-2)k_0 + n\sqrt{k_0^2 - 4k_1\left(\frac{n-1}{n-2}\right)}\right) > 0, \tag{50}$$

and $(l_0, l_1) = \left(a, -\frac{n}{n-2}a + k_0\right)$ satisfies Equations (40), (42) and (44).

Furthermore, if $l_0 = a^*$ then by (42),

$$l_1 = 2\left(\frac{-k_0^2 + k_1\frac{n^2}{n-2}}{(n-2)k_0 + n\sqrt{k_0^2 - 4k_1\left(\frac{n-1}{n-2}\right)}}\right) \le 0. \tag{51}$$

So $(l_0, l_1) = \left(a^*, -\frac{n}{n-2}a^* + k_0\right)$ does not satisfy Equations (40), (42) and (44).

So for this case, we obtain an unique solution (l_0, l_1) for Equations (40), (42) and (44) with l_0, l_1 constant positive functions on $\mathbb{R}_+$.

4) If $\frac{n-2}{n^2}k_0^2 < k_1 \le \frac{n-2}{4(n-1)}k_0^2$ then the polynomial Equation (45) has the

positive solutions a and a^* respectively in (48) and (49), with by (42) we have

$$-\frac{n}{n-2}a+k_0=\frac{1}{2(n-1)}\left((n-2)k_0+n\sqrt{k_0^2-4k_1\left(\frac{n-1}{n-2}\right)}\right)>0$$

$$-\frac{n}{n-2}a^*+k_0=2\left(\frac{-k_0^2+k_1\frac{n^2}{n-2}}{(n-2)k_0+n\sqrt{k_0^2-4k_1\left(\frac{n-1}{n-2}\right)}}\right)>0.$$

So each $(l_0,l_1)\in\left\{\left(a,-\frac{n}{n-2}a+k_0\right);\left(a^*,-\frac{n}{n-2}a^*+k_0\right)\right\}$ satisfies Equations (40), (42) and (44) with l_0,l_1 constant positive functions on $\mathbb{R}_+$.

Conversely if (l_0,l_1) is a pair of positive functions satisfying cases 3. or 4. then l_0 and l_1 are constant functions and therefore Equations (38), (39) and (40) are satisfied. This completes the proof.

□

Remark

1) If $k_0=\frac{2(n-1)}{n-2}$ and $k_1=\frac{n-1}{n-2}$ in (33) and (34) we obtain the trivial case $\bar{G}=G$.

2) Sasaki and Cheeger-Gromoll metrics are examples of Kaluza-Klein metrics on tangent bundles satisfying the hypothesis $(\alpha_1+\alpha_0)'=0$ of Theorem 3.7.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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