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Rational Energy Decay Rate of a Wave Equation: The Case of Dimension ≥ 2

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Abstract

We apply the multiplier method to obtain the rational energy decay rate of the energy of wave equation in case $n \geq 2$, under an assumption on the potential energy.

Keywords

Wave Equation, Decay Rate, Multiplier Method

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$, and let ν denote the outward unit normal vector to $\partial\Omega$. Given a point $x_0 \in \mathbb{R}^n \setminus \bar{\Omega}$, set $m(x) := x - x_0$,

$$\Gamma_0 := \left\{x \in \partial\Omega : m(x)\cdot\nu(x) \leq 0\right\} \text{ and } \Gamma = \partial\Omega \setminus \Gamma_0,$$

and assume that $m \cdot \nu > 0$ on $\bar{\Gamma}$. We are going to study the long time behavior of the solutions of the following system:

$$\begin{cases} y_{tt} - \Delta y = 0 \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma_0, \\ y_{tt} + \partial_\nu y + y_t = 0 \text{ on } \Gamma, \\ y(0) = y_0,\ \ y_t(0) = y_1,\ \ y_t(0)\big|_\Gamma = w_0. \end{cases} \tag{1}$$

Its decay rate has been investigated by various techniques in the past; see, e.g., [1] and [2]. Our method will be based on a theorem of Haraux; see [3].

First, we study the well-posedness of (1). We set

$$E := \left\{y \in H^1(\Omega), y\big|_{\Gamma_0} = 0\right\},$$

and we introduce the Hilbert space

$$H := E \times L^2(\Omega) \times L^2(\Gamma)$$

with the inner product

$$\langle (y,z,\xi),(a,b,c) \rangle := \int_\Omega (\nabla y \nabla a + zb)\,\mathrm{d}v + \int_\Gamma \xi c\,\mathrm{d}\Gamma.$$

Proposition 1.1. *The system* (1) *is well-posed in H.*

Proof. Let us introduce the operators

$$A\left(y,z,z|_\Gamma\right) := \left(z,\Delta y,-\partial_\nu y\right) \text{ and } B\left(y,z,z|_\Gamma\right) := \left(0,0,z|_\Gamma\right)$$

with

$$D(A) := \left\{\left(y,z,z|_\Gamma\right) \in H : y \in H^2(\Omega) \text{ and } z \in E\right\} \text{ and } D(B) := H.$$

Setting $u := \left(y,z,z|_\Gamma\right)$ with $z := y_t$ we have

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \left(y_t,z_t,z_t|_\Gamma\right)\\ &= \left(z,\Delta y,-\partial_\nu y - z|_\Gamma\right)\\ &= \left(z,\Delta y,-\partial_\nu y\right) + \left(0,0,z|_\Gamma\right)\\ &= Au + Bu,\end{aligned}$$

and a simple computation shows that

$$\left((A+B)u,u\right) = (Au,u) + (Bu,u) = 0 - \int_\Gamma \left(z|_\Gamma\right)^2 \mathrm{d}\Gamma = -\int_\Gamma \left(z|_\Gamma\right)^2 \mathrm{d}\Gamma.$$

Using the techniques in ([4], Page 141), we get $R\left(I-(A+B)\right) = H$, and then applying Theorem 1.2.3 in ([5], Page 3), we conclude that the operator $A+B$ generates a C_0 semigroup of contraction $S(t)$. □

The main purpose of this paper is to prove the following result concerning the energy of the solutions.

Theorem 1.2. *Let us define the energy by the formula*

$$E(t) = \frac{1}{2}\left\{\int_\Omega \left((\nabla y)^2 + y_t^2\right)\mathrm{d}v + \int_\Gamma y_t^2|_\Gamma \,\mathrm{d}\Gamma\right\},$$

and assume the following assumption on the potential energy of smooth solutions:

$$\int_\Omega \|\nabla y\|^2 \,\mathrm{d}v \le C_1(\Omega)\int_\Gamma \|\nabla y\|^2 \,\mathrm{d}\Gamma \le C_2(\Omega)\int_\Gamma \left|\partial_\nu y\right|^2 \mathrm{d}\Gamma,$$

with suitable constants $C_1(\Omega)$*, and* $C_2(\Omega)$*. If* $(y_0,y_1,w_0) \in D(A)$*, then there exists a constant M such that*

$$E(t) \le E(0)\frac{2M}{M+t}$$

for every $t \ge 0$.

We prove Theorem 1.2 by the multiplier method in the following two sections, first for $n=2$ and then for $n \ge 3$.

2. Proof of Theorem 1.2 for *n* = 2

Taking the derivative of $E(t)$, we obtain $E_t(t) = -\int_\Gamma y_t^2(x,t)\,\mathrm{d}\Gamma$, so that the energy is a decreasing function.

Since $E(t)=\frac{1}{2}\|u\|^2$, we can consider the energy of higher order $E^1(t)=\frac{1}{2}\|u_t\|^2$. We multiply the equality $y_{tt}=\Delta y$ by $m\cdot\nabla y E(t)$, then we integrate by parts for t with $0\le S\le t\le T$, and finally we use Rellich's formula for $v\in\Omega$ to obtain the following equality:

$$\frac{n}{2}\int_\Omega\int_S^T y_t^2 E(t)\,\mathrm{d}t\mathrm{d}v-\frac{n-2}{2}\int_S^T\int_\Omega E(t)(\nabla y)^2\,\mathrm{d}v\mathrm{d}t$$
$$=\frac{1}{2}\int_\Gamma\int_S^T \nu\cdot m y_t^2 E(t)\,\mathrm{d}t\mathrm{d}\Gamma+\int_S^T\int_\Omega E_t(t)y_t m\cdot\nabla y\,\mathrm{d}v\mathrm{d}t$$
$$-\int_S^T\int_\Gamma E(t)\left(\frac{1}{2}m\cdot\nu(\nabla y)^2-\partial_\nu y m\cdot\nabla y\right)\mathrm{d}\Gamma\mathrm{d}t-\left[\int_\Omega y_t m\cdot\nabla y E(t)\,\mathrm{d}v\right]_S^T.$$

Since $n=2$ in this section, we have

$$\int_\Omega\int_S^T y_t^2 E(t)\,\mathrm{d}t\mathrm{d}v=\frac{1}{2}\int_\Gamma\int_S^T \nu\cdot m y_t^2 E(t)\,\mathrm{d}t\mathrm{d}\Gamma+\int_S^T\int_\Omega E_t(t)y_t m\cdot\nabla y\,\mathrm{d}v\mathrm{d}t$$
$$-\int_S^T\int_\Gamma E(t)\left(\frac{1}{2}m\cdot\nu(\nabla y)^2-\partial_\nu y m\cdot\nabla y\right)\mathrm{d}\Gamma\mathrm{d}t$$
$$-\left[\int_\Omega y_t m\cdot\nabla y E(t)\,\mathrm{d}v\right]_S^T.$$

Now we majorize all the terms on the right hand side of the above equality:

$$0\le\int_\Gamma\int_S^T \nu\cdot m y_t^2 E(t)\,\mathrm{d}t\mathrm{d}\Gamma=\int_S^T E(t)\int_\Gamma \nu\cdot m y_t^2 E(t)\,\mathrm{d}\Gamma\mathrm{d}t$$
$$\le E(S)\int_S^T\int_\Gamma \nu\cdot m y_t^2\,\mathrm{d}\Gamma\mathrm{d}t\le E(S)\|m\|_\infty\int_S^T -E_t(t)\,\mathrm{d}t$$
$$=E(S)\|m\|_\infty\left(E(S)-E(T)\right)\le E(S)\|m\|_\infty E(0).$$

We note that by the Cauchy-Schwarz inequality we have

$$\left|\int_\Omega y_t m\cdot\nabla y\,\mathrm{d}v\right|\le\|m\|_\infty E(t)$$

for all $t\ge 0$, so that

$$\left|\int_S^T\int_\Omega E_t(t)y_t m\cdot\nabla y\,\mathrm{d}v\mathrm{d}t-\left[\int_\Omega y_t m\cdot\nabla y E(t)\,\mathrm{d}v\right]_S^T\right|\le 3\|m\|_\infty E(S)E(0).$$

Using the inequality $|ab|\le\frac{a^2+b^2}{2}$ hence we obtain the estimate

$$\left|\int_\Gamma\left(\frac{1}{2}m\cdot\nu(\nabla y)^2-\partial_\nu y m\cdot\nabla y\right)\mathrm{d}\Gamma\right|\le A\int_\Gamma|\partial_\nu y|^2\frac{\|m\|}{2m\cdot\nu}\mathrm{d}\Gamma\le C\int_\Gamma|\partial_\nu y|^2\,\mathrm{d}\Gamma$$

with some constants A and C, and this implies the following relations:

$$\left|\int_S^T\int_\Gamma E(t)\left(\frac{1}{2}m\cdot\nu(\nabla y)^2-\partial_\nu y m\cdot\nabla y\right)\mathrm{d}\Gamma\mathrm{d}t\right|$$
$$\le C\int_S^T E(t)\int_\Gamma|\partial_\nu y|^2\,\mathrm{d}\Gamma\mathrm{d}t\le 2C\int_S^T E(t)\int_\Gamma\left(y_t^2+y_{tt}^2\right)\mathrm{d}t\mathrm{d}\Gamma$$
$$\le 2CE(S)\left(\int_S^T\left(\int_\Gamma y_t^2\mathrm{d}\Gamma+\int_\Gamma y_{tt}^2\mathrm{d}\Gamma\right)\mathrm{d}t\right)$$
$$\le 2CE(S)\left(\int_S^T -E_t(t)\,\mathrm{d}t+\int_S^T -E_t^1(t)\,\mathrm{d}t\right)$$
$$\le 2CE(S)\left(E(S)-E(T)+E^1(S)-E^1(T)\right)\le 2CE(S)\left(E(0)+E^1(0)\right).$$

In passing, we have obtained the estimate

$$\int_S^T E(t)\int_\Gamma y_t^2 \mathrm{d}\Gamma\mathrm{d}t \le E(S)E(0).$$

Using the assumption on potential energy and the above inequalities we obtain with some constant M that

$$\int_S^{+\infty} E^2(t)\mathrm{d}t \le ME(0)E(S)$$

for all $S \ge 0$. Now applying Haraux's theorem (see [2] or [3]) we conclude that

$$E(t) \le E(0)\frac{2M}{M+t}$$

for all $t \ge 0$.

3. Proof of Theorem 1.2 for $n \ge 3$

For $n \ge 3$ we have to modify the proof of the case $n = 2$ because one of the terms in Rellich's formula does not vanish any more.

Taking the derivative of $E(t)$, we have $E_t(t) = -\int_\Gamma y_t^2(x,t)\mathrm{d}\Gamma$: so the energy is a decreasing function. We note that $E(t) = \frac{1}{2}\|u\|^2$, so we can consider the energy of high order: $E^1(t) = \frac{1}{2}\|u_t\|^2$. So we begin by the equality $y_{tt} = \Delta y$, that we multiply by $m\nabla yE(t)$, then we integrate by parts for t, with $0 \le S \le t \le T$, and we use Rellich's formula for $v \in \Omega$, to obtain

$$\begin{aligned}
&\frac{n}{2}\int_\Omega\int_S^T y_t^2 E(t)\mathrm{d}t\mathrm{d}v \\
&= \frac{n-2}{2}\int_S^T\int_\Omega E(t)(\nabla y)^2 \mathrm{d}v\mathrm{d}t + \frac{1}{2}\int_\Gamma\int_S^T v\cdot my_t^2 E(t)\mathrm{d}t\mathrm{d}\Gamma \\
&\quad + \int_S^T\int_\Omega E_t(t)y_t m\cdot\nabla y\mathrm{d}v\mathrm{d}t - \int_S^T\int_\Gamma E(t)\left(\frac{1}{2}m\cdot v(\nabla y)^2 - \partial_v ym\cdot\nabla y\right)\mathrm{d}\Gamma\mathrm{d}t \\
&\quad - \left[\int_\Omega y_t m\cdot\nabla yE(t)\mathrm{d}v\right]_S^T.
\end{aligned} \tag{2}$$

Now we majorize all terms on the right hand side of the above equality:

$$\begin{aligned}
0 &\le \int_\Gamma\int_S^T v\cdot my_t^2 E(t)\mathrm{d}t\mathrm{d}\Gamma = \int_S^T E(t)\int_\Gamma v\cdot my_t^2 E(t)\mathrm{d}\Gamma\mathrm{d}t \\
&\le E(S)\int_S^T\int_\Gamma v\cdot my_t^2\mathrm{d}\Gamma\mathrm{d}t \le E(S)\|m\|_\infty\int_S^T -E_t(t)\mathrm{d}t \\
&\le E(S)\|m\|_\infty\left(E(S)-E(T)\right) \le E(S)\|m\|_\infty E(0).
\end{aligned}$$

We note that by the Cauchy-Schwarz inequality we have

$$\left|\int_\Omega y_t m\cdot\nabla y\mathrm{d}v\right| \le \|m\|_\infty E(t), \quad \forall t \ge 0,$$

so that

$$\left|\int_S^T\int_\Omega E_t(t)y_t m\cdot\nabla y\mathrm{d}v\mathrm{d}t - \left[\int_\Omega y_t m\cdot\nabla yE(t)\mathrm{d}v\right]_S^T\right| \le 3\|m\|_\infty E(S)E(0).$$

Using the inequality $|ab| \le \frac{a^2+b^2}{2}$ hence we obtain the inequality

$$\left|\int_\Gamma\left(\frac{1}{2}m\cdot\nu(\nabla y)^2-\partial_\nu y m\cdot\nabla y\right)\mathrm{d}\Gamma\right|\le A\int_\Gamma|\partial_\nu y|^2\frac{\|m\|}{2m\cdot\nu}\mathrm{d}\Gamma\le C\int_\Gamma|\partial_\nu y|^2\,\mathrm{d}\Gamma$$

for some constants *A* and *C*, and therefore

$$\begin{aligned}&\left|\int_S^T\int_\Gamma E(t)\left(\frac{1}{2}m\cdot\nu(\nabla y)^2-\partial_\nu y m\cdot\nabla y\right)\mathrm{d}\Gamma\mathrm{d}t\right|\\&\le C\int_S^T E(t)\int_\Gamma|\partial_\nu y|^2\,\mathrm{d}\Gamma\mathrm{d}t\le 2C\int_S^T E(t)\int_\Gamma\left(y_t^2+y_{tt}^2\right)\mathrm{d}t\mathrm{d}\Gamma\\&\le 2CE(S)\left(\int_S^T\left(\int_\Gamma y_t^2\mathrm{d}\Gamma+\int_\Gamma y_{tt}^2\mathrm{d}\Gamma\right)\mathrm{d}t\right)\\&\le 2CE(S)\left(\int_S^T-E_t(t)\mathrm{d}t+\int_S^T-E_t^1(t)\mathrm{d}t\right)\\&\le 2CE(S)\left(E(S)-E(T)+E^1(S)-E^1(T)\right)\le 2CE(S)\left(E(0)+E^1(0)\right).\end{aligned}$$

In passing, we have obtained the estimate

$$\int_S^T E(t)\int_\Gamma y_t^2\mathrm{d}\Gamma\mathrm{d}t\le E(S)E(0).$$

Using the assumption on the potential energy and the above inequalities hence we infer that

$$\int_S^{+\infty}E^2(t)\mathrm{d}t\le ME(0)E(S).$$

for all $S>0$, with some constant *M*. Now by applying Haraux's theorem we conclude that

$$E(t)\le E(0)\frac{2M}{M+t},\quad\forall t\ge 0.$$

4. Conclusion

Under some a priori assumptions on the potential energy, we have obtained a polynomial decay rate of the solutions of the wave equation with dynamic boundary feedback by the multiplier method.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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