Journal of Applied Mathematics and Physics, 2022, 10, 2820-2836
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

On the Meromorphic Solutions of Fermat-Type Differential Equations

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How to cite this paper: Liu, D.F. and Pan, B. (2022) On the Meromorphic Solutions of Fermat-Type Differential Equations. *Journal of Applied Mathematics and Physics*, **10**, 2820-2836.
https://doi.org/10.4236/jamp.2022.109188

Received: August 22, 2022
Accepted: September 27, 2022
Published: September 30, 2022

Abstract

In this paper, we investigate the meromorphic solutions of the Fermat-type differential equations $f'(z)^n + f(z+c)^m = \mathrm{e}^{Az+B}$ $(c \neq 0)$ over the complex plane $\mathbb{C}$ for positive integers m,n, and A,B,c are constants. Our results improve and extend some earlier results given by Liu *et al.* Moreover, some examples are presented to show the preciseness of our results.

Keywords

Fermat-Type Equations, Differential Equations, Nevanlinna Theory

1. Introduction and Main Results

Throughout this paper, we concentrate on such meromorphic functions that are nonconstant and meromorphic in the whole complex plane $\mathbb{C}$. Then it is assumed that the reader is familiar with the fundamental notation and terminology of Nevanlinna's value distribution theory (see [1] [2] [3] [4]) and in particular with the most usual of symbols: $T(r,f)$, $N(r,f)$, $\bar{N}(r,f)$, $m(r,f)$ and the order $\rho(f)$ and so on. Meanwhile, we denote by $\mathcal{S}(f)$ the family of all meromorphic functions α such that $T(\alpha,f)=o\{T(r,f)\}$, where $r\to+\infty$ outside of a possible exceptional set of finite logarithmic measure. Moreover, we also include all constant functions in $\mathcal{S}(f)$.

The following equation

$$f^n(z)+g^n(z)=1 \tag{1.1}$$

can be regarded as the Fermat diophantine equations $x^n+y^n=1$ over function fields, where n is a positive integer. Montel [5] obtained that Equation (1.1) has no nonconstant entire solutions when $n>2$. Gross [6] proved that Equation (1.1) has no nonconstant meromorphic solutions when $n>3$. For $n=2$,

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Gross [7] showed that all meromorphic solutions of Equation (1.1) of the form $f(z)=\frac{1-\alpha^2(z)}{1+\alpha^2(z)}$ and $g(z)=\frac{2\alpha(z)}{1+\alpha^2(z)}$, where $\alpha(z)$ is a nonconstant meromorphic function. For $n=3$, Baker [8] proved that the only nonconstant meromorphic solutions of Equation (1.1) are the functions $f=\frac{1}{2}\left\{1+\frac{\wp'(u)}{\sqrt{3}}\right\}\Big/\wp(u)$ and $g=\frac{\eta}{2}\left\{1-\frac{\wp'(u)}{\sqrt{3}}\right\}\Big/\wp(u)$ for a nonconstant entire function u and a cubic root η of unity, where $\wp$ denotes the Weierstrass $\wp$ function. Further, Yang [9] investigated a generalization of the Fermat-type equation

$$f^n(z)+g^m(z)=1 \tag{1.2}$$

and obtained that Equation (1.2) has no nonconstant entire solutions when $\frac{1}{m}+\frac{1}{n}<1$. For more detail, we refer the reader to the work of Hu, Li and Yang [10].

As we known, Halburd-Korhonen [11] and Chiang-Feng [12] independently proved the difference analogue of the logarithmic derivative lemma in 2006. Afterwards, a number of papers have focused on entire solutions of complex difference equations and differential-difference equations. For some works related to partial differential equations of Fermat type, see [13]-[18].

In 2012, Liu *et al.* [13] investigated the entire solutions of the Fermat-type differential equation of the

$$f'(z)^n+f(z+c)^m=1 \tag{1.3}$$

and obtained the following results.

Theorem A ([13]) Equation (1.3) has no transcendental entire solutions with finite order, provided that $m\neq n$, where n,m are positive integers, $c(\neq 0)$ is a constant.

Theorem B ([13]) The transcendental entire solutions with finite order of the equation $f'(z)^2+f(z+c)^2=1$ must satisfy $f(z)=\sin(z+B\mathrm{i})$, where B is a constant and $c=2k\pi$ or $c=(2k+1)\pi$, k is an integer.

For $m=n=1$, Liu [13] gave examples to illustrate the existence of the solutions of the Equation (1.3).

Example A $f(z)=1+\mathrm{e}^z$ is a solution of the equation $f'(z)+f(z+c)=1$, where $\mathrm{e}^c=-1$.

Example B $f(z)=1+\sin z$ is a solution of the equation $f'(z)+f(z+c)=1$, where $c=\frac{3\pi}{2}$.

Since no attempts, till now, have so far been made by any researchers investigating the form of the solution of Equation (1.3) When $m=n=1$. Naturally, we pose the following questions.

Question 1. Can we get the forms of the solutions of Equation (1.3) when $m=n=1$.

In addition, we recall the definition of exponential polynomial, an exponential polynomial of order $q \ge 1$, which is an entire function of the form:

$$f(z) = P_1(z)\mathrm{e}^{Q_1(z)} + P_2(z)\mathrm{e}^{Q_2(z)} + \cdots + P_k(z)\mathrm{e}^{Q_k(z)}, \tag{1.4}$$

where $P_i(z)$ and $Q_i(z)$ are polynomials in z for $1 \le j \le k$, such that $q = \max\{\deg(Q_j) : 1 \le j \le k\}$. Following Steinmetz [19], such a function can be written in the form:

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}, \tag{1.5}$$

where $\omega_1, \omega_2, \cdots, \omega_m$ $(1 \le k \le m)$ are m distinct finite nonzero complex numbers, while $H_j(z) \not\equiv 0$ $(j = 1, 2, \cdots, m)$ is either an exponential polynomial of degree less than q or an ordinary polynomial in z for $0 \le m \le j$.

In this paper, we pay our attention to the above question and prove the following some theorems that improve and extend Theorem B.

Theorem 1.1. Let $f(z)$ be the exponential polynomial solutions of the equation

$$f'(z) + f(z+c) = 1, \tag{1.6}$$

where $c \in \mathbb{C} \setminus \{0\}$, one of the following conclusions hold:

1) If $c = -\frac{1}{e}$, then $f(z) = 1 + (az+b)e^{-ez} + \sum_{i=1}^{m} b_i e^{w_i z}$, where $a, b, b_i \in \mathbb{C}$, w_i satisfy $\mathrm{e}^{-\frac{1}{e}w_i} + w_i = 0\ (w_i \ne -\mathrm{e})$.

2) If $c \ne -\frac{1}{e}$, then $f(z) = 1 + \sum_{i=1}^{m} b_i e^{w_i z}$, where $b_i \in \mathbb{C}$, w_i satisfy $e^{w_i c} + w_i = 0$.

Example 1.1. If $c = -\frac{1}{e}$ in (1.6), then the exponential polynomial solutions of the following equation

$$f'(z) + f\left(z - \frac{1}{e}\right) = 1$$

must satisfy $f(z) = 1 + (az+b)e^{-ez} + \sum_{i=1}^{m} b_i e^{w_i z}$, where $a, b, b_i \in \mathbb{C}$, $e^{-\frac{1}{e}w_i} + w_i = 0\ (w_i \ne -e)$. It is easy to obtain that the above equation has a solution $f(z) = 1 + (1+z)e^{-ez}$.

Example 1.2. If $c = -\frac{\pi}{2} + 2k\pi$ in (1.6), then the exponential polynomial solutions of the following equation

$$f'(z) + f\left(z - \frac{\pi}{2} + 2k\pi\right) = 1$$

must satisfy $f(z) = 1 + \sum_{i=1}^{m} b_i e^{w_i z}$, where $b_i \in \mathbb{C}$, $e^{-\frac{\pi}{2}w_i} + w_i = 0$. It is easy to obtain that the above equation has a solution $f(z) = 1 + a_1 e^{\mathrm{i}z} + a_2 e^{-\mathrm{i}z}$, where a_1, a_2 are constants.

Further, we study the solutions of Fermat-type differential equation

$$f'(z)^n + f(z+c)^m = e^{Az+B}, \tag{1.7}$$

where m,n are positive integers, $c \in \mathbb{C}\setminus\{0\}$, $A,B,c \in \mathbb{C}$. First of all, one fact needs to be clear. For $n=m$, the general trivial solutions of Equation (1.7) is $f(z) = de^{\frac{Az+B}{n}}$, where $d^n\left(\left(\frac{A}{n}\right)^n + e^{Ac}\right) = 1$. In this paper, we study the solution of nontrivial solutions and prove the following results.

Theorem 1.2. Let $f(z)$ be the exponential polynomial solutions of the equation

$$f'(z) + f(z+c) = e^{Az+B}, \tag{1.8}$$

where $A,B,c \in \mathbb{C}$ and $c\setminus\{0\}$, one of the following conclusions hold:

1) If $A+e^{Ac} \neq 0$ and $c = -\frac{1}{e}$, then $f(z) = \mathrm{e}^{Az+B}\left(\frac{1}{A+\mathrm{e}^{Ac}} + (az+b)\mathrm{e}^{(-e-A)z} + \sum_{i=1}^{m} b_i \mathrm{e}^{w_i z}\right)$, where $a,b,b_i \in \mathbb{C}$, w_i satisfy $e^{-\frac{1}{e}(A+w_i)} + A + w_i = 0$. If $A+e^{Ac} \neq 0$ and $c \neq -\frac{1}{e}$, then $f(z) = e^{Az+B}\left(\frac{1}{A+e^{Ac}} + \sum_{i=1}^{m} b_i e^{w_i z}\right)$, where $b_i \in \mathbb{C}$, w_i satisfy $e^{(A+w_i)c} + A + w_i = 0$.

2) If $A+e^{Ac} = 0$, $c = -\frac{1}{e}$ and $Ac \neq 1$ (namely $A \neq -e$), then $f(z) = e^{Az+B}\left(a_0 + \frac{1}{1+ce^{Ac}}z + (az+b)e^{(-e-A)z} + \sum_{i=1}^{m} b_i e^{w_i z}\right)$, where $a_0,a,b,b_i \in \mathbb{C}$, w_i satisfy $e^{-\frac{1}{e}(A+w_i)} + A + w_i = 0$. If $A+e^{Ac} = 0$, $c \neq -\frac{1}{e}$ and $Ac \neq 1$, then $f(z) = e^{Az+B}\left(a_0 + \frac{1}{1+ce^{Ac}}z + \sum_{i=1}^{m} b_i e^{w_i z}\right)$, where $a_0,b_i \in \mathbb{C}$, w_i satisfy $e^{(A+w_i)c} + A + w_i = 0$.

3) If $A+e^{Ac} = 0$, $c = -\frac{1}{e}$ and $A = -e$ (namely $Ac = 1$), then $f(z) = e^{Az+B}\left(a_0 + a_1 z + \frac{1}{c^2 e^{Ac}}z^2 + \sum_{i=1}^{m} b_i e^{w_i z}\right)$, where $a_0,a_1,b_i \in \mathbb{C}$, w_i satisfy $e^{-\frac{1}{e}(-e+w_i)} - e + w_i = 0$.

Example 1.3. If $c = \pi\mathrm{i}$, $A=2$, $B=1$ in (1.8), then the exponential polynomial solutions of the following equation

$$f'(z) + f(z+\pi\mathrm{i}) = e^{2z+1}$$

must satisfy $f(z) = e^{2z+1}\left(\frac{1}{3} + \sum_{i=1}^{m} b_i e^{w_i z}\right)$, where $b_i \in \mathbb{C}$, $e^{(2+w_i)\pi\mathrm{i}} + 2 + w_i = 0$. It is easy to obtain that the above equation has a solution $f(z) = e^{2z+1}\left(\frac{1}{3} + e^{-z}\right)$.

Example 1.4. If $c=-\frac{1}{e}$, $A=1$, $B=-1$ in (1.8), then the exponential polynomial solutions of the following equation

$$f'(z)+f\left(z-\frac{1}{e}\right)=e^{z-1}$$

must satisfy $f(z)=e^{z-1}\left(\frac{1}{1+e^{-\frac{1}{e}}}+(az+b)e^{(-e-1)z}+\sum_{i=1}^{m}b_ie^{w_iz}\right)$, where $a,b,b_i\in\mathbb{C}$, $e^{(1+w_i)\left(-\frac{1}{e}\right)}+1+w_i=0$. It is easy to obtain that the above equation has a solution $f(z)=e^{z-1}\left(\frac{1}{1+e^{-\frac{1}{e}}}+(1+z)e^{(-e-1)z}\right)$.

Example 1.5. If $c=\pi\mathrm{i}$, $A=1$, $B=1$ in (1.8), then the exponential polynomial solutions of the following equation

$$f'(z)+f(z+\pi\mathrm{i})=e^{z+1}$$

must satisfy $f(z)=e^{z+1}\left(a_0+\frac{1}{1-\pi i}z+\sum_{i=1}^{m}b_ie^{w_iz}\right)$, where $a_0,b_i\in\mathbb{C}$, $e^{(1+w_i)\pi i}+1+w_i=0$. It is easy to obtain that the above equation has a solution $f(z)=e^{z+1}\left(1+\frac{1}{1-\pi i}z\right)$.

Example 1.6. If $c=-\frac{1}{e}$, $A=-e$, $B=1$ in (1.8), then the exponential polynomial solutions of the following equation

$$f'(z)+f\left(z-\frac{1}{e}\right)=e^{-ez+1}$$

must satisfy $f(z)=e^{-ez+1}\left(a_0+a_1z+ez^2+\sum_{i=1}^{m}b_ie^{w_iz}\right)$, where $a_0,a_1,b_i\in\mathbb{C}$, $e^{(-e+w_i)\left(-\frac{1}{e}\right)}-e+w_i=0$. It is easy to obtain that the above equation has a solution $f(z)=e^{-ez+1}\left(1+z+ez^2\right)$.

Theorem 1.3. The transcendental entire solutions of

$$f'(z)^2+f(z+c)^2=e^{Az+B} \tag{1.9}$$

must satisfy

$$f(z)=\frac{e^{h(z-c)}-e^{-h(z-c)}}{2i}e^{\frac{Ac}{2}}e^{\frac{Az+B}{2}},$$

where $h(z)=az+b$, $a^2=\frac{A^2}{4}-e^{Ac}$ and $c=\frac{\ln\frac{a-\frac{A}{2}}{\mathrm{i}}+2k\pi\mathrm{i}}{\frac{A}{2}-a}$.

Remark 1. Since for a particular choice of $A=B=0$ in Theorem 1.3, then

$e^{Az+B}=1$, $a^2=\frac{A^2}{4}-e^{Ac}=-1$, namely $a=\pm\mathrm{i}$. If $a=\mathrm{i}$, then $c=2k\pi$, $f(z)=\sin(z-B\mathrm{i})$, if $a=-\mathrm{i}$, then $c=(2k+1)\pi$, $f(z)=\sin(z+B\mathrm{i})$, k is an integer. The result obtained is the same as Theorem B, and Theorem 1.3 omit the condition of the finite order. Therefore, Theorem 1.3 improves and extends Theorem B.

Theorem 1.4. The following equation

$$f'(z)^n+f(z+c)^n=e^{Az+B} \tag{1.10}$$

has no nontrivial meromorphic solutions, where $n\geq 3$ is positive integers and $c\in\mathbb{C}\setminus\{0\}$.

Theorem 1.5 Let m,n be positive integers satisfying $\frac{1}{m}+\frac{1}{n}<\frac{2}{3}$, then Equation (1.7) has no nontrivial meromorphic solutions with $\rho(f)\neq 1$.

2. Some Lemmas

Lemma 2.1. (see [2]) Let $f_j(z)(j=1,2,\cdots,n)(n\geq 2)$ be meromorphic functions, $g_j(z)(j=1,2,\cdots,n)$ be entire functions satisfying following conditions:

1) $\sum_{j=1}^{n}f_j(z)e^{g_j(z)}\equiv 0$;

2) for $1\leq j<k\leq n$, g_j-g_k are not constants;

3) for $1\leq j\leq n, 1\leq h<k\leq n$,

$$T(r,f_j)=o\left\{T\left(r,e^{g_h-g_k}\right)\right\}, r\to\infty, r\notin E.$$

where $E\subset(1,\infty)$, E is a possible exceptional set of finite logarithmic measure, then $f_j(z)\equiv 0(j=1,2,\cdots,n)$.

Lemma 2.2. (see [2]) Let $f_1(z)$, $f_2(z)$, $f_3(z)$ be meromorphic functions and satisfying $\sum_{i=1}^{3}f_i(z)\equiv 1$, f_1 be nonconstant and

$$\sum_{j=1}^{3}N\left(r,\frac{1}{f_j}\right)+2\sum_{j=2}^{3}\overline{N}(r,f_j)<(\lambda+o(1))T(r), r\in I$$

where $0\leq\lambda<1$, $T(r)=\max_{1<j<3}\{T(r,f_j)\}$, and I has infinite linear measure, then $f_2(z)\equiv 1$ or $f_3(z)\equiv 1$.

Lemma 2.3. (see [2]) Let $f(z)$ be a nonconstant meromorphic function in the complex plane and $R(f)=\frac{P(f)}{Q(f)}$, where $P(f)=\sum_{k=0}^{p}a_kf^k$ and $Q(f)=\sum_{j=0}^{q}b_jf^j$ are two mutually prime polynomials in f. If the coefficients $\{a_k(z)\}$, $\{b_j(z)\}$ are small functions of f and $a_p(z)\not\equiv 0$, $b_q(z)\not\equiv 0$, then

$$T(r,R(f))=\max\{p,q\}\cdot T(r,f)+S(r,f).$$

Lemma 2.4. (see [2]) Let $h(z)$ be a nonconstant entire function and $f(z)=e^{h(z)}$, then $T(r,h')=o(T(r,f))$.

Lemma 2.5. (see [9]) Let $a(z)$, $b(z)$, $f(z)$, $g(z)$ be meromorphic functions and satisfying $T(r,a(z))=o(T(r,f))$, $T(r,b(z))=o(T(r,g))$. If

$\max\{m,n\} > 3$, $\min\{m,n\} \geq 3$, then the following equation

$$a(z)f^n(z) + b(z)g^m(z) = 1$$

has no meromorphic solution.

Lemma 2.6. (see [11]) Let $f(z)$ be a nonconstant meromorphic function with $\rho(f) < +\infty$, $c \in \mathbb{C}$, then for $\varepsilon > 0$ we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O\left(r^{\rho-1+\varepsilon}\right),$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.7. (see [12]) Let $f(z)$ be a nonconstant meromorphic function with $\rho(f) < +\infty$, $c \in \mathbb{C}$, then for $\varepsilon > 0$, we have

$$T(r, f(z+c)) = T(r, f(z)) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r),$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.8. (see [20]) Let $f(z)$ be a nonconstant meromorphic function, then $\rho(f) = \rho(f')$ and $\mu(f) = \mu(f')$.

Lemma 2.9 (see [21]) Let $f(z)$ be a nonconstant meromorphic function and $f_1(z) = f(az+b)$, $a \neq 0$, then $\rho(f) = \rho(f_1)$.

3. Proofs of Theorems

Proof of Theorem 1.1

Assume that $f(z)$ is the exponential polynomial solutions of (1.6), substituting (1.5) into (1.6) yields

$$\begin{aligned} &f'(z) + f(z+c) \\ &= H_0'(z) + H_0(z+c) + \left[H_1'(z) + \omega_1 q z^{q-1} H_1(z) + H_1(z+c) e^{\omega_1(z+c)^q - \omega_1 z^q}\right] e^{\omega_1 z^q} \\ &\quad + \cdots + \left[H_m'(z) + \omega_m q z^{q-1} H_m(z) + H_m(z+c) e^{\omega_m(z+c)^q - \omega_m z^q}\right] e^{\omega_m z^q} \\ &= 1. \end{aligned} \tag{3.1}$$

Let

$$f_i = \begin{cases} H_0'(z) + H_0(z+c),\ i = 0, \\ H_i'(z) + \omega_i q z^{q-1} H_i(z) + H_i(z+c) e^{\omega_i(z+c)^q - \omega_i z^q},\ 1 \leq i \leq m. \end{cases} \tag{3.2}$$

$$g_i = \begin{cases} 0,\ i = 0, \\ \omega_i z^q,\ 1 \leq i \leq m. \end{cases}$$

Further, we get

$$g_j - g_k = \begin{cases} \omega_k z^q,\ j = 0, 1 \leq k \leq m, \\ \omega_j z^q - \omega_k z^q, 1 \leq j < k \leq m. \end{cases} \tag{3.3}$$

From (3.2) and (3.3), for $0 \leq i \leq m$, $0 \leq j < k \leq m$, we have

$$T(r, f_i) = o\left\{T\left(r, e^{g_j - g_k}\right)\right\}, r \to \infty.$$

Combining (3.1) with Lemma 2.1, we know that

$$\begin{cases} H_0'(z)+H_0(z+c)-1\equiv 0, \\ H_1'(z)+\omega_1 qz^{q-1}H_1(z)+H_1(z+c)e^{\omega_1(z+c)^q-\omega_1 z^q}\equiv 0, \\ \vdots \\ H_i'(z)+\omega_i qz^{q-1}H_i(z)+H_i(z+c)e^{\omega_i(z+c)^q-\omega_i z^q}\equiv 0, \\ \vdots \\ H_m'(z)+\omega_m qz^{q-1}H_m(z)+H_m(z+c)e^{\omega_m(z+c)^q-\omega_m z^q}\equiv 0. \end{cases} \tag{3.4}$$

We deduce from (3.4) that $q=1$. If not, suppose that $q\ge 2$, we have

$$H_i'(z)+\omega_i qz^{q-1}H_i(z)+H_i(z+c)e^{\omega_i(z+c)^q-\omega_i z^q}\equiv 0,(i=1,2,\cdots,m).$$

Namely

$$e^{\omega_i(z+c)^q-\omega_i z^q}\equiv -\left(\frac{H_i'(z)}{H_i(z+c)}+\omega_i qz^{q-1}\frac{H_i(z)}{H_i(z+c)}\right),(i=1,2,\cdots,m). \tag{3.5}$$

Using logarithmic derivative Lemma and Lemma 2.6 in (3.5), then for $\varepsilon>0$, we have

$$\begin{aligned} m\left(r,e^{\omega_i(z+c)^q-\omega_i z^q}\right) &= m\left(r,\frac{H_i'(z)}{H_i(z+c)}+\omega_i qz^{q-1}\frac{H_i(z)}{H_i(z+c)}\right) \\ &= m\left(r,\frac{H_i'(z)}{H_i(z)}\frac{H_i(z)}{H_i(z+c)}+\omega_i qz^{q-1}\frac{H_i(z)}{H_i(z+c)}\right) \\ &\le m\left(r,\frac{H_i'(z)}{H_i(z)}\right)+2m\left(r,\frac{H_i(z)}{H_i(z+c)}\right)+m\left(r,\omega_i qz^{q-1}\right)+\log 2 \\ &\le S\left(r,H_i(z)\right)+O\left(r^{q-2+\varepsilon}\right)+O(\log r) \\ &\le o\left(r^{q-1}\right), \end{aligned}$$

which implies that $q=1$. Thus, we have

$$f(z)=H_0(z)+H_1(z)e^{\omega_1 z}+\cdots+H_m(z)e^{\omega_m z},$$

where $H_0(z),H_1(z),\cdots,H_m(z)$ are polynomials.

Assume that

$$H_0(z)=a_k z^k+a_{k-1}z^{k-1}+\cdots+a_1 z+a_0.$$

Further, we have

$$\begin{aligned} &H_0'(z)=ka_k z^{k-1}+(k-1)a_{k-1}z^{k-2}+\cdots+a_1, \\ &H_0(z+c)=a_k(z+c)^k+a_{k-1}(z+c)^{k-1}+\cdots+a_1(z+c)+a_0. \end{aligned} \tag{3.6}$$

Substituting (3.6) into $H_0'(z)+H_0(z+c)-1\equiv 0$ yields

$$\begin{cases} a_k=0, \\ C_k^1(1+c)a_k+a_{k-1}=0, \\ C_k^2c^2a_k+C_{k-1}^1(1+c)a_{k-1}+a_{k-2}=0, \\ \vdots \\ C_k^ic^ia_k+C_{k-1}^{i-1}c^{i-1}a_{k-1}+\cdots+C_{k-i+1}^1(1+c)a_{k-i+1}+a_{k-i}=0, \\ \vdots \\ C_k^{k-1}c^{k-1}a_k+C_{k-1}^{k-2}c^{k-2}a_{k-1}+\cdots+C_2^1(1+c)a_2+a_1=0, \\ c^ka_k+c^{k-1}a_{k-1}+\cdots+(1+c)a_1+a_0=1, \end{cases} \tag{3.7}$$

where $C_k^i = \dfrac{k!}{i!(k-i)!}$.

Assume that

$$H_i(z) = a_{t_i} z^{t_i} + a_{t_i-1} z^{t_i-1} + \cdots + a_{1_i} z + b_i,\ (i = 1, 2, \cdots, m).$$

Further, we get

$$\begin{aligned} &H_i'(z) = t_i a_{t_i} z^{t_i-1} + (t_i - 1) a_{t_i-1} z^{t_i-2} + \cdots + 2a_{2_i} z + a_{1_i}, \\ &H_i(z+c) = a_{t_i}(z+c)^{t_i} + a_{t_i-1}(z+c)^{t_i-1} + \cdots + a_{1_i}(z+c) + b_i \quad (i = 1, 2, \cdots, m). \end{aligned} \tag{3.8}$$

Substituting (3.8) into $H_i'(z) + \omega_i H_i(z) + H_i(z+c) e^{\omega_i c} \equiv 0$ yields

$$\begin{cases} \left(e^{w_i c} + w_i\right) a_{t_i} = 0, \\ \left(ce^{w_i c} + 1\right) t_i a_{t_i} + \left(e^{w_i c} + w_i\right) a_{t_i-1} = 0, \\ C_{t_i}^2 c^2 e^{w_i c} a_{t_i} + \left(ce^{w_i c} + 1\right)(t_i - 1) a_{t_i-1} + \left(e^{w_i c} + w_i\right) a_{t_i-2} = 0, \\ \qquad \vdots \\ C_{t_i}^j c^j e^{w_i c} a_{t_i} + \cdots + C_{t_i-j+2}^2 c^2 e^{w_i c} a_{t_i-j+2} + \left(ce^{w_i c} + 1\right)(t_i - j + 1) a_{t_i-j+1} \\ + \left(e^{w_i c} + w_i\right) a_{t_i-j} = 0, \\ \qquad \vdots \\ C_{t_i}^{t_i-1} c^{t_i-1} e^{w_i c} a_{t_i} + \cdots + C_3^2 c^2 e^{w_i c} a_{3_i} + \left(ce^{w_i c} + 1\right) 2a_{2_i} + \left(e^{w_i c} + w_i\right) a_{1_i} = 0, \\ c^{t_i} e^{w_i c} a_{t_i} + \cdots + \cdots + c^2 e^{w_i c} a_{2_i} + \left(ce^{w_i c} + 1\right) a_{1_i} + \left(e^{w_i c} + w_i\right) b_i = 0. \end{cases} \tag{3.9}$$

From (3.7), we can deduce that $a_k = a_{k-1} = a_{k-2} = \cdots = a_1 = 0$ and $a_0 = 1$, which implies that $H_0(z) = 1$.

From (3.9), we can conclude the following

1) If $e^{w_i c} + w_i \neq 0$, we obtain $a_{t_i} = a_{t_i-1} = \cdots = a_{1_i} = b_i = 0$, then $H_i(z) = 0$, which contradicts that $f(z)$ is nonconstant.

2) If $e^{w_i c} + w_i = 0$ and $w_i c \neq 1$, we obtain $a_{t_i} = a_{t_i-1} = \cdots = a_{1_i} = 0$, then $H_i(z) = b_i$, where $b_i \in \mathbb{C}$.

3) If $e^{w_i c} + w_i = 0$ and $w_i c = 1$, we obtain $a_{t_i} = a_{t_i-1} = \cdots = a_{2_i} = 0$, then $H_i(z) = a_{1_i} z + b_i$, where $a_{1_i}, b_i \in \mathbb{C}$.

From the above discussion, we can get that the case of the exponential polynomial solutions of Equation (1.6) is as follows.

Case 1. If $c = -\dfrac{1}{e}$, then the exponential polynomial solutions of Equation (1.6) must satisfy $f(z) = 1 + (az + b) e^{-ez} + \sum_{i=1}^m b_i e^{w_i z}$, where $a, b, b_i \in \mathbb{C}$, w_i satisfy $e^{-\frac{1}{e} w_i} + w_i = 0 \left(w_i \neq -e\right)$.

Case 2. If $c \neq -\dfrac{1}{e}$, then the exponential polynomial solutions of Equation (1.6) must satisfy $f(z) = 1 + \sum_{i=1}^m b_i e^{w_i z}$, where $b_i \in \mathbb{C}$, w_i satisfy $e^{w_i c} + w_i = 0$.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2

We rewrite (1.8) as follows

$$e^{-(Az+B)}f'(z)+e^{-(Az+B)}f(z+c)=1. \tag{3.10}$$

Assume that $f(z)$ is the exponential polynomial solutions of (3.10) and let

$$f(z)=e^{Az+B}\left(H_0(z)+H_1(z)e^{\omega_1 z^q}+\cdots+H_m(z)e^{\omega_m z^q}\right).$$

Thus, we have

$$\begin{aligned} f'(z)&=Ae^{Az+B}\left(H_0(z)+H_1(z)e^{\omega_1 z^q}+\cdots+H_m(z)e^{\omega_m z^q}\right)\\ &\quad+e^{Az+B}\left(H_0'(z)+H_1'(z)e^{\omega_1 z^q}+H_1(z)\omega_1 qz^{q-1}e^{\omega_1 z^q}+\cdots\right.\\ &\quad\left.+H_m'(z)e^{\omega_m z^q}+H_m(z)\omega_m qz^{q-1}e^{\omega_m z^q}\right),\\ f(z+c)&=e^{Ac}e^{Az+B}\left(H_0(z+c)+H_1(z+c)e^{\omega_1(z+c)^q}+\cdots\right.\\ &\quad\left.+H_m(z+c)e^{\omega_m(z+c)^q}\right). \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10) yields

$$\begin{aligned} &e^{-(Az+B)}f'(z)+e^{-(Az+B)}f(z+c)\\ &=AH_0(z)+H_0'(z)+e^{Ac}H_0(z+c)\\ &\quad+\left(AH_1(z)+H_1'(z)+H_1(z)\omega_1 qz^{q-1}+e^{Ac}H_1(z+c)e^{\omega_1(z+c)^q-\omega_1 z^q}\right)e^{\omega_1 z^q}+\cdots\\ &\quad+\left(AH_m(z)+H_m'(z)+H_m(z)\omega_m qz^{q-1}+e^{Ac}H_m(z+c)e^{\omega_m(z+c)^q-\omega_m z^q}\right)e^{\omega_m z^q}\\ &=1. \end{aligned} \tag{3.12}$$

Let

$$f_i=\begin{cases} AH_0(z)+H_0'(z)+e^{Ac}H_0(z+c)-1,\ i=0,\\ AH_i(z)+H_i'(z)+H_i(z)\omega_i qz^{q-1}+e^{Ac}H_i(z+c)e^{\omega_i(z+c)^q-\omega_i z^q},\ 1\le i\le m. \end{cases} \tag{3.13}$$

$$g_i=\begin{cases} 0,\ i=0,\\ \omega_i z^q,\ 1\le i\le m. \end{cases}$$

Further, we have

$$g_j-g_k=\begin{cases} \omega_k z^q,\ j=0,1\le k\le m,\\ \omega_j z^q-\omega_k z^q,1\le j<k\le m. \end{cases} \tag{3.14}$$

From (3.13) and (3.14), for $0\le i\le m$, $0\le j<k\le m$, we have

$$T(r,f_i)=o\left\{T\left(r,e^{g_j-g_k}\right)\right\},\quad r\to\infty.$$

Combining Lemma 2.1 with (3.12), we know that

$$\begin{cases} AH_0(z)+H_0'(z)+e^{Ac}H_0(z+c)-1\equiv 0,\\ AH_1(z)+H_1'(z)+H_1(z)\omega_1 qz^{q-1}+e^{Ac}H_1(z+c)e^{\omega_1(z+c)^q-\omega_1 z^q}\equiv 0,\\ \qquad\vdots\\ AH_i(z)+H_i'(z)+H_i(z)\omega_i qz^{q-1}+e^{Ac}H_i(z+c)e^{\omega_i(z+c)^q-\omega_i z^q}\equiv 0,\\ \qquad\vdots\\ AH_m(z)+H_m'(z)+H_m(z)\omega_m qz^{q-1}+e^{Ac}H_m(z+c)e^{\omega_m(z+c)^q-\omega_m z^q}\equiv 0. \end{cases} \tag{3.15}$$

We deduce from (3.4) that $q=1$. If not, suppose that $q\geq 2$, we have

$$\left(A+\omega_i qz^{q-1}\right)H_i(z)+H_i'(z)+e^{Ac}H_i(z+c)e^{\omega_i(z+c)^q-\omega_i z^q}\equiv 0,\ (i=1,2,\cdots,m).$$

Namely

$$e^{\omega_i(z+c)^q-\omega_i z^q}\equiv -\frac{1}{e^{Ac}}\left(\frac{H_i'(z)}{H_i(z+c)}+\left(A+\omega_i qz^{q-1}\right)\frac{H_i(z)}{H_i(z+c)}\right),\ (i=1,2,\cdots,m). \quad (3.16)$$

Using logarithmic derivative Lemma and Lemma 2.6 in (3.16), then for $\forall\varepsilon>0$, we have

$$\begin{aligned}
&m\left(r,e^{\omega_i(z+c)^q-\omega_i z^q}\right)\\
&=m\left(r,\frac{H_i'(z)}{H_i(z+c)}+\left(A+\omega_i qz^{q-1}\right)\frac{H_i(z)}{H_i(z+c)}\right)+O(1)\\
&=m\left(r,\frac{H_i'(z)}{H_i(z)}\frac{H_i(z)}{H_i(z+c)}+\left(A+\omega_i qz^{q-1}\right)\frac{H_i(z)}{H_i(z+c)}\right)+O(1)\\
&\leq m\left(r,\frac{H_i'(z)}{H_i(z)}\right)+2\,m\left(r,\frac{H_i(z)}{H_i(z+c)}\right)+m\left(r,A+\omega_i qz^{q-1}\right)+O(1)\\
&\leq S\left(r,H_i(z)\right)+O\left(r^{q-2+\varepsilon}\right)+O(\log r)\leq o\left(r^{q-1}\right),
\end{aligned}$$

which implies that $q=1$. Thus, we have

$$f(z)=e^{Az+B}\left(H_0(z)+H_1(z)e^{\omega_1 z}+\cdots+H_m(z)e^{\omega_m z}\right),$$

where $H_0(z),H_1(z),\cdots,H_m(z)$ are polynomials.

Let

$$H_0(z)=a_k z^k+a_{k-1}z^{k-1}+\cdots+a_1 z+a_0. \quad (3.17)$$

Further, we have

$$\begin{aligned}
&H_0'(z)=ka_k z^{k-1}+(k-1)a_{k-1}z^{k-2}+\cdots+a_1,\\
&H_0(z+c)=a_k(z+c)^k+a_{k-1}(z+c)^{k-1}+\cdots+a_1(z+c)+a_0.
\end{aligned} \quad (3.18)$$

Substituting (3.17), (3.18) into $AH_0(z)+H_0'(z)+e^{Ac}H_0(z+c)-1\equiv 0$ yields

$$\begin{cases}
\left(A+e^{Ac}\right)a_k=0,\\
C_k^1\left(1+ce^{Ac}\right)a_k+\left(A+e^{Ac}\right)a_{k-1}=0,\\
C_k^2 e^{Ac}c^2 a_k+C_{k-1}^1\left(1+ce^{Ac}\right)a_{k-1}+\left(A+e^{Ac}\right)a_{k-2}=0,\\
\qquad\vdots\\
C_k^i e^{Ac}c^i a_k+C_{k-1}^{i-1}e^{Ac}c^{i-1}a_{k-1}+\cdots+C_{k-i+1}^1\left(1+ce^{Ac}\right)a_{k-i+1}+\left(A+e^{Ac}\right)a_{k-i}=0,\\
\qquad\vdots\\
C_k^{k-1}e^{Ac}c^{k-1}a_k+C_{k-1}^{k-2}e^{Ac}c^{k-2}a_{k-1}+\cdots+C_2^1\left(1+ce^{Ac}\right)a_2+\left(A+e^{Ac}\right)a_1=0,\\
e^{Ac}c^k a_k+e^{Ac}c^{k-1}a_{k-1}+\cdots+e^{Ac}c^2 a_2+\left(1+ce^{Ac}\right)a_1+\left(A+e^{Ac}\right)a_0=1,
\end{cases} \quad (3.19)$$

where $C_k^i=\dfrac{k!}{i!(k-i)!}$.

Let

$$H_i(z)=a_{t_i}z^{t_i}+a_{t_i-1}z^{t_i-1}+\cdots+a_{1_i}z+b_i,\ \ (i=1,2,\cdots,m). \tag{3.20}$$

Further, we have

$$\begin{aligned}&H_i'(z)=t_ia_{t_i}z^{t_i-1}+(t_i-1)a_{t_i-1}z^{t_i-2}+\cdots+2a_{2_i}z+a_{1_i},\\&H_i(z+c)=a_{t_i}(z+c)^{t_i}+a_{t_i-1}(z+c)^{t_i-1}+\cdots+a_{1_i}(z+c)+b_i\ \ (i=1,2,\cdots,m).\end{aligned} \tag{3.21}$$

Substituting (3.20), (3.21) into $(A+\omega_i)H_i(z)+H_i'(z)+e^{(A+\omega_i)c}H_i(z+c)\equiv 0$ yields

$$\begin{cases}\left(e^{(A+w_i)c}+A+w_i\right)a_{t_i}=0,\\ \left(ce^{(A+w_i)c}+1\right)t_ia_{t_i}+\left(e^{(A+w_i)c}+A+w_i\right)a_{t_i-1}=0,\\ C_{t_i}^2c^2e^{(A+w_i)c}a_{t_i}+\left(ce^{(A+w_i)c}+1\right)(t_i-1)a_{t_i-1}+\left(e^{(A+w_i)c}+A+w_i\right)a_{t_i-2}=0,\\ \qquad\vdots\\ C_{t_i}^jc^je^{(A+w_i)c}a_{t_i}+\cdots+C_{t_i-j+2}^2c^2e^{(A+w_i)c}a_{t_i-j+2}\\ +\left(ce^{(A+w_i)c}+1\right)(t_i-j+1)a_{t_i-j+1}+\left(e^{(A+w_i)c}+A+w_i\right)a_{t_i-j}=0,\\ \qquad\vdots\\ C_{t_i}^{t_i-1}c^{t_i-1}e^{(A+w_i)c}a_{t_i}+\cdots+C_3^2c^2e^{(A+w_i)c}a_3+C_2^1\left(ce^{(A+w_i)c}+1\right)a_2\\ +\left(e^{(A+w_i)c}+A+w_i\right)a_1=0,\\ c^{t_i}e^{(A+w_i)c}a_{t_i}+\cdots+c^2e^{(A+w_i)c}a_{2_i}+\left(ce^{(A+w_i)c}+1\right)a_{1_i}+\left(e^{(A+w_i)c}+A+w_i\right)b_i=0.\end{cases} \tag{3.22}$$

From (3.19), we can conclude the following

(i) If $A+e^{Ac}\neq 0$, we obtain $a_k=a_{k-1}=\cdots=a_2=a_1=0$, $a_0=\dfrac{1}{A+e^{Ac}}$, then $H_0(z)=\dfrac{1}{A+e^{Ac}}$.

(ii) If $A+e^{Ac}=0$ and $Ac\neq 1$, we obtain $a_k=a_{k-1}=\cdots=a_2=0$, $a_1=\dfrac{1}{1+ce^{Ac}}$, then $H_0(z)=\dfrac{1}{1+ce^{Ac}}z+a_0$, where $a_0\in\mathbb{C}$.

(iii) If $A+e^{Ac}=0$ and $Ac=1$, we obtain $a_k=a_{k-1}=\cdots=a_3=0$, $a_2=\dfrac{1}{c^2e^{Ac}}$, then $H_0(z)=\dfrac{1}{c^2e^{Ac}}z^2+a_1z+a_0$, where $a_0,a_1\in\mathbb{C}$.

From (3.22), we can conclude the following

(i) If $e^{(A+w_i)c}+A+w_i\neq 0$, we obtain $a_{t_i}=a_{t_i-1}=\cdots=a_{1_i}=b_i=0$, then $H_i(z)=0$, which contradicts that $f(z)$ is nonconstant.

(ii) If $e^{(A+w_i)c}+A+w_i=0$ and $(A+w_i)c\neq 1$, we obtain $a_{t_i}=a_{t_i-1}=\cdots=a_{1_i}=0$, then $H_i(z)=b_i$, where $b_i\in\mathbb{C}$.

(iii) If $e^{(A+w_i)c}+A+w_i=0$ and $(A+w_i)c=1$, we obtain $a_{t_i}=a_{t_i-1}=\cdots=a_{2_i}=0$, then $H_i(z)=a_{1_i}z+b_i$, where $a_{1_i},b_i\in\mathbb{C}$.

From the above discussion, we can get that the case of the exponential polynomial solutions of Equation (1.8) is as follows.

Case 1. If $A+e^{Ac}\neq 0$ and $c=-\frac{1}{e}$, then $f(z)=e^{Az+B}\left(\frac{1}{A+e^{Ac}}+(az+b)e^{(-e-A)z}+\sum_{i=1}^{m}b_ie^{w_iz}\right)$, where $a,b,b_i\in\mathbb{C}$, w_i satisfy $e^{-\frac{1}{e}(A+w_i)}+A+w_i=0$. If $A+e^{Ac}\neq 0$ and $c\neq -\frac{1}{e}$, then $f(z)=e^{Az+B}\left(\frac{1}{A+e^{Ac}}+\sum_{i=1}^{m}b_ie^{w_iz}\right)$, where $b_i\in\mathbb{C}$, w_i satisfy $e^{(A+w_i)c}+A+w_i=0$.

Case 2. If $A+e^{Ac}=0$, $c=-\frac{1}{e}$ and $Ac\neq 1$ (namely $A\neq -e$), then $f(z)=e^{Az+B}\left(\frac{1}{1+ce^{Ac}}z+a_0+(az+b)e^{(-e-A)z}+\sum_{i=1}^{m}b_ie^{w_iz}\right)$, where $a_0,a,b,b_i\in\mathbb{C}$, w_i satisfy $e^{-\frac{1}{e}(A+w_i)}+A+w_i=0$. If $A+e^{Ac}=0$, $c\neq -\frac{1}{e}$ and $Ac\neq 1$, then $f(z)=e^{Az+B}\left(\frac{1}{1+ce^{Ac}}z+a_0+\sum_{i=1}^{m}b_ie^{w_iz}\right)$, where $a_0,b_i\in\mathbb{C}$, w_i satisfy $e^{(A+w_i)c}+A+w_i=0$.

Case 3. If $A+e^{Ac}=0$, $c=-\frac{1}{e}$ and $A=-e$ (namely $Ac=1$), then $f(z)=e^{Az+B}\left(\frac{1}{c^2e^{Ac}}z^2+a_0+a_1z+\sum_{i=1}^{m}b_ie^{w_iz}\right)$, where $a_0,a_1,b_i\in\mathbb{C}$, w_i satisfy $e^{-\frac{1}{e}(-e+w_i)}-e+w_i=0$.

Thus, we complete the proof of Theorem 1.2.

Proof of Theorem 1.3

Assume that $f(z)$ is a transcendental entire solution of (1.9), we rewrite (1.9) as follows

$$\left(e^{-\frac{Az+B}{2}}f'(z)\right)^2+\left(e^{-\frac{Az+B}{2}}f(z+c)\right)^2=1, \tag{3.23}$$

then

$$\left(e^{-\frac{Az+B}{2}}f'(z)+\mathrm{i}e^{-\frac{Az+B}{2}}f(z+c)\right)\left(e^{-\frac{Az+B}{2}}f'(z)-\mathrm{i}e^{-\frac{Az+B}{2}}f(z+c)\right)=1.$$

It then follows that $e^{-\frac{Az+B}{2}}f'(z)+\mathrm{i}e^{-\frac{Az+B}{2}}f(z+c)$, $e^{-\frac{Az+B}{2}}f'(z)-\mathrm{i}e^{-\frac{Az+B}{2}}f(z+c)$ have no zeros. With Welerstrass factorization theorem for entire functions, we have

$$\begin{cases}e^{-\frac{Az+B}{2}}f'(z)+\mathrm{i}e^{-\frac{Az+B}{2}}f(z+c)=e^{h(z)},\\ e^{-\frac{Az+B}{2}}f'(z)-\mathrm{i}e^{-\frac{Az+B}{2}}f(z+c)=e^{-h(z)},\end{cases} \tag{3.24}$$

where $h(z)$ is a entire function. From (3.24), we get

$$\begin{cases} f'(z)=\dfrac{e^{h(z)}+e^{-h(z)}}{2}e^{\frac{Az+B}{2}}, \\ f(z+c)=\dfrac{e^{h(z)}-e^{-h(z)}}{2\mathrm{i}}e^{\frac{Az+B}{2}}. \end{cases} \tag{3.25}$$

From (3.25), on the one hand, we obtain $f(z)=\dfrac{e^{h(z-c)}-e^{-h(z-c)}}{2\mathrm{i}}e^{\frac{Ac}{2}}e^{\frac{Az+B}{2}}$, on the other hand, we have

$$\mathrm{i}\left(e^{h(z+c)}+e^{-h(z+c)}\right)e^{\frac{Ac}{2}}=\left(h'(z)+\frac{A}{2}\right)e^{h(z)}+\left(h'(z)-\frac{A}{2}\right)e^{-h(z)}. \tag{3.26}$$

Next we divide our discussion into two cases.

Case 1. If $h(z)$ is a constant, then

$$\mathrm{i}\left(e^{h(z)}+e^{-h(z)}\right)e^{\frac{Ac}{2}}=\frac{A}{2}e^{h(z)}-\frac{A}{2}e^{-h(z)},$$

from the above identity, we can deduce that $h(z)=\dfrac{1}{2}\left(\ln\dfrac{\frac{A}{2}+\mathrm{i}e^{\frac{Ac}{2}}}{\frac{A}{2}-\mathrm{i}e^{\frac{Ac}{2}}}+2k\pi\mathrm{i}\right)$.

Case 2. If $h(z)$ is a non-constant entire function, then from (3.26), we get

$$-e^{2h(z+c)}+\frac{h'(z)+\frac{A}{2}}{\mathrm{i}e^{\frac{Ac}{2}}}e^{h(z+c)+h(z)}+\frac{h'(z)-\frac{A}{2}}{\mathrm{i}e^{\frac{Ac}{2}}}e^{h(z+c)-h(z)}=1. \tag{3.27}$$

Denote

$$f_1=-e^{2h(z+c)},\ f_2=\frac{h'(z)+\frac{A}{2}}{\mathrm{i}e^{\frac{Ac}{2}}}e^{h(z+c)+h(z)},\ f_3=\frac{h'(z)-\frac{A}{2}}{\mathrm{i}e^{\frac{Ac}{2}}}e^{h(z+c)-h(z)}.$$

Obviously, f_1 is a nonconstant and by Lemma 2.4 and 2.9, we can obtain that $T(r,h'(z))=o\left(T\left(r,e^{2h(z+c)}\right)\right)$. Thus, we obtain

$$\sum_{j=1}^{3}N\left(r,\frac{1}{f_j}\right)+2\sum_{j=2}^{3}\overline{N}(r,f_j)=o\left(T\left(r,e^{2h(z+c)}\right)\right). \tag{3.28}$$

Combining (3.28) and Lemma 2.2, we get $f_2(z)\equiv 1$ or $f_3(z)\equiv 1$.

Now two subcases will be considered in the following.

Subcase 2.1. If $f_2(z)\equiv 1$, then from (3.28), we have

$$e^{h(z+c)+h(z)}=\frac{\mathrm{i}e^{\frac{Ac}{2}}}{h'(z)+\frac{A}{2}}=\frac{h'(z)-\frac{A}{2}}{\mathrm{i}e^{\frac{Ac}{2}}}.$$

Further, we get

$$-e^{Ac}=\left(h'(z)\right)^2-\frac{A^2}{4},$$

which implies that $h(z)$ must be a polynomial and $\deg(h)=1$. Note that

$e^{h(z+c)+h(z)}$ is a transcendental entire function and $\frac{h'(z)-\frac{A}{2}}{\mathrm{i}e^{\frac{Ac}{2}}}$ is a constant, which is a contradiction.

Subcase 2.2. If $f_3(z)\equiv 1$, then from (3.28), we have

$$e^{h(z+c)-h(z)}=\frac{\mathrm{i}e^{\frac{Ac}{2}}}{h'(z)-\frac{A}{2}}=\frac{h'(z)+\frac{A}{2}}{\mathrm{i}e^{\frac{Ac}{2}}}.$$

From the above identity, we can get that

$$-e^{Ac}=\left(h'(z)\right)^2-\frac{A^2}{4}, \tag{3.29}$$

which implies that $h(z)$ must be a polynomial and $\deg(h)=1$. Assume that $h(z)=az+b$, from (3.29), we have $a^2=\frac{A^2}{4}-e^{Ac}$ and $c=\frac{\ln\frac{a-\frac{A}{2}}{\mathrm{i}}+2k\pi\mathrm{i}}{\frac{A}{2}-a}$.

This completes the proof of Theorem 1.3.

Proof of Theorem 1.4

Now we divide our discussion into two cases.

Case 1. Assume that $f(z)$ is a nonconstant entire solutions of (1.10), we rewrite (1.10) as follows

$$\left(e^{-\frac{Az+B}{n}}f'(z)\right)^n+\left(e^{-\frac{Az+B}{n}}f(z+c)\right)^n=1. \tag{3.30}$$

Denote $F=e^{-\frac{Az+B}{n}}f'(z)$, $G=e^{-\frac{Az+B}{n}}f(z+c)$, from the references [[5], Theorem3], we get the equation $F^2+G^2=1$ has no nonconstant entire function solution when $n>2$. Thus, Equation (3.30) has no nontrivial entire function solution when $n>2$.

Case 2. Assume that $f(z)$ is a meromorphic solutions with at least one pole of (1.10), we rewrite (1.10) as follows

$$f'(z-c)^n=e^{Az+B}-f(z)^n. \tag{3.31}$$

Suppose that z_0 is p multiplicity pole of $f(z)$. From (3.31), we get z_0-c is p multiplicity pole of $f'(z)$, which implies that z_0-c is $p-1$ multiplicity pole of $f(z)$. Thus we get z_0-2c is $p-1$ multiplicity pole of $f'(z)$, which implies that z_0-2c is $p-2$ multiplicity pole of $f(z)$. Sequential recurrence, we can get that z_0-pc is 1 multiplicity pole of $f'(z)$, this contradiction with $f(z)$ is a meromorphic function with at least one pole.

This completes the proof of Theorem 1.4.

Proof of Theorem 1.5

Assume that $f(z)$ is a meromorphic solutions of (1.7), we rewrite (1.7) as follows

$$e^{-Az-B}f'(z)^n+e^{-Az-B}f(z+c)^m=1. \tag{3.32}$$

Next we discuss the following two cases.

Case 1. If $\rho(f(z))>1$, then by lemma 2.8 and lemma 2.9, we can obtain that $\rho(f(z))=\rho(f'(z))=\rho(f(z+c))>1$. This means that $T(r,e^{-Az-B})=o(T(r,f'(z)))$ and $T(r,e^{-Az-B})=o(T(r,f(z+c)))$. This, combining with lemma 2.5, we can get that the Equation (3.32) has no nontrivial meromorphic solutions when $\frac{1}{m}+\frac{1}{n}<\frac{2}{3}$.

Case 2. If $\rho(f(z))<1$, then by lemma 2.8 and lemma 2.9, we get $\rho(f(z))=\rho(f'(z))=\rho(f(z+c))<1$. This means that $T(r,f'(z))=T(r,f(z+c))=O(T(r,f(z)))=o(r)$. Now, comparing the characteristic functions of both side of (1.7), by lemma 2.3, we have $T(r,f'(z)^n+f(z+c)^m)=o(r)$, and we know that $T(r,e^{Az+B})=O(r)$, which is a contradiction. Thus Equation (1.7) has no nontrivial meromorphic solutions with $\rho(f)<1$.

This completes the proof of Theorem 1.5.

Funding

This work is supported by the National Natural Science Foundation of China (Grant No. 11801291) and the Natural Science Foundation of Fujian Province, China (Grant Nos. 2020R0039; 2019J05047; 2019J01672).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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