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Optimal Management of the Pension Contribution Rate Using the H_∞-Control Theory

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Abstract

In this paper, we construct a general model for a pension fund when there are time delays in the valuation process. Actually, we use the standard structure of the basic reserve equation in order to rebuild a more sophisticated approach based on the theory of H_∞ control. Our model evaluates the incomplete information from the delayed fund valuations—due to the oscillatory pattern for benefit claims and investment experience of the past years—within the context of uncertainty additionally to the randomness which certainly exists. So, we construct estimations for the optimal proposed contribution rates based on a feedback mechanism which is a robust stabilization controller, using typical linear matrix inequalities. Finally, a numerical application is fully investigated to obtain further insight into the problem.

Keywords

Defined Benefit Pension Schemes, Pension Contributions and Fund Levels, H_∞-Control, Linear Matrix Inequalities

1. Introduction

The determination of the contribution rate is one of the basic concerns of a Pension Fund, on a constant basis when a defined benefit plan is considered. Nowadays, the typical deterministic stationary approach has already been swapped by the dynamic approach using the standard tools of stochastic control theory. In that context, pension funds have been designated by the means of stochastic differential or difference equations while the choice of the pension contribution has been established as an optimization problem under a certain objective function.

Of course, there are several issues and modelling technicalities such as: the stochastic nature of the parameters, the inherent delays of the available information, the uncertainty of the economic environment and many others. A pension fund, most probably starting with an initial capital, continuously receives contributions from the active members (workers) while continuously paying benefits (pensions) to retirees (pensioners) and always targets a positive (but not huge) balance in any future time.

Past research papers have investigated similar problems of a pension fund allowing for stochastic variables or delays or other technicalities. In this paper, we adapt the basic modelling structure as appears in Zimbidis and Haberman (refer to [1]) while we transform the discrete framework into a continuous one. Furthermore, we incorporate the new modelling concept of the uncertainty. These models may be manipulated using the Newton-like methods. Such methods may be found in the H_∞-control theory. So, we actually use tools of H_∞-control theory in order to obtain an analytic solution to our model

The paper is organized as follows: Section 2 contains a short introductory guide to (H infinity) H_∞-control and Linear Matrix Inequalities (LMI's). Section 3 describes the basic structure and the respective system of delayed differential equations for the fund level and the contribution rate. Section 4 presents the theoretical solution of the model while providing a detailed numerical application. Finally, Section 5 concludes the paper.

2. H_∞-Control and Linear Matrix Inequalities

Control theory and especially optimal control theory have played an important role in many scientific areas and practical problems over the past decades. In the last two or three decades, control theory has been fully applied to actuarial problems. A new direction of research for control theory in the very last years is the H_∞-control. Actually, H_∞-control is optimal control design when considering the worst exogenous input for a closed loop system. So, H_∞-control offers an ideal framework to investigate problems under uncertain (but somehow bounded) parameters and conditions. Below, we provide a short note on the relevant theory as regards the H_∞-control (refer to [2] for more details) and linear matrix inequalities—(LMIs) (refer to [3] for more details) that is the powerful tool for solving the respective stability problems. Before going further, we formalize the notation for the matrices *i.e.* Let A represents a matrix, then $A \succ (\succeq, \prec, \preceq) 0$ denotes that A is symmetric positive definite (symmetric positive semi-definite, symmetric negative definite, symmetric negative semi-definite).

2.1. H_∞-Control (or H Infinity Control)

We assume an uncertain linear stochastic delayed and controlled differential system,

$$\begin{aligned} \mathrm{d}x(t) = & \left(A(t)x(t) + A_d(t)x(t-\tau(t)) + B(t)u(t) + B_\upsilon \upsilon(t)\right)\mathrm{d}t \\ & + \left(E(t)x(t) + E_d(t)x(t-\tau(t)) + E_\upsilon \upsilon(t)\right)\mathrm{d}W(t),\ t \geq 0 \end{aligned} \tag{1}$$

$$z(t) = Cx(t) + C_d x(t - \tau(t)) + Du(t),\ t \geq 0 \tag{2}$$

$$z(t) = \phi(t), \quad t \in [-\mu, 0] \tag{3}$$

where

$x(t) \in \mathbb{R}^n$ is the state variable.

$u(t) \in \mathbb{R}^m$ is the controlled input variable.

$\upsilon(t) \in \mathbb{R}^p$ is the disturbance input variable (zero or positive valued & square integrable).

$z(t) \in \mathbb{R}^q$ is the controlled output variable.

$\tau(t)$ is time-varying bounded delay time satisfying the following conditions

$$0 \leq \tau(t) \leq \mu,\ \frac{\mathrm{d}}{\mathrm{d}t}\tau(t) \leq h < 1.$$

$\phi(t)$ is any given initial data in the $\mathbb{R}^n$-valued family $\boldsymbol{F}_0$ measurable stochastic process $\zeta(s)$ with $\sup_{-\mu \leq s \leq 0} E|\zeta(s)|^2 < \infty$, where $E[\ldots]$ stands for the expectation operator with respect to the given probability measure P.

$\Omega = \{\Omega(t);\ t \geq 0\}$ is a scalar Brownian motion defined on a complete probability space (Ω, F, P) with a natural filtration $\{\boldsymbol{F}_t\}_{t\geq 0}$.

B_υ, E_υ, C, C_d, D are known constant matrices

$A(t)$, $A_d(t)$, $B(t)$, $E(t)$, $E_d(t)$ are matrix-valued functions with time-varying uncertainties as

$$A(t) = A + \Delta A(t),\ A_d(t) = A_d + \Delta A_d(t),\ B(t) = B + \Delta B(t),$$
$$E(t) = E + \Delta E(t),\ E_d(t) = E_d + \Delta E_d(t)$$

where A, A_d, B, E and E_d are known real constant matrices.

While $\Delta A(t), \Delta A_d(t), \Delta B(t), \Delta E(t)$ and $\Delta E_d(t)$ are unknown matrices representing time-varying parameters uncertainties. These uncertainties are norm-bounded and may be described as follows

$$\left[\Delta A(t)\ \ \Delta A_d(t)\ \ \Delta B(t)\ \ \Delta E(t)\ \ \Delta E_d(t)\right] = HF(t)\left[N_a\ \ N_{ad}\ \ N_b\ \ N_e\ \ N_{ed}\right]$$

where H, N_a, N_{ad}, N_b, N_e and N_{ed} are known real constant matrices and $F(t)$ is an unknown matrix function with Lebesgue measurable elements, satisfying the condition

$$F^\tau(t) \cdot F(t) \preceq I$$

where $F^\tau(t)$ stands for the transpose matrix of $F(t)$.

Furthermore, we provide the formal definitions for robust stability and robust performance for the system (1)-(3).

Definition 1, (refer to Boyd *et al.*, [3]): The system (1)-(3) with $u(t) = 0$, $\upsilon(t) = 0$ is said to be robust stochastically stable if there exists a positive constant ρ such that

$$\lim_{T\to\infty} E\left[\int_0^T x^\tau(t) \cdot x(t)\,\mathrm{d}t\right] < \rho \sup_{-\mu \leq s \leq 0} E|\zeta(s)|^2 \tag{4}$$

for all admissible uncertainties $\Delta A(t), \Delta A_d(t), \Delta E(t)$ and $\Delta E_d(t)$.

Definition 2, (refer to Boyd *et al.*, [3]): Given a scalar $\gamma > 0$, the unforced stochastic system (1)-(3) with $u(t) = 0$ is said to be robust stochastically stable with disturbance attenuation γ if it is robust stochastically stable in the sense of definition 1 and under zero initial conditions,

$$E\left[\int_0^\infty z^\tau(t)\cdot z(t)\mathrm{d}t\right] < \gamma^2 \cdot \int_0^\infty W^\tau(t)\cdot W(t)\mathrm{d}t \tag{5}$$

Below, we provide two basic theorems that present the solutions to the robust stabilization problem for system (1)-(3). The first theorem corresponds to the special case of a zero disturbance input variable, $\upsilon(t) = 0$. The second theorem solves the general format of the problem. So, we have the following theorems 1 and 2:

Theorem 1, (refer to Chen *et al.*, [2]): Consider the system (1)-(3) with $\upsilon(t) = 0$. Then for given scalars $\mu > 0$, $h < 1$, this system is robust stochastically stabilizable for any time-delay $\tau(t)$ satisfying $0 \le \tau(t) \le \mu, \frac{\mathrm{d}}{\mathrm{d}t}\tau(t) \le h$ if for some prescribed scalar δ, there exist matrices $X \succ 0, Y, Z, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{K}, \bar{Q} \succ 0, \bar{S} \succ 0, \bar{R} \succ 0$ and scalars $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$. Such that the following Linear Matrix Inequalities (LMIs) (6) and (7) hold:

$$\begin{bmatrix} \Omega_1 & \Omega_2 & 0 & X & \mu Z^\tau & L_{11} & L_{21} \\ \Omega_2^\tau & \Omega_3 & -(1-\delta)A_d\bar{S} & 0 & \mu Y^\tau & 0 & 0 \\ 0 & -(1-\delta)\bar{S}^\tau A_d^\tau & -(1-h)\bar{S} & 0 & 0 & L_{12} & L_{22} \\ X^\tau & 0 & 0 & \bar{S} & 0 & 0 & 0 \\ \mu Z & \mu Y & 0 & 0 & -\mu\bar{Q} & 0 & 0 \\ L_{11}^\tau & 0 & L_{12}^\tau & 0 & 0 & \Phi_1 & 0 \\ L_{21}^\tau & 0 & L_{22}^\tau & 0 & 0 & 0 & \Phi_2 \end{bmatrix} \prec 0 \tag{6}$$

$$\begin{bmatrix} \bar{W}_1 & \bar{W}_2 & 0 \\ \bar{W}_2^\tau & \bar{W}_3 & \delta A_d\bar{Q} \\ 0 & \delta\bar{Q}^\tau A_d^\tau & \bar{Q} \end{bmatrix} \succeq 0 \tag{7}$$

(Note: some of the ZEROS in (6) and (7) correspond to blocks of zeros).

Where

$$\Omega_1 = Z + Z^\tau + \mu\bar{W}_1$$

$$\Omega_2 = X\left(A^\tau + \delta A_d^\tau\right) + \bar{K}^\tau B^\tau + Y - Z^\tau + \mu\bar{W}_2$$

$$\Omega_3 = -Y - Y^\tau + \varepsilon_1 HH^\tau + \delta^2 A_d\bar{R}A_d^\tau + \mu\bar{W}_3$$

$$\Phi_1 = diag\left\{\varepsilon_1 I, \varepsilon_2 I, \frac{\mu}{1-h}\varepsilon_3 I\right\}$$

$$\Phi_2 = diag\left\{X - \varepsilon_2 HH^\tau, \frac{\mu}{1-h}\left(\bar{R} - \varepsilon_3 HH^\tau\right)\right\}$$

$$L_{11} = \begin{bmatrix} XN_a^\tau + \bar{K}N_b^\tau & XN_e^\tau & \frac{\mu}{1-h}XN_e^\tau \end{bmatrix}$$

$$L_{12} = \begin{bmatrix} \bar{S}N_{ad}^{\tau} & \bar{S}N_{ed}^{\tau} & \frac{\mu}{1-h}\bar{S}N_{ed}^{\tau} \end{bmatrix}$$

$$L_{21} = \begin{bmatrix} XE^{\tau} & \frac{\mu}{1-h}XE^{\tau} \end{bmatrix}$$

$$L_{22} = \begin{bmatrix} \bar{S}E_{d}^{\tau} & \frac{\mu}{1-h}\bar{S}E_{d}^{\tau} \end{bmatrix}$$

and the stabilizing control law is described as

$$u(t) = \bar{K} \cdot X^{-1} \cdot x(t) \tag{8}$$

Theorem 2, (refer to Chen *et al.*, [2]): Consider the system (1)-(3). Then for given scalars $\mu > 0$, $h < 1$, this system is robust stochastically stabilizable with disturbance attenuation $\gamma > 0$ for any time-delay $\tau(t)$ satisfying $0 \le \tau(t) \le \mu$, $\frac{\mathrm{d}}{\mathrm{d}t}\tau(t) \le h$ if for some prescribed scalar δ, there exist matrices $X \succ 0, Y, Z, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{K}, \bar{Q} \succ 0, \bar{S} \succ 0, \bar{R} \succ 0$ and scalars $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$ such that LMI (7) and the following LMI holds:

$$\begin{bmatrix} \Omega_1 & \Omega_2 & 0 & 0 & XC^{\tau} + \bar{K}D^{\tau} & L_{11} & L_{21} & X & \mu Z^{\tau} \\ \Omega_2^{\tau} & \Omega_3 & (1-\delta)A_d\bar{S} & B_{\upsilon} & 0 & 0 & 0 & 0 & \mu Y^{\tau} \\ 0 & (1-\delta)\bar{S}^{\tau}A_d & -(1-h)\bar{S} & 0 & \bar{S}C_d^{\tau} & L_{12} & L_{22} & 0 & 0 \\ 0 & B_{\upsilon}^{\tau} & 0 & -\gamma^2 I & 0 & 0 & L_{23} & 0 & 0 \\ X^{\tau}C + D\bar{K}^{\tau} & 0 & C_d\bar{S}^{\tau} & 0 & -I & 0 & 0 & 0 & 0 \\ L_{11}^{\tau} & 0 & L_{12}^{\tau} & 0 & 0 & -\Phi_1 & 0 & 0 & 0 \\ L_{21}^{\tau} & 0 & L_{22}^{\tau} & L_{23}^{\tau} & 0 & 0 & -\Phi_2 & 0 & 0 \\ X^{\tau} & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{S} & 0 \\ \mu Z & \mu Y^{\tau} & 0 & 0 & 0 & 0 & 0 & 0 & -\mu\bar{Q} \end{bmatrix} \prec 0 \tag{9}$$

(some of the ZEROS in (9) correspond to blocks of zeros).

Where Ω_1, Ω_2, Ω_3, Φ_1, Φ_2, L_{11}, L_{12}, L_{21}, L_{22}, are defined in previous theorem 1 and $L_{23} = \begin{bmatrix} E_{\upsilon}^{\tau} & \frac{\mu}{1-h}E_{\upsilon}^{\tau} \end{bmatrix}$. Then, the stabilizing control law is described as

$$u(t) = \bar{K} \cdot X^{-1} \cdot x(t) \tag{10}$$

2.2. Linear Matrix Inequalities (LMI's)

The general format of a typical Linear Matrix Inequality is the following

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \succ 0 \tag{11}$$

where $x = (x_1, x_2, x_3, \cdots, x_m) \in \mathbb{R}^m$ is the variable to be determined, while $F_i = F_i^{\tau} \in \mathbb{R}^{n \times n}, i = 0, 1, 2, \cdots, m$, additionally, we can state that the LMI above may be easily transformed to a set of n inequalities in x.

The relationship of LMIs and dynamic systems has been early recognized by Lyapunov (refer to [4]). He demonstrated that the stability of the basic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t)$$

is obtained when there is a matrix P such that the following conditions hold

$$P \succ 0 \text{ and } A^{\tau}P + PA \prec 0$$

Now, we may easily transform the conditions above into the standard format of a LMI using a matrix of matrices as below

$$\begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & -\left(A^{\tau}P + PA\right) \end{bmatrix} \succ 0$$

Furthermore, Yakubovich (refer to [5]) was the first who formally established the importance of the LMI's in the solution of the control problems.

The solution for LMI's has been initially based on the ellipsoid algorithm and on the interior-point methods. These two basic approaches have been further exploited over the last years and also inspire other similar ones. In this paper, we use a quite recent algorithm proposed by Orsi *et al.* (refer to [6]) and Rami *et al.* (refer to [7]) that is based on the classical alternating projection method. Actually, they use this algorithm to solve "convex feasibility problems where the constraints are given by the intersection of two convex cones in a Hilbert space". Then, as an application, they derive the solution of an LMI problem calculating a sequence of matrix eigenvalue-eigenvector decompositions.

Below, we provide a short description of the relevant algorithm that is fully described in the papers mentioned above, aiming to find $x \in \mathbb{R}^m$ satisfying a strict LMI (11) (or the non-strict version where the symbol "$\succ$" is replaced by "$\succeq$"). Before providing the formal description of the algorithm, we establish two special functionals "$\widehat{\ }$" and "$\breve{\ }$" from $\mathcal{S}^n$ the set of real symmetric matrices to $\mathbb{R}^p$ and vice versa, where $p = \dfrac{n(n+1)}{2}$, as below:

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \cdots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} \end{pmatrix}$$

$$\to \widehat{S} = \left(s_{11}, s_{12}, s_{13}, \cdots, s_{1n}, s_{22}, s_{23}, \cdots, s_{2n}, s_{33}, \cdots, s_{3n}, \cdots, s_{nn}\right)^{\tau}$$

$$\breve{S} = \left(s_{11}, s_{12}, s_{13}, \cdots, s_{1n}, s_{22}, s_{23}, \cdots, s_{2n}, s_{33}, \cdots, s_{3n}, \cdots, s_{nn}\right)^{\tau}$$

$$\to S = \begin{pmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \cdots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \cdots & s_{nn} \end{pmatrix}$$

Algorithm for solving the strict LMI (11)

Data: A set of $m + 1$ symmetric matrices $F_i = F_i^{\tau} \in \mathbb{R}^{n \times n}, i = 0,1,2,\cdots,m$

Initialization: Choose initial conditions and calculate the basic matrix G^{-1}, as below

1): Choose values for the parameters ρ, t such that $\rho > 0$ and $t \in (0,2)$ (e.g. $\rho = 1$ & $t = 1.99$);

2): Choose any $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}^m$;

3): Choose any real symmetric matrix $S \in \mathbb{R}^{n\times n}$ (that is a slack variable);

4): Define matrix Q as $Q := \left[\hat{F}_0, \hat{F}_1, \cdots, \hat{F}_m\right] \in \mathbb{R}^{p\times(m+1)}$, $p = \dfrac{n(n+1)}{2}$;

5): Calculate matrix $G := Q \cdot Q^\tau + \dfrac{1}{2} diag\left(\hat{I} + \underline{1}\right)$,

where I is the n x n identity matrix and $\underline{1} \in \mathbb{R}^p$ while $\underline{1} = (1,1,\cdots,1)^\tau$;

6): Calculate the inverse matrix G^{-1}.

Then we may start the algorithm described with the following steps:

Step 1: Calculate and replace x_0 via the formula

$x_0 := (1-t)\cdot x_0 + t \cdot \max\{\rho, x_0\}$.

Step 2: Find an eigenvalue-eigenvector decomposition of S.

Let $S = V \cdot D \cdot V^\tau$ with $D = diag(d_1, d_2, \cdots, d_n)$ the respective matrix of eigenvalues while V the respective matrix of eigenvectors.

Step 3: Define a new matrix

$\bar{D} = diag\left(\max\{\rho, d_1\}, \max\{\rho, d_2\}, \cdots, \max\{\rho, d_n\}\right)$.

Step 4: Calculate and replace matrix S via the formula $S = V \cdot \left((1-t)D + t\bar{D}\right) \cdot V^\tau$.

Step 5: Calculate $a := G^{-1} \cdot \left(Q \cdot \left[x_0; x^\tau\right]^\tau - \hat{S}\right)$.

Step 6: Calculate and replace x_0 and x via the formula

$\left[x_0; x^\tau\right]^\tau := \left[x_0; x^\tau\right]^\tau - Q^\tau \cdot \alpha$.

Step 7: Define $T := -0.5 \cdot \breve{a}$.

Step 8: Redefine matrix T by duplicating all the elements of the first diagonal.

Step 9: Replace matrix S via the formula $S := S - T$.

Step 10: If $x_0 > 0$ and the minimum eigenvalue of S is greater than zero then the solution is x/x_0 otherwise we return to step 1 and continue the algorithm.

3. The Proposed Model for the Pension Fund

We consider a typical pension fund with an initial reserve plus 1) incoming contributions determined through a standard contribution rate and a feedback mechanism according to past experience of benefits paid and 2) outcoming benefits paid, which are driven by a drifted Brownian motion. The reserve is invested in a safe asset with a variable yet uncertain force of interest. Then, the reserve of the company obeys the following stochastic differential equation.

d$R(t)$ = [Reserve at time $(t + \mathrm{d}t)$] − [Reserve at time (t)] = [Investment income earned in $(t, t + \mathrm{d}t)$] + [Contributions in $(t, t + \mathrm{d}t)$] − [Benefits in $(t, t + \mathrm{d}t)$] * (Benefits may also include the element of administration expenses).

Or using standard notation the reserve obeys the following relationship

$$\mathrm{d}R(t) = r(t)R(t)\mathrm{d}t + p^*(t)\mathrm{d}t - \left[m(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)\right] \tag{12}$$

while the contribution rate is established via the following feedback mechanism

$$p^*(t) = p(t) - f \cdot R(t - l(t)) \quad (13)$$

where

$R(t)$: is the reserve at time *t*;

$r(t)$: is the force of interest at time *t*;

$p^*(t)$: actual contribution rate at time *t*;

$p(t)$: standard contribution rate at time *t*;

f: feedback or profit sharing factor;

$l(t)$: time delay for the valuation settlement of the Reserve values at time *t*;

$m(t)$: average benefit rate at time *t*;

$\sigma(t)$: volatility benefit rate at time *t*, further assume that $\sigma(t) = \lambda \cdot m(t)$;

$W(t)$: standard Brownian motion.

Combining Equations (12) and (13) and after some algebra we finally obtain the system

$$\mathrm{d}R(t) = \left[r(t)R(t) - f \cdot R(t - l(t)) + p(t) - m(t)\right]\mathrm{d}t + \left[-\lambda m(t)\right]\mathrm{d}W(t) \quad (14)$$

$$p^*(t) = p(t) - f \cdot R(t - l(t)) \quad (15)$$

The parameter (*f*)—the feedback factor—normally lies in the interval of [0, 1]. Additionally, supporting Equation (13) we also assume that there is no distribution of surplus to anyone else (e.g. the tax authorities) except to the members (actives or retirees) of the pension fund.

4. The General Theoretical Solution and Numerical Application

4.1. The Theoretical Solution

As regards the theoretical solution, it is straight-forward when applying Theorem 2. We must only match the notation and symbols accordingly. So, if we assume the following

$$x(t) = R(t),\ u(t) = p(t),\ \upsilon(t) = m(t),\ z(t) = p^*(t)$$

$$E(t) = 0,\ E_d(t) = 0,\ C = 0,\ N_{ad} = 0,\ N_b = 0,\ N_e = 0,\ N_{ed} = 0$$

$$\Delta A_d(t) = 0,\ \Delta B(t) = 0,\ \Delta E(t) = 0,\ \Delta E_d(t) = 0$$

$$\Delta A(t) = HF(t)N_a$$

where H, N_a are known real constant matrices and *F*(*t*) is an unknown matrix function with Lebesgue measurable elements, satisfying the condition $F^\tau(t) \cdot F(t) \preceq I$. The optimal solution is obtained via the application of LMIs (9) and (7).

The detailed numerical solution is calculated in the following subsection.

4.2. Numerical Application

Now, we apply the theoretical solution and solve our problem, assuming the following

$$x(t)=R(t),\ \ u(t)=p(t),\ \ \upsilon(t)=m(t),\ \ z(t)=p^{*}(t)$$

$$A(t)=r(t)=r+\Delta r(t),\ \ A_{d}(t)=-f,\ \ B(t)=1,\ \ B_{\upsilon}(t)=-1$$

$$E(t)=0,\ \ E_{d}(t)=0,\ \ E_{\upsilon}=-\lambda,\ \ C=0,\ \ C_{d}=-f,\ \ D=1$$

$$N_{ad}=0,\ \ N_{b}=0,\ \ N_{e}=0,\ \ N_{ed}=0$$

Substituting in matrices (9) and (7) and since our problem has only one dimension (there is no meaning for the symbol of transpose $(.)^{\tau}$), we obtain,

$$\begin{bmatrix}
2Z+\mu\bar{W}_1 & \boldsymbol{g}' & 0 & 0 & \bar{K} & XN_{\alpha} & 0 & 0 & 0 & 0 & X & \mu Z\\
\boldsymbol{g}' & \boldsymbol{g} & -(1-\delta)f\bar{S} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu Y\\
0 & -(1-\delta)f\bar{S} & -(1-h)\bar{S} & 0 & -f\bar{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & -\gamma^{2} & 0 & 0 & 0 & 0 & -\lambda & -\lambda\frac{\mu}{1-h} & 0 & 0\\
\bar{K} & 0 & -f\bar{S} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
XN_{a} & 0 & 0 & 0 & 0 & -\varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\mu}{1-h}\varepsilon_3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & -X+\varepsilon_2H^{2} & 0 & 0 & 0\\
0 & 0 & 0 & -\lambda\frac{\mu}{1-h} & 0 & 0 & 0 & 0 & 0 & -\frac{\mu}{1-h}\left(\bar{R}-\varepsilon_3H^{2}\right) & 0 & 0\\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{S} & 0\\
\mu Z & \mu Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu\bar{Q}
\end{bmatrix}\prec 0$$

$$\boldsymbol{g}=-2Y+\varepsilon_1H^{2}+\delta^{2}f^{2}\bar{R}+\mu\bar{W}_3,\ \ \boldsymbol{g}'=X(r-\delta f)+\bar{K}+Y-Z+\mu\bar{W}_2$$

$$\begin{bmatrix}
\bar{W}_1 & \bar{W}_2 & 0\\
\bar{W}_2 & \bar{W}_3 & -\delta f\bar{Q}\\
0 & -\delta f\bar{Q} & \bar{Q}
\end{bmatrix}\succeq 0$$

Furthermore assuming

$$A=r,\ \ E_{\upsilon}=-\lambda,\ \ C_{d}=-f,\ N_{a}=\psi,$$
$$\mu=1,\ \ h=0.5,\ \ \gamma=1,\ \ H=1,\ \ F(t)=1,\ \ \delta=1$$

We obtain

$$\Xi=\begin{bmatrix}
2Z+\bar{W}_1 & \boldsymbol{g}' & 0 & 0 & \bar{K} & \psi X & 0 & 0 & 0 & 0 & X & Z\\
\boldsymbol{g}' & \boldsymbol{g} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y\\
0 & 0 & -0.5\bar{S} & 0 & -f\bar{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -\lambda & -2\lambda & 0 & 0\\
\bar{K} & 0 & -f\bar{S} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\psi X & 0 & 0 & 0 & 0 & -\varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\varepsilon_3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & -X+\varepsilon_2 & 0 & 0 & 0\\
0 & 0 & 0 & -2\lambda & 0 & 0 & 0 & 0 & 0 & -2\left(\bar{R}-\varepsilon_3\right) & 0 & 0\\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{S} & 0\\
Z & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{Q}
\end{bmatrix}\prec 0$$

$$\boldsymbol{g} = -2Y + \varepsilon_1 + f^2\bar{R} + \bar{W}_3, \ \ \boldsymbol{g}' = X\left(r - f\right) + \bar{K} + Y - Z + \bar{W}_2$$

$$\Xi' = \begin{bmatrix} \bar{W}_1 & \bar{W}_2 & 0 \\ \bar{W}_2 & \bar{W}_3 & -f\bar{Q} \\ 0 & -f\bar{Q} & \bar{Q} \end{bmatrix} \geq 0,$$

$$X > 0, Y, Z, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{K}, \bar{Q} > 0, \bar{S} > 0, \bar{R} > 0, \varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$$

Now, we may transfer our problem assuming $m = 13$ and $n = 15$

$$x = \left(x_1, x_2, x_3, \cdots, x_{m=13}\right) = \left(\bar{K}, X, Y, Z, \bar{Q}, \bar{R}, \bar{S}, \bar{W}_1, \bar{W}_2, \bar{W}_3, \varepsilon_1, \varepsilon_2, \varepsilon_3\right)$$

while the 14 matrices with dimensions (15 × 15) will derived from the following LMI

$$\begin{bmatrix} -\Xi & \mathbf{O} \\ \mathbf{O} & \Xi' \end{bmatrix} = F_0 + x_1 \cdot F_1 + x_2 \cdot F_2 + x_3 \cdot F_3 + \cdots + x_{13} \cdot F_{13} \succ 0$$

The solution of the last LMI will also be a solution for our problem. So, we have the following 14 matrices.

$\boldsymbol{F_0}$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \lambda & 2\lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$\boldsymbol{F_1}\ (\bar{K})$

$$\begin{bmatrix}
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$\boldsymbol{F_2}\ (X)$

$$\begin{bmatrix}
0 & f-r & 0 & 0 & 0 & -\psi & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
f-r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$\boldsymbol{F_3}\ (Y)$

$$\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$\boldsymbol{F_4}\ (Z)$

$$\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$\boldsymbol{F_5}\ (Q)$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -f \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -f & 1
\end{bmatrix}$$

$F_6\,(R)$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -f^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$F_7\,(S)$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$F_8\,(W_1)$

$$\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$F_9\,(W_2)$

$$\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$F_{10}\,(W_3)$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$F_{11}\,(\varepsilon_1)$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$F_{12}\,(\varepsilon_2)$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$F_{13}\,(\varepsilon_3)$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Furthermore, we assume seven different scenarios as regards the following parameters:

1) The central value for the investment rate r;

2) The uncertainty level for the investment rate ψ;

3) The volatility factor λ.

While for the feedback factor we run through all the potential range from zero to unity using a step of 0.1 (so actually we run all seven scenarios for f = 0%, 10%, 20%, …, 100%).

Scenarios	f = 0%	f = 10%	f = 20%	f = 30%	f = 40%
ψ = 0.04 - r = 0.03 - λ = 3	−52.3%	−47.2%	−42.7%	−38.6%	−34.7%
ψ = 0.04 - r = 0.06 - λ = 3	−54.0%	−48.7%	−44.0%	−39.8%	−35.9%
ψ = 0.08 - r = 0.03 - λ = 3	−52.5%	−47.4%	−42.8%	−38.7%	−34.9%
ψ = 0.08 - r = 0.06 - λ = 3	−54.2%	−48.9%	−44.2%	−40.0%	−36.1%
ψ = 0.04 - r = 0.03 - λ = 5	−52.3%	−47.1%	−42.6%	−38.5%	No convergence
ψ = 0.04 - r = 0.06 - λ = 5	−54.0%	−48.7%	−44.0%	−39.8%	
ψ = 0.08 - r = 0.06 - λ = 5	−54.2%	−48.9%	−44.2%	−40.0%	

Scenarios	f = 50%	f = 60%	f = 70%	f = 80%	f = 90%	f = 100%
ψ = 0.04 - r = 0.03 - λ = 3	*The algorithm for the solution of the respective LMI does not converge (using a set of 1000 iterations)*					
ψ = 0.04 - r = 0.06 - λ = 3						
ψ = 0.08 - r = 0.03 - λ = 3						
ψ = 0.08 - r = 0.06 - λ = 3						
ψ = 0.04 - r = 0.03 - λ = 5						
ψ = 0.04 - r = 0.06 - λ = 5						
ψ = 0.08 - r = 0.06 - λ = 5						

As we observe, there is no viable solution when the feedback delay factor f exceeds the critical value 50%. Additionally, when the volatility (see the parameter λ) or/and the uncertainty level of the investment rate (see the parameter ψ) are increased then the viable solutions are restricted even more. Under these scenarios (5, 6 and 7) there is no viable solution even for f = 40%. That is directly comparable with the results of Zimbidis & Haberman (1993) who found similar values for instability level.

The results are also presented in the following **Figure 1**.

5. Conclusions—Further Research

Closing this paper, we present a short resume. The new modeling concept introduced by this research project is the introduction of uncertainty into a pension fund. The framework of uncertainty is further enhanced assuming also some kind of delay. The H-infinity control theory is employed as the basic tool in order to handle the application. Furthermore, H_∞-control leads to Linear Matrix Inequalities (LMIs) since the basic stability condition relies on a system of LMIs. After appointing the general solution of the model, we focus on the numerical application which is fully investigated by solving the respective LMI using an iterative algorithm (up to 1000 iterations).

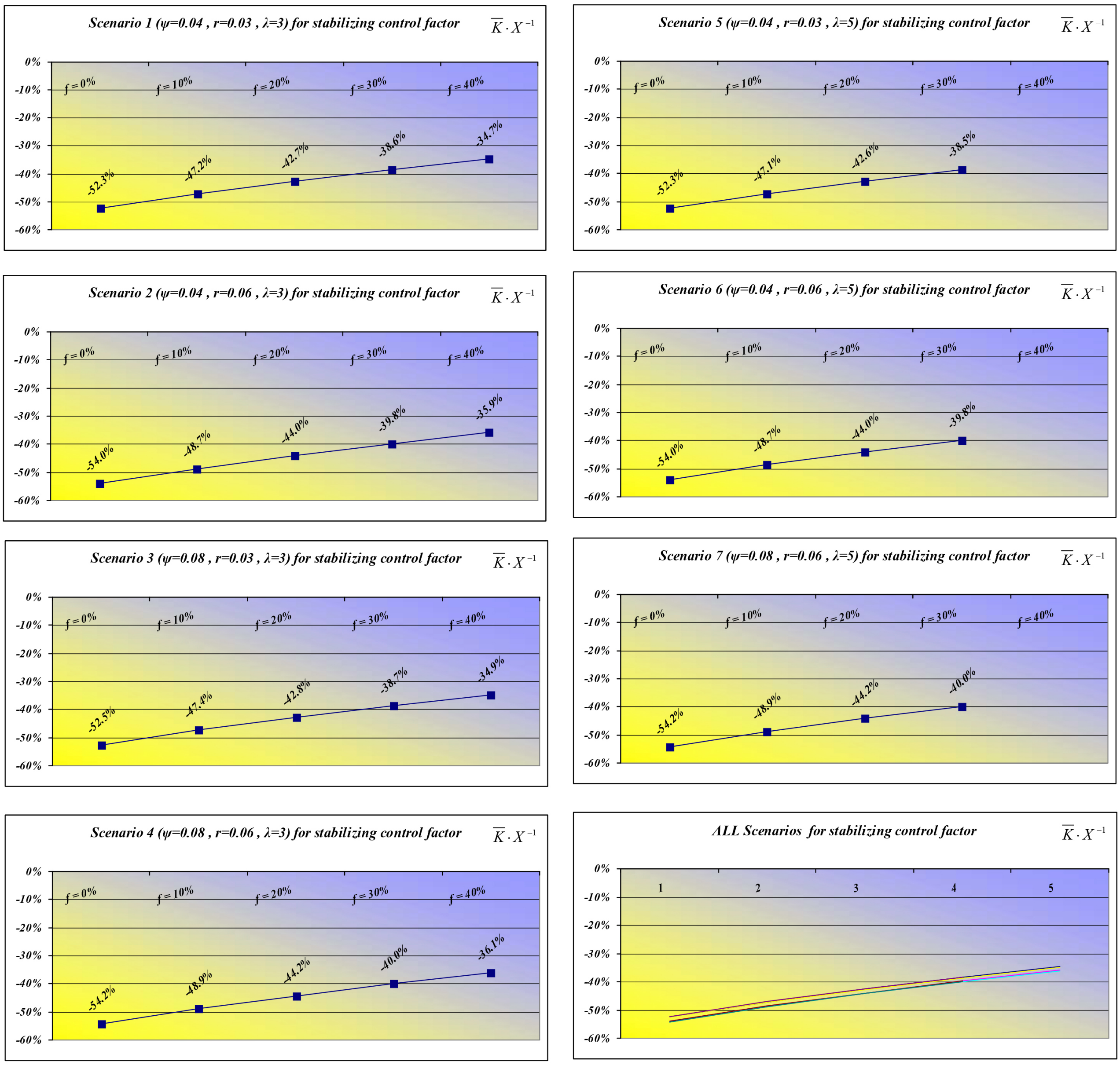

Figure 1. Results for the stabilization control factor under different scenarios.

The numerical results of our application coincide with those of Zimbidis & Haberman (1993) supporting that the feedback factor f (f represents the amount of delayed information integrated into the system) must be restricted below 50% or even lower when considering models with high uncertainty or volatility levels, otherwise the system is not robust stochastically stable. It is also interesting to note the solution for the trivial case where there is no delay factor (f = 0%). Then, the robust stabilization factor $\overline{K}\cdot X^{-1}$ is slightly higher than 50% (from 52% up to 54%). It is also worth noticing (see the last diagram of **Figure 1**) that all the solutions are almost parallel and be arranged within a narrow zone-path starting from the interval [52%, 54%] for f = 0% and ending at interval [34%, 36%] for f = 40%.

Finally, further research may be also carried forward in the near future using the tools of *H*-infinity control and Linear Matrix Inequality theory into certain problems in pension funds.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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