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Traveling Wave Solution of the Modified Benjamin-Bona-Mahony Equation

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Abstract

In this paper, the ansatze method is implemented to study the exact solutions for the modified Benjamin-Bona-Mahony equation (mBBM). The singular-shaped traveling wave solution, the Bell-shape is traveling wave solution, the kink-shaped traveling wave solution and the periodic traveling wave solution is obtained. With the assist of computational software MATLAB, the graphical exemplifications of solutions are illustrated of the two-dimension (2D) and three-dimension (3D) plots.

Keywords

Modified Benjamin-Bona-Mahony Equation, Ansatze Method, Traveling Wave Solution, MATLAB

1. Introduction

Nonlinear evolution equations are extensively used as models to describe intricate physical phenomena in diverse fields of the sciences, peculiarly in fluid mechanics, solid state physics, plasma physics, and chemical physics [1]. Furthermore, the study of exact solutions of nonlinear evolution equations plays a significant part in soliton theory. In the past, many available methods have been established to obtain exact solutions of nonlinear evolution equations, for instance the Hirota's bilinear transformation method [2] [3], the tanh-function method [4] [5], the Exp-function method [6]-[13], the (G'/G)-expansion method [14]-[21], discrete Galerkin approximations method [22], the Jacobi elliptic function method [23], the homogeneous balance method [24], the modified simple equation method [25] [26] and so on.

The Benjamin-Bona-Mahony (BBM) equation

$$u_t + u_x + u^n u_x + u_{xxt} = 0 \tag{1}$$

is the most famous model in physical applications. This equation model long waves in a nonlinear dispersive system. The solution of the BBM equation exhibits definite soliton-like behavior that is not explainable by any known theory [27]. The BBM equation is used in the analysis of the surface waves of long wavelength in liquids, hydromagnetic waves in cold plasma, acoustic-gravity waves in compressible fluids and acoustic waves in anharmonic crystals. Where $n = 2$ in (1), the BBM equation is called the modified Benjamin-Bona-Mahony equation (mBBM) [28]. In the article, we apply the ansatze method to study the exact traveling wave solutions of the following mBBM equation [29].

$$u_t + u_x + au^2u_x + bu_{xxt} = 0, \tag{2}$$

Here a and b are nonzero constants, $u(x,t)$ is an unknown function, with respect to the spatial variable x and temporal variable t.

Kamruzzaman Khan and M.Ali Akbar [30] apply the $(-\Phi(\xi))$-expansion method to find the exact solitary wave solutions of mBBM equation, including hyperbolic function solutions, trigonometric function solutions and rational solutions. Kamruzzaman Khan, M. Ali Akbar and Md. Nur Alam [29] applies the MSE method to find the traveling wave solutions of mBBM equation. A K Gupta, J Hazarika [31] investigate the exact solutions by using Kudryashov method, Furthermore, we use four-term approximate solution of OHAM and compare it with the solution of the Kudryashov method.

The article is prepared as follows: In Section 2, we discuss the exact traveling wave solutions of modified Benjamin-Bona-Mahoney equation, In Section 3, we provide the dynamic behaviors of the traveling wave solutions of mBBM equation, and the short conclusions are given in Section 4.

2. Solutions of the Modified Benjamin-Bona-Mahoney Equation

Using the Following Traveling Wave Transformation

$$u(x,t) = u(\xi) = u(x + \omega t), \tag{3}$$

Equation (2) can be reduced into the following ordinary differential equation (ODE).

$$(\omega + 1)u' + au^2u' + b\omega u''' = 0. \tag{4}$$

Integrating (4) once with respect to ξ, and choosing constant of integration as zero, we can obtain the following ODE.

$$(\omega + 1)u + \frac{1}{3}au^3 + b\omega'' = 0. \tag{5}$$

2.1. The Hyperbolic Function Solution

Suppose that the solutions of Equation (5) have the following form.

$$u(\xi) = \sum_{i=1}^{m} \sinh^{i-1}\alpha\left(B_i \sinh\alpha + A_i \cosh\alpha\right) + A_0. \tag{6}$$

Balancing the highest nonlinear terms u^3 and the highest derivative term u'' in Equation (5), we obtain $3m = m + 2$, which gives $m = 1$.

By (6) we can write as

$$u(\xi) = B_1 \sinh\alpha + A_1 \cosh\alpha + A_0. \tag{7}$$

where A_0, A_1 and B_1 are undetermined constants. Let be the ansatze

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\xi} = \sinh\alpha. \tag{8}$$

By Substituting (7), (8) into Equation (5), this yields the polynomial of $\sinh^i\alpha\cosh^i\alpha\ (i = 0,1,2,3)$, setting the coefficients of each power $\sinh^i\alpha\cosh^i\alpha\ (i = 0,1,2,3)$ to zero, we can obtain the following algebraic equations with respect to A_0, A_1, B_1 and ω.

$$\begin{cases} (\omega+1)A_0 + \frac{1}{3}aA_0^3 + aA_0^2A_1 = 0, \\ (\omega+1)B_1 + aA_1^2B_1 + aA_0^2B_1 + b\omega B_1 = 0, \\ aA_0B_1^2 + aA_0^2A_1 = 0, \\ \frac{1}{3}aB_1^3 + aA_1^2B_1 + 2b\omega B_1 = 0, \\ (\omega+1)A_1 - aA_1B_1^2 + aA_0^2A_1 - 2b\omega A_1 = 0, \\ \frac{1}{3}aA_1^3 + aA_1B_1^2 + 2b\omega A_1 = 0, \\ 2aA_0A_1B_1 = 0. \end{cases} \tag{9}$$

Solving the above algebraic Equations (9), we can obtain the following three sets of solutions,

$$\text{Case 1.}\quad A_0 = 0, A_1 = 0, B_1 = \pm\sqrt{\frac{-6b\omega}{a}}, \omega = \frac{-1}{1+b}; \tag{10}$$

$$\text{Case 2.}\quad A_0 = 0, B_1 = 0, A_1 = \pm\sqrt{\frac{-6b\omega}{a}}, \omega = \frac{-1}{1-2b}; \tag{11}$$

$$\text{Case 3.}\quad A_0 = 0, A_1 = \pm\sqrt{\frac{-(\omega+1)-b\omega}{a}}, B_1 = \pm\sqrt{\frac{(\omega+1)-2b\omega}{a}}, \omega = \frac{-2}{2-b}. \tag{12}$$

By applying the method of separating into variables to solve Equation (8), with choosing constant of integration to zero, we obtain:

$$\sinh\alpha = -\operatorname{csch}\xi, \cosh\alpha = -\coth\xi. \tag{13}$$

Using (10), (13), (3) and (7), we have:

$$u_{1,2}(\xi) = \mp\sqrt{\frac{-6b\omega}{a}}\operatorname{csch}\xi, \tag{14}$$

where $\xi = x - \frac{1}{1+b}t$, u_1 takes "+", u_2 takes "−".

Using (11), (13), (3) and (7), we have:

$$u_{3,4}(\xi) = \mp\sqrt{\frac{-6b\omega}{a}}\coth\xi, \tag{15}$$

where $\xi = x - \frac{1}{1-2b}t$, u_3 takes "+", u_4 takes "−".

Using (12), (13), (3) and (7), we have:

$$u_{5,6,7,8}(\xi) = \mp\sqrt{\frac{(\omega+1)-2b\omega}{a}}\operatorname{csch}\xi \mp\sqrt{\frac{-(\omega+1)-b\omega}{a}}\coth\xi, \tag{16}$$

where $\xi = x - \frac{2}{2-b}t$, u_5 takes "+, +", u_6 takes "−, −", u_7 takes "+, −", u_8 takes "−, +".

By the above analysis, we can obtain the solutions of Equation (2), as follows:

Proposition 1.

1) If $ab(1+b) > 0$, Equation (2) have the following hyperbolic function solutions

$$u_{1,2}(x,t) = \mp\sqrt{\frac{6b}{a(1+b)}}\operatorname{csch}\left(x - \frac{t}{1+b}\right),$$

2) If $ab(2b-1) < 0$, Equation (2) have the following hyperbolic function solutions

$$u_{3,4}(x,t) = \mp\sqrt{\frac{6b}{a(1-2b)}}\coth\left(x - \frac{t}{1-2b}\right),$$

3) If $ab(b-2) < 0$, Equation (2) have the following hyperbolic function solutions

$$u_{5,6,7,8}(x,t) = \mp\sqrt{\frac{3b}{a(2-b)}}\left(\operatorname{csch}\left(x + \frac{2}{b-2}t\right) \mp \coth\left(x + \frac{2}{b-2}t\right)\right).$$

Suppose that the solutions of Equation (5) have the following form.

$$u(\xi) = \sum_{i=1}^{m}\sin^{i-1}\alpha\left(B_i\sin\alpha + A_i\cos\alpha\right) + A_0. \tag{17}$$

Balancing the highest nonlinear terms u^3 and the highest derivative terms u'' in Equation (5), we obtain $m = 1$.

By (17) we can write as

$$u(\xi) = B_1\sin\alpha + A_1\cos\alpha + A_0, \tag{18}$$

where A_0, A_1 and B_1 are undetermined constants. Let be the ansatze

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\xi} = \sin\alpha. \tag{19}$$

By Substituting (18), (19) into Equation (5), this yields the polynomial of $\sinh^i\alpha\cosh^i\alpha$ $(i = 0,1,2,3)$, setting the coefficients of each power
$\sinh^i\alpha\cosh^i\alpha$ $(i = 0,1,2,3)$ to zero, we can obtain the following algebraic equations with respect to A_0, A_1, B_1 and ω.

$$
\begin{cases}
(\omega+1)A_0+\frac{1}{3}aA_0^3+aA_0A_1^2=0,\\
(\omega+1)B_1+aA_1^2B_1+aA_0^2B_1+b\omega B_1=0,\\
aA_0B_1^2-aA_0A_1^2=0,\\
\frac{1}{3}aB_1^3-aA_1^2B_1-2b\omega B_1=0,\\
(\omega+1)A_1+aA_1B_1^2+aA_0^2A_1-2b\omega A_1=0,\\
\frac{1}{3}aA_1^3-aA_1B_1^2+2b\omega A_1=0,\\
2aA_0A_1B_1=0.
\end{cases} \tag{20}
$$

Solving the above algebraic Equations (20), we can obtain the following two sets of solutions,

$$\text{Case 1.}\quad A_0=0, A_1=0, B_1=\pm\sqrt{\frac{6b\omega}{a}}, \omega=\frac{-1}{1+b} \tag{21}$$

$$\text{Case 2.}\quad A_0=0, B_1=0, A_1=\pm\sqrt{\frac{-6b\omega}{a}}, \omega=\frac{-1}{1-2b}. \tag{22}$$

By applying the method of separating into variables to Equation (19), with choosing constant of integration to zero, we obtain:

$$\sin\alpha=\sec h\xi, \cos\alpha=\pm\tanh\xi. \tag{23}$$

Using (21), (23), (3) and (18), we have:

$$u_{1,2}(\xi)=\pm\sqrt{\frac{6b\omega}{a}}\operatorname{sech}\xi, \tag{24}$$

where $\xi=x-\frac{1}{1+b}t$, u_1 takes "+", u_2 takes "−".

Using (22), (23), (3) and (18), we have:

$$u_{3,4}(\xi)=\pm\sqrt{\frac{-6b\omega}{a}}\tanh\xi, \tag{25}$$

where $\xi=x-\frac{1}{1-2b}t$, u_3 takes "+", u_4 takes "−".

By the above analysis, we can obtain the solutions of Equation (2), as follows:

Proposition 2.

1) If $ab(1+b)<0$, Equation (2) have the following hyperbolic function solutions

$$u_{1,2}(x,t)=\mp\sqrt{\frac{-6b}{a(1+b)}}\operatorname{sech}\left(x-\frac{t}{1+b}\right),$$

2) If $ab(1-2b)>0$, Equation (2) have the following hyperbolic function solutions

$$u_{3,4}(x,t)=\mp\sqrt{\frac{6b}{a(1-2b)}}\tanh\left(x-\frac{t}{1-2b}\right).$$

2.2. The Trigonometric Function Solutions

For (6) and (7), Let be the ansatze

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\xi} = \cosh\alpha. \tag{26}$$

By Substituting (7), (26) into Equation (5), this yields the polynomial of $\sinh^i\alpha\cosh^i\alpha\ (i = 0,1,2,3)$, setting the coefficients of each power $\sinh^i\alpha\cosh^i\alpha\ (i = 0,1,2,3)$ to zero, we can obtain the following algebraic equations with respect to A_0, A_1, B_1 and ω.

$$\begin{cases} (\omega+1)A_0 + \frac{1}{3}aA_0^3 + aA_0^2A_1 = 0, \\ (\omega+1)B_1 + aA_1^2B_1 + aA_0^2B_1 + 2b\omega B_1 = 0, \\ aA_0B_1^2 + aA_0^2A_1 = 0, \\ \frac{1}{3}aB_1^3 + aA_1^2B_1 + 2b\omega B_1 = 0, \\ (\omega+1)A_1 - aA_1B_1^2 + aA_0^2A_1 - b\omega A_1 = 0, \\ \frac{1}{3}aA_1^3 + aA_1B_1^2 + 2b\omega A_1 = 0, \\ 2aA_0A_1B_1 = 0. \end{cases} \tag{27}$$

Solving the above algebraic Equations (27), we can obtain the following three sets of solutions,

$$\text{Case 1.}\quad A_0 = 0, A_1 = 0, B_1 = \pm\sqrt{\frac{-6b\omega}{a}}, \omega = \frac{-1}{1+2b}; \tag{28}$$

$$\text{Case 2.}\quad A_0 = 0, B_1 = 0, A_1 = \pm\sqrt{\frac{-6b\omega}{a}}, \omega = \frac{-1}{1-b}; \tag{29}$$

$$\text{Case 3.}\quad A_0 = 0, A_1 = \pm\sqrt{\frac{-(\omega+1)-2b\omega}{a}}, B_1 = \pm\sqrt{\frac{(\omega+1)-b\omega}{a}}, \omega = \frac{-2}{2+b}. \tag{30}$$

By applying the method of separating into variables to Equation (26), with choosing constant of integration to zero, we obtain:

$$\sinh\alpha = -\cot\xi, \cosh\alpha = -\csc\xi. \tag{31}$$

Using (28), (31), (3) and (7), we have:

$$u_{1,2}(\xi) = \mp\sqrt{\frac{-6b\omega}{a}}\cot\xi, \tag{32}$$

where $\xi = x - \frac{1}{1+2b}t$, u_1 takes "+", u_2 takes "−".

Using (29), (31), (3) and (7), we have:

$$u_{3,4}(\xi) = \mp\sqrt{\frac{-6b\omega}{a}}\csc\xi, \tag{33}$$

where $\xi = x - \frac{1}{1-b}t$, u_3 takes "+", u_4 takes "−".

Using (30), (31), (3) and (7), we have:

$$u_{5,6,7,8}(\xi) = \mp\sqrt{\frac{(\omega+1)-2b\omega}{a}}\cot\xi \mp \sqrt{\frac{-(\omega+1)-b\omega}{a}}\csc\xi, \tag{34}$$

where $\xi = x - \frac{2}{2+b}t$, u_5 takes "+, +", u_6 takes "−, −", u_7 takes "+, −", u_8 takes "−, +".

By the above analysis, we can obtain the solutions of Equation (2), as follows:

Proposition 3.

1) If $ab(1+2b) > 0$, Equation (2) have the following trigonometric function solutions

$$u_{1,2}(x,t) = \mp\sqrt{\frac{6b}{a(1+2b)}}\cot\left(x - \frac{t}{1+2b}\right),$$

2) If $ab(b-1) < 0$, Equation (2) have the following trigonometric function solutions

$$u_{3,4}(x,t) = \mp\sqrt{\frac{6b}{a(1-b)}}\csc\left(x - \frac{t}{1-b}\right),$$

3) If $ab(b+2) > 0$, Equation (2) have the following trigonometric function solutions

$$u_{5,6,7,8}(x,t) = \mp\sqrt{\frac{5b}{a(2+b)}}\cot\left(x - \frac{2}{2+b}t\right) \mp \sqrt{\frac{b}{a(2+b)}}\csc\left(x - \frac{2}{2+b}t\right).$$

3. Physical and Graphical Explanation

The solution (14) is the singular-shape traveling wave solution. **Figure 1** shows the shape of the exact singular solution u_1 of Equation (2) (only shows the shape of solution (14) with wave speed $a = 1$, $b = 1$, $\omega = -\frac{1}{2}$ into the interval $-5 \le x, t \le 5$). By **Figure 1**, we know that the solution u_1 have a valley and a peak; at the same time, the highest points and lowest points are nonsmooth. The solution (15) is the singular-shape traveling wave solution. **Figure 2** shows the shape of the exact singular solution u_3 of Equation (2) (only shows the shape of solution (15) with wave speed $a = -2$, $b = 1$, $\omega = 1$ into the interval $-20 \le x, t \le 20$). By **Figure 2**, we know that the solution u_3 have a valley and a peak and these are not smooth. The solution (16) is the singular-shape traveling wave solution. **Figure 3** shows the shape of the exact singular solution u_5 of Equation (2) (only shows the shape of solution (16) with wave speed $a = 2$, $b = 1$, $\omega = -2$ into the interval $-10 \le x, t \le 10$). By **Figure 3**, we know that the solution u_5 have a valley and a peak, meantime, the highest points and lowest points are nonsmooth, the valleys and peaks are shorter compared to **Figure 1**.

The solution (24) is the Bell-shape traveling wave solution. **Figure 4** shows the shape of the exact Bell-shape isolated solution u_1 of Equation (2) (only shows the shape of solution (24)) with wave speed $a = -1$, $b = 1$, $\omega = 1$ into

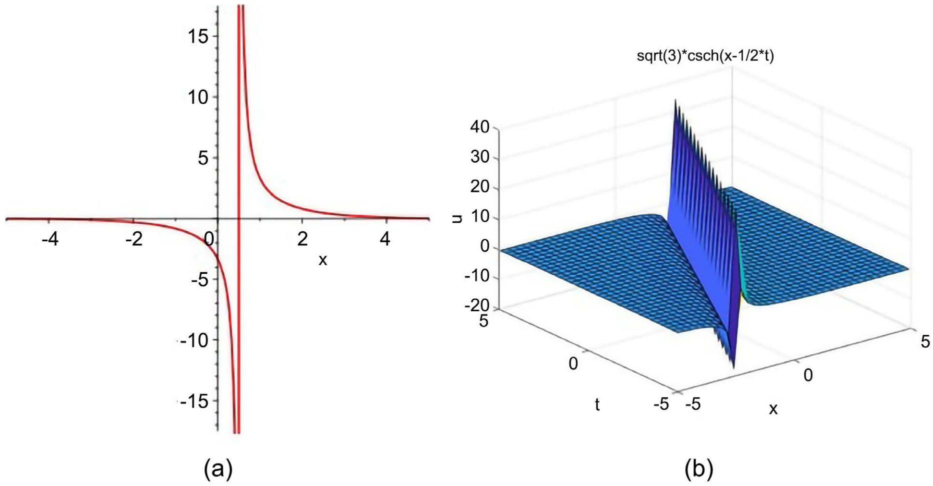

Figure 1. The solution $u = u_1(x,t)$ in (14) with $a = 1$, $b = 1$, $\omega = -\frac{1}{2}$, $-5 \le x,t \le 5$: (a) the 2D plot for $t = 1$; (b) the 3D plot.

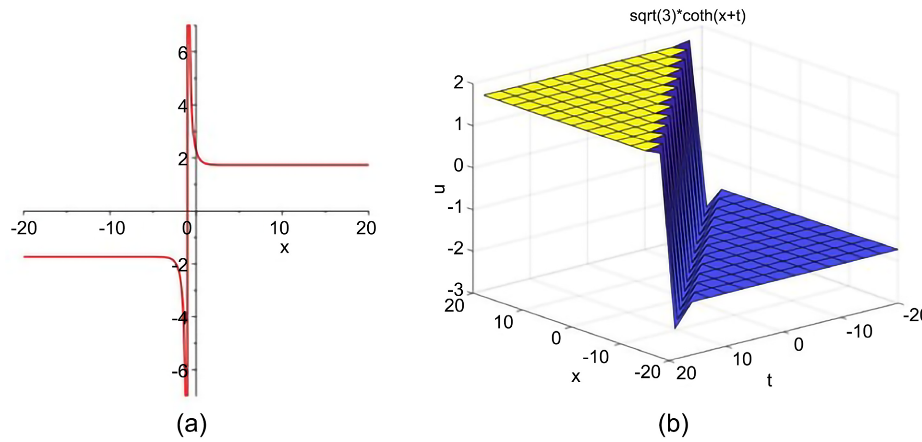

Figure 2. The solution $u = u_3(x,t)$ in (15) with $a = -2$, $b = 1$, $\omega = 1$, $-20 \le x,t \le 20$: (a) the 2D plot for $t = 1$; (b) the 3D plot.

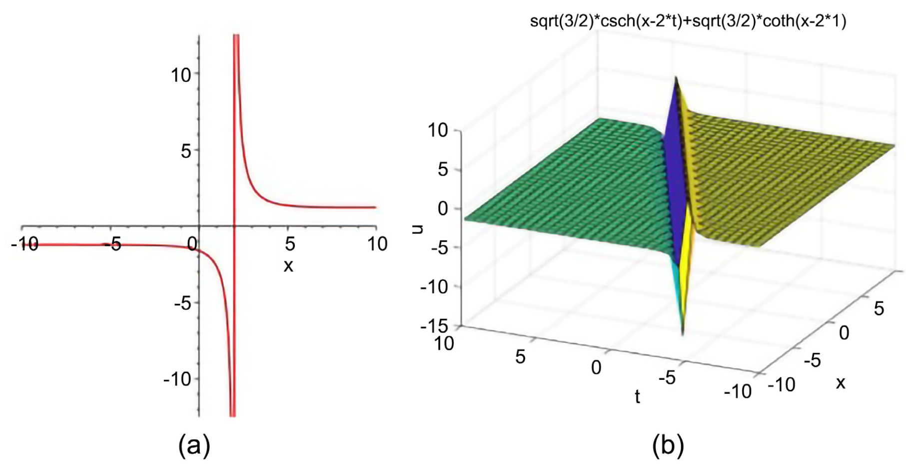

Figure 3. The solution $u = u_5(x,t)$ in (16) with $a = 2$, $b = 1$, $\omega = -2$, $-10 \le x,t \le 10$: (a) the 2D plot for $t = 1$; (b) the 3D plot.

the interval $-5 \le x,t \le 5$). By **Figure 4**, we know that the solution u_1 approaches the x-axis infinitely and smooth everywhere. The solution (25) is the kink-shape traveling wave solution. **Figure 5** shows the shape of the exact Kink-type solution u_3 of Equation (2) (only shows the shape of solution (25) with wave speed $a = -1$, $b = 1$, $\omega = 1$ into the interval $-20 \le x,t \le 20$). By **Figure 5**, we know that the solution u_3 is continuous and monotonically increasing in the minddle.

The solution (32) is the periodic traveling wave solution. **Figure 6** shows the shape of the exact singular solution u_1 of Equation (2) (only shows the shape of solution (32) with wave speed $a = 3$, $b = 2$, $\omega = -\frac{1}{5}$ into the interval $-10 \le x,t \le 10$). By **Figure 6**, we know that the solution u_1 have multiple valley

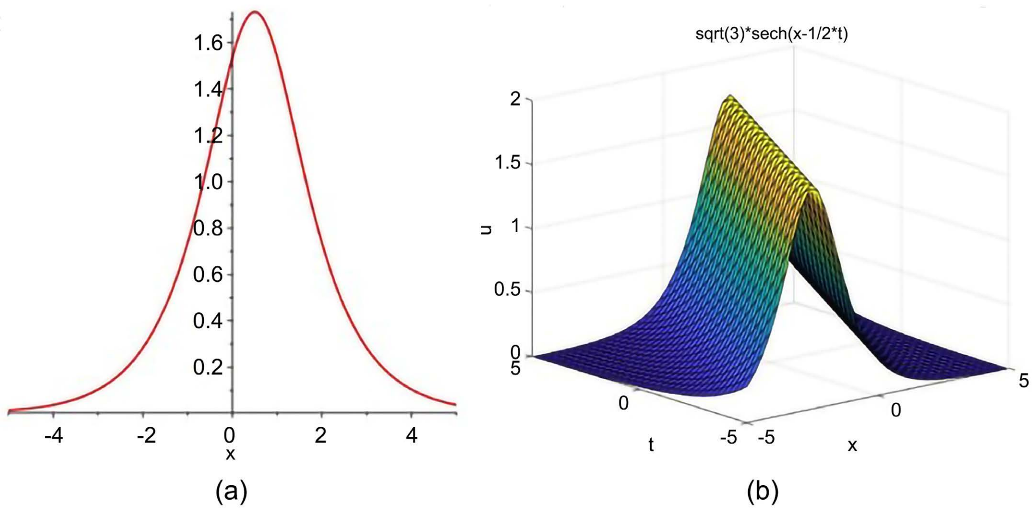

Figure 4. The solution $u = u_1(x,t)$ in (24) with $a = -1$, $b = 1$, $\omega = -\frac{1}{2}$, $-5 \le x,t \le 5$: (a) the 2D plot for $t = 1$; (b) the 3D plot.

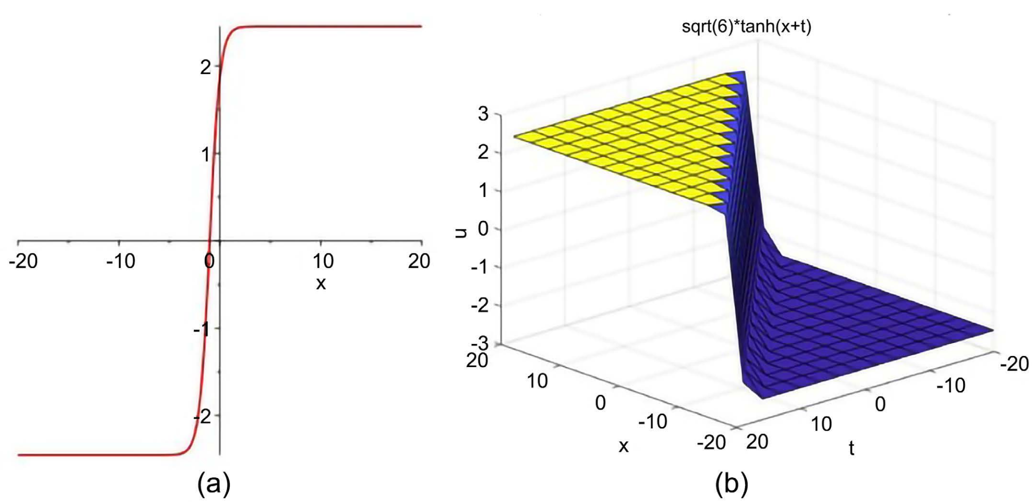

Figure 5. The solution $u = u_3(x,t)$ in (25) with $a = -1$, $b = 1$, $\omega = 1$, $-20 \le x,t \le 20$: (a) the 2D plot for $t = 1$; (b) the 3D plot.

and multiple peaks and varies in height. The solution (33) is the periodic traveling wave solution. **Figure 7** shows the shape of the exact singular solution u_3 of Equation (2) (only shows the shape of solution (33) with wave speed $a = -1$, $b = -2$, $\omega = -\frac{1}{3}$ into the interval $-10 \le x, t \le 10$). By **Figure 7**, we know that the solution u_3 have multiple valley and multiple peaks and varies in height. The solution (34) is the periodic traveling wave solution. **Figure 8** shows the shape of the exact singular solution u_5 of Equation (2) (only shows the shape of solution (34) with wave speed $a = 1$, $b = 1$, $\omega = -\frac{2}{3}$ into the interval $-4 \le x \le 4$, $-5 \le t \le 5$). By **Figure 8**, we know that the solution u_5 have multiple valley and multiple peaks and varies in height.

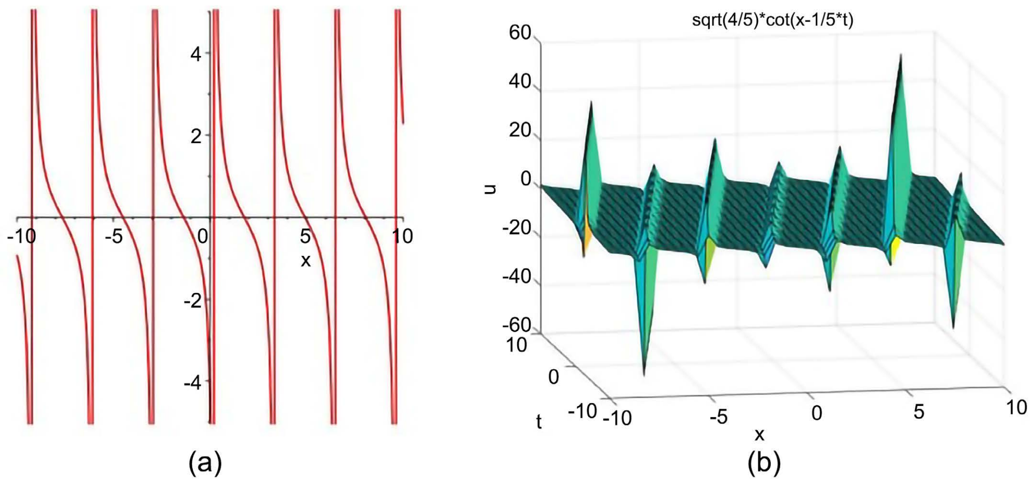

Figure 6. The solution $u = u_1(x,t)$ in (32) with $a = 3$, $b = 2$, $\omega = -\frac{1}{5}$, $-10 \le x, t \le 10$: (a) the 2D plot for $t = 1$; (b) the 3D plot.

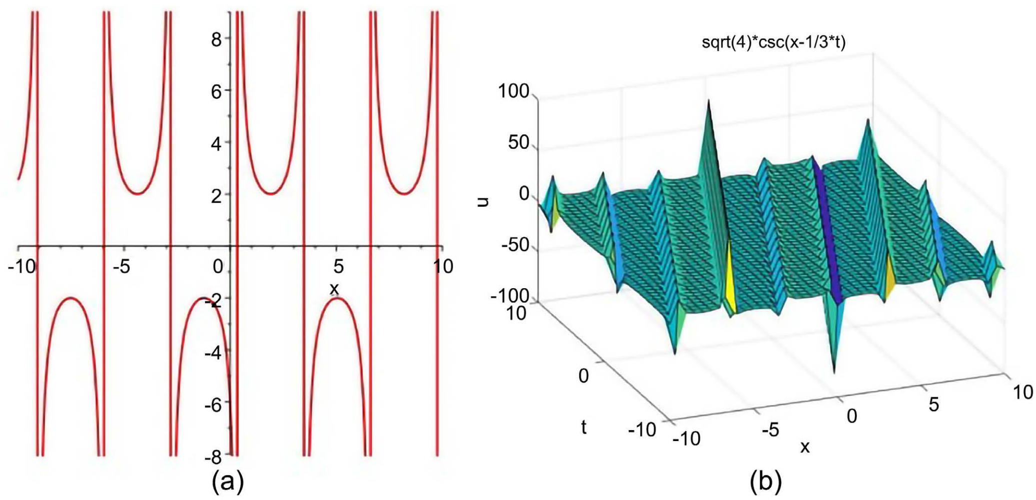

Figure 7. The solution $u = u_3(x,t)$ in (33) with $a = -1$, $b = -2$, $\omega = -\frac{1}{3}$, $-10 \le x, t \le 10$: (a) the 2D plot for $t = 1$;(b) the 3D plot.

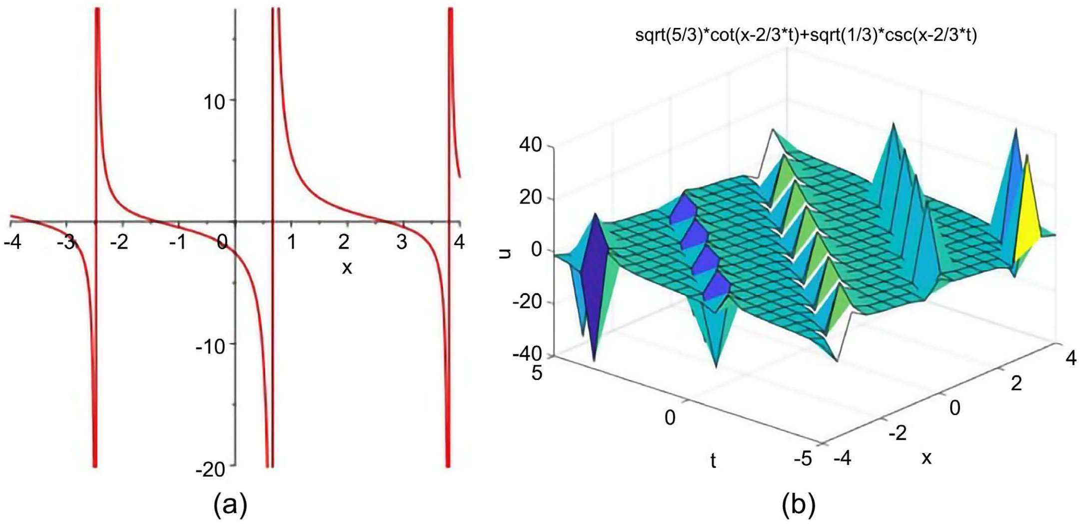

Figure 8. The solution $u = u_5(x,t)$ in (34) with $a = 1$, $b = 1$, $\omega = -\frac{2}{3}$, $-4 \le x \le 4$, $-5 \le t \le 5$: (a) the 2D plot for $t = 1$; (b) the 3D plot.

4. Conclusion

In this paper, we apply the ansatze method to find the exact solutions of the mBBM equation. Including the singular-shape traveling wave solution, the Bell-shape traveling wave solution, the kink-shape traveling wave solution and the periodic traveling wave solution, the dynamic behaviors of solutions are given.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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