Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2022, 10, 2780-2795
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The History of the Derivation of Euler's Number

Mohsen Aghaeiboorkheili[1*], John Giuna Kawagle[2]

[1]Department of Mathematics and Computer Science, University of Technology, Lae, Papua New Guinea
[2]Department of Civil Engineering, University of Technology, Lae, Papua New Guinea
Email: *mohsen.aghaeiboorkheili@pnguot.ac.pg

How to cite this paper: Aghaeiboorkheili, M. and Kawagle, J.G. (2022) The History of the Derivation of Euler's Number. *Journal of Applied Mathematics and Physics*, **10**, 2780-2795.
https://doi.org/10.4236/jamp.2022.109185

Received: August 10, 2022
Accepted: September 24, 2022
Published: September 27, 2022

Abstract

This paper summarizes the works of numerous inspiring hard-working mathematicians in chronological order who have toiled the fields of mathematics to bring forward the harvest of Eulers number, also known as Napiers number or more infamously, e.

Keywords

Euler Number, Napier Number, History of Mathematics

1. Introduction

The number e, rather famously known as Euler's number, is one of the irrational numbers, similar to the infamous π. It is a relatively young irrational number, compared to π, having first shown up in calculations in the mid-16th century [1].

In the years that followed, the very intriguing number e remained hidden in early mathematicians' complex calculations, and its existence and important properties were shrouded for mathematicians at that time were on the verge to find other less abstract-more practical physical and mathematical solutions to more pressing problems such as navigation [2].

It was in the early 18th century that Leonhard Euler first mentioned the number e in one of his letters. That is the first time that e had appeared and thanks go to Leonhard Euler for also identifying e's unique properties that makes it an irrational number [3].

Over the years that followed e was further studied but not with as much enthusiasm as mathematicians had for π. Euler's number, among other irrational constants, is nature's way of indicating that the curiosity of men and men will power to verge into the unknown is unlimited [4].

For ages, the mysteries of the universe have been hidden in the strange and intriguing "fabric" of mathematics. Apart from the few mathematically inclined persons in this world, mathematics seems rather like an abstract phenomenon but it is more intertwined into every individual's daily life then one might expect.

It is without a doubt that irrational constants could be the most fascinating wonder of mathematics. As a simple definition, one can say that irrational constants are which can never have a finite representation in digits, having their significant figures expand into infinity. In other words, they go on forever [5]. Some examples include the famous π, the golden ratio and even simply the square roots of some numbers, for example square root of two. However, the constant this piece will be mainly discussing is one that is intertwined into business, computing and almost every other subject derived from mathematics-Euler's constant is represented as e.

2. The Number e First Appears in Mathematics through the Efforts of John Napier in His Work on Logarithm—1618

John Napier of Merchiston (see **Figure 1**) was a Scottish landowner. He is also well known for being a mathematician, keen physicist, and also an astronomer. John Napier is best known to have discovered logarithms and work with logarithmic calculations [6]. During his time, he saw that the calculations involving incredibly large numbers, and very small numbers were a hefty task. Being a practical man who had a strong opinion that mathematics always had to be applied in real life, began by developing a system of logarithms which became the earliest logarithm known.

In essence, logarithms today simplify extensive calculations by allowing the lengthy task of multiplication to be simplified down to just being able to add two numbers to obtain the same value. In other words, a shortcut for calculating

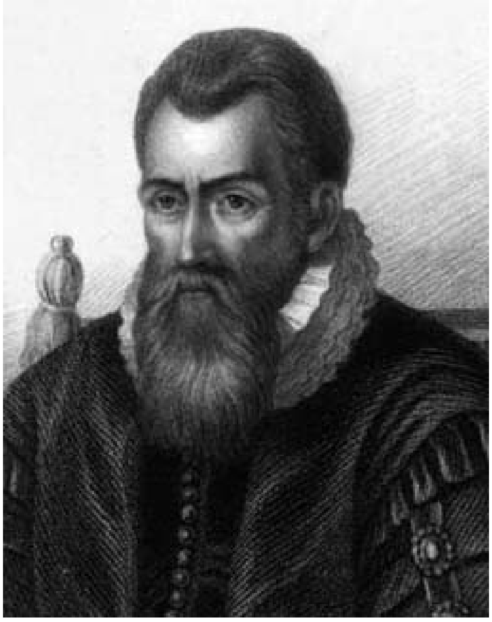

Figure 1. John Napier (1550-1670).

exponents. John Napier's aim was essentially this, so he went on to develop the first system of logarithm which follows the procedure described briefly below.

Associated to every number was a representative numeral. Napier termed this representative numeral at first as an "artificial number" (which he after some time labeled as "logarithm"). The main idea was that when these two representative numbers were "added" (or subtracted) it would give a result which was the equivalent to the result obtained when the two original numbers were multiplied or divided (see Figure 2).

Napier's original work on logarithms is displayed in his publication of *Mirifici Logarithmorum Canonis Descriptio* in the year 1614. The constant e first time grand appearance in this piece of work, but no formal recognition of its uniqueness which stem from its properties were provided. Napier's original logarithms are not as much alike as the logarithm used today. These old logarithms were to the base 1/e and calculations also involved a constant (10^7). Napier defined the logarithms he had found as a ratio of two distances in a geometric form. The logarithms used today however, are based on the concept of exponents. Napier's calculations may not be widely used today, but a clear understanding of Napier's work can help one grasp the concept of the early relation of e with Napier's work and logarithms (see Figure 3).

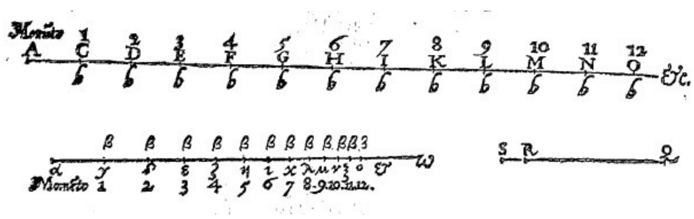

Figure 2. The two original parallel lines that John Napier used with moving particles.

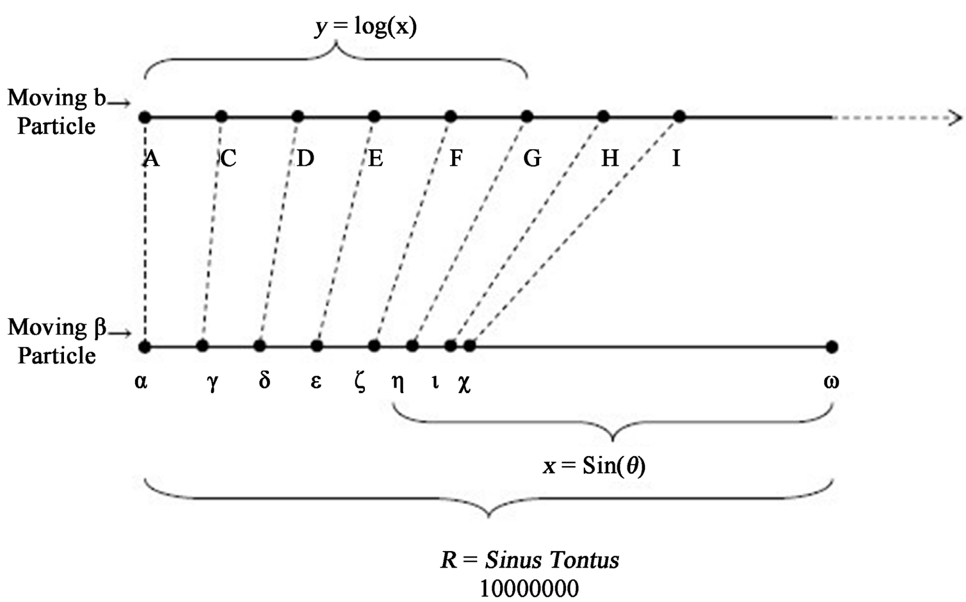

Figure 3. The relation between the two lines and the log and sine functions. The two particles and their line of travel can be clearly seen.

Napier based his ideas of the logarithm in a, what is primarily, a kinematic framework. He began by imagining two particles, one he set moving with a constant velocity on an infinite line to cover equal distances in regular time intervals. The other particle he set moving, at the same time and velocity as the first, but the length of the line this particle had to travel was finite. Its velocity at a particular point was proportional to the distance left to cover in its path [7].

3. Henry Briggs, a Professor at Gresham College in London, Began Investigating Napier's Work and Identifies a Numerical Operation of Logarithm of e to the Base 10—1624

Henry Briggs (see **Figure 4**) was a professor at the Gresham College in London. After Napier published his work on logarithm, Briggs begin studying it and saw a relation which led to the discovery of the base 10 logarithm.

Briggs was very interested in the application of logarithm in navigation, a field of major importance for England at that time. His major work was that he made calculations of a numerical operation to the base 10 logarithm of e. However, Briggs never made any specific referencing to any constants such as e [8].

4. Saint Vincent Computes the Area under a Rectangular Hyperbola—1647

Grgoire de Saint-Vincent (see **Figure 5**) was a famous Flemish mathematician. His study and work, mainly on the quadrature of the hyperbola is what he is remembered for in the history of mathematics. A rectangular hyperbola is any curve given by

$$xy = k$$

on the Cartesian axis. Saint-Vincent found that the area formed under a rectangular hyperbola between the interval $[p,q]$ is the same as over another interval $[r,s]$ when

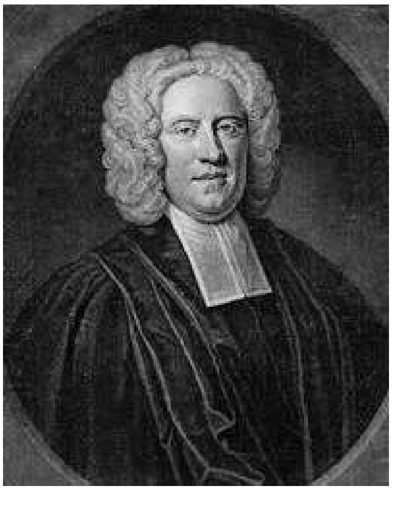

Figure 4. The British mathematician Henry Briggs (1561-1630).

Figure 5. Gregoire Saint Vincent (1584-1667).

$$\frac{p}{q} = \frac{r}{s}$$

Whether or not he recognized the connection with logarithms is still uncertain as no direct evidence of any of his calculations are provided. Even if he did include the extensive calculations, the probability was low that he would come across the number e plainly [9].

5. Huygens Understood the Relation between the Rectangular Hyperbola That Saint Vincent Discovered to the Logarithm—1661

The famous Dutch Christian Huygens (see **Figure 7**) was a highly respected mathematician, inventor and appreciated astronomer of his time. Many consider him to be one of the greatest scientists of all time and a significant figure in the scientific revolution. Fascinating enough, he was also a student of Saint-Vincent. Huygens in his study of the rectangular hyperbola, established the relation between the logarithms that Napier had found and the rectangular hyperbola that Saint Vincent discovered.

It was mentioned previously that a rectangular hyperbola is any curve given by $xy = k$ on the Cartesian axis. A useful property that Huygens paid attention to was that from the hyperbola was that the area under a rectangular hyperbola is the same between interval $[p,q]$ as it was between the interval $[r,s]$ when

$$\frac{p}{q} = \frac{r}{s},$$

Huygens examined clearly the relation between the area under the rectangular hyperbola

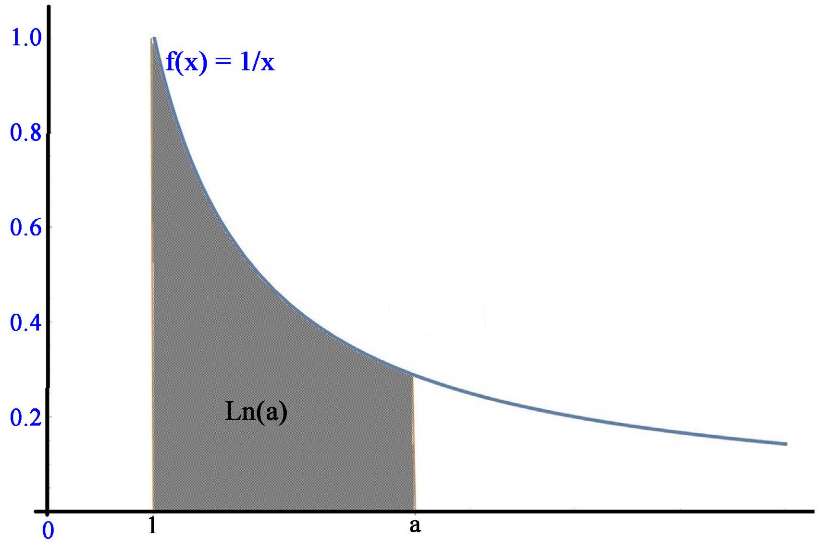

Figure 6. Illustrated here is a rectangular hyperbolic curve. Highlighted in gray is the area under the curve $f(x) = 1/x$ from 1 to *a*. If *a* is less than 1, the area from *a* to 1 is counted as negative.

$$xy = 1$$

and the natural logarithm which is essentially log to the base e (see **Figure 6**). From this property, one can define a function $A(x)$, which is the area under the above curve from 1 to *x*, to be defined by the property that

$$A(xy) = A(x) + A(y).$$

This functional property is what makes logarithms unique, and it is common practice to call such functions $A(x)$ a logarithm. It is worth noting that when the rectangular hyperbola $xy = 1$ is chosen, one obtains the logarithm to base e-also known as the natural logarithm.

Again, from this it is clear to identify the logarithm to base 10 of e, which was calculated to 17 decimal places by Huygens. Finally, for the first-time in history, e has made a major appearance. However, it is just considered as another constant in Huygens's work and does not have much significance as an irrational constant. The sole reason being that the unique properties of e have not been discovered yet. Once again, it is a close call to finding the irrational number, but e remains unrecognized still, and shrouded by other ideologies [5].

6. Nicolaus Mercator Publishes *Logarithmotechnia*, A Brilliant Piece of Work Which Contains the Series Expansion of log(1 + *x*)—1668

Nikolaus Mercator (see **Figure 8**) was a German mathematician, well known for his published work on logarithms-*Logarithmotechnia* (see **Figure 9**). In Mercator's work *Logarithmotechnia*, Mercator labels all the logarithms to the base e with the term "natural logarithm" for the first time. Again, sadly the unique number e remains just another constant and is not appreciated for its uniqueness. It

Figure 7. Christiaan Huygens (1929-1695).

Figure 8. Nicolaus Mercator (1620-1687).

20894

LOGARITHMO-TECHNIA:

SIVE

Methodus construendi

LOGARITHMOS

Nova, accurata, & facilis;

SCRIPTO

Antehàc Communicata, Anno Sc. 1667.

Nonis Augusti: Cui nunc accedit.

Vera Quadratura Hyperbolæ,

&

Inventio summæ Logarithmorum.

AUCTORE NICOLAO MERCATORE

Holsato, è Societate Regia.

HUIC ETIAM JUNGITUR

MICHAELIS ANGELI RICCII Exercitatio

Geometrica de Maximis & Minimis; hic ob Argumenti

praestantiam & Exemplarium raritatem recusa.

LONDINI,

Typis Guilielmi Godbid, & Impensis Mosis Pitt Bibliopolæ, in

vico vulgò vocato Little Britain. Anno M. DC. LXVIII.

Figure 9. The original work-*Logarithmotechnia* of Mercator.

waits elusively just around the corner for another brilliant mind to bring it into light. He used the series expansion of $\log(x+1)$.

7. Jacob Bernoulli Finally "Discovers" the Number e through, not Logarithm, But Instead Compound Interest—1683

The famous Bernoulli (see Figure 10) family had many successful mathematicians but Jacob Bernoulli was the most successful of them. His multiple contributions to calculus make him well known in the mathematical community. It will be shown that he was also the first mathematician to determine the fundamental properties of the irrational mathematical constant e.

The fascinating thing about the discovery of e, is that though extensive work and research was put into logarithms, the number e was not discovered by previous mathematicians in the process of these logarithmic calculations. In fact, it was in an entirely different field; through Jacob Bernoulli's work on compound interest that e became "discovered".

Jacob Bernoulli stumbled across the number e in 1683, while studying a question on compound interest.

His main research was based on the problem of compound interest and, in probing the properties of continuous compound interest. He was after the answer to the limit of

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

The limit had to lie between two and three according to his calculations using the binomial theorem [10]. An expansion of working out similar to his is explained below.

Imagine a bank account has an initial deposit of $1.00 total paying 100 percent in interest annually. The value of the account will be valued at $2.00 at the end of the year after the first interest has been credited. The question in Bernoulli's mind was what would happen if the interest given by the bank had been credited more frequently throughout the following financial year?

Bernoulli calculated that if the interest had been credited twice in any specific year, the interest rate 50% for a time period of 6 months. As a result, the $1 in the opening of the account is multiplied by 1.5 twice in a year, giving the bank

Figure 10. Jacob Bernoulli (1655-1705).

account a value of:

$$\$1.00 \times 1.5^2 = \$2.25$$

If a similar situation was taken but now with the interest being compounded quarterly, it would yield

$$\$1.00 \times 1.25^4 = \$2.4114\cdots$$

Taking a scenario where the interest was compounded monthly would yield

$$\$1 \times \left(1 + \frac{1}{12}\right)^{12} = \$2.613035\cdots$$

If n compounding time frames and intervals are taken, the interest for each interval can be calculated by dividing 100% by n time frames which yields at the end of a financial year a total account valued at (1 + 1/n) raised to n,

$$\$1.00 \times \left(1 + \frac{1}{n}\right)^n .$$

Bernoulli saw that as n approached a very high value, the sequence reached a particular number. When the compounding interval was taken weekly, the account at the end of the year was valued at $2.692597. When the time frame was decreased even farther to a daily interval, it yielded $2.714567 at the end of the fiscal year.

After taking very small-time intervals, Bernoulli found that the sequence limit approached a value of $2.7182818 [7]. This, in the history of mathematics can be taken as the first approximation of e. In addition, this can also be said to be the first time in mathematical history, a number was defined by the limiting process, and this process defines e. Bernoulli did not realize that his work was related to the work on logarithms at that time though.

8. Leibniz Writes a Letter to Huygens, and Formally States the Value of the Constant e, But Uses the Letter *b* Instead—1690

Gottfried Wilhelm Leibniz (see **Figure 11**) was a prominent German mathematician, logician and philosopher of his time. He was also a physics student of Christiaan Huygens.

Leibniz wrote a letter in the year 1690 to another mathematician Huygens. In this particular letter Leibniz had denoted the letter b to represent the value which is now known as e. This can be seen to be the first time that e shows up as a unique number on its own. At last, after decades of computation by numerous mathematicians, the number e had been given a label or name. Despite not being the present name that is used today, at the least, it was recognized.

Leibniz came across the constant e in his work in calculus and the process of integration. His work was done at the same time at which Isaac Newton was around and there is still debate to which mathematician is the father of calculus.

Figure 11. Gottfried Wilhelm Leibniz (1646-1716).

Figure 12. Johann Bernoulli (1667-1748).

9. Johann Bernoulli Commences His Work the Calculus Involving the Exponential Function—1697

Johann Bernoulli (see Figure 12) was a Swiss mathematician of the well renowned Bernoulli family. His major work includes the application of calculus to mechanical problems.

Earlier on, it was stated that there were problems arising from the fact that log was not considered by the mathematical community as a function. It was Johann Bernoulli that began studying the calculus of the exponential function in 1697 which was published in his work *Principia calculi exponentialium seu percurrentium*. This work uses the lengthy process of integrating term by term to calculate many exponential series. This calculus relates the number e and logarithms. Unusually in the history of mathematics, a single family, the Bernoulli's, produced half a dozen outstanding mathematicians over a couple of generations at the end of the 17th and start of the 18th Century (see Figure 13).

Johann Bernoulli is truly a significant mathematician, having made many discoveries not only in the calculus relating to exponential functions, but also in other fields such as solving the *brachystochrone problem* in which the fastest path had t be determined for the descent of a particle between two points in a gravitational field; among other mathematical contributions [10].

10. Leonhard Euler Goes on to Designate the Variable e to Represent the Constant—1731

Leonhard Euler (see **Figure 14**) still remains one of the most admired Swiss mathematician, geographer, engineer and astronomer, having contributed to science and mathematics, numerous discoveries.

A large amount of mathematical contributions was given by Leonhard Euler to the mathematical community that it does not amaze one that much, to find that the notation that is used today-e-to represent the irrational constant is a contribution by Euler. However, it may not be entirely true that the letter e was

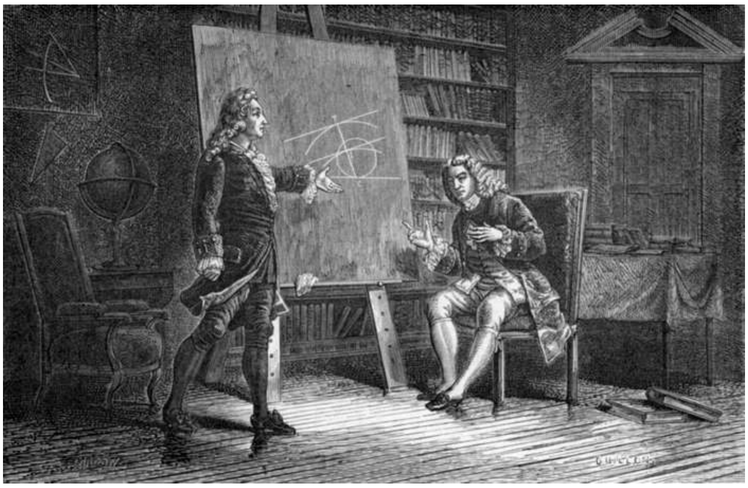

Figure 13. Johann Bernoulli and Jacob Bernoulli working on mathematical problems.

Figure 14. Leonhard Euler (1707-1783).

used because it was the first letter of his name. It may not even be the case that the e was derived from the first letter of the word "exponential". At that time Euler had already assigned the notation *a* to another constant in his study and he used the letter e, which was the vowel after *a* to denote and represent Euler's number e. The reason, whatever it may be, led to the allocation of e to the constant as it was used in a letter that Euler wrote and sent to Goldbach in the year 1731 [2].

11. Leonhard Euler Goes on to Make Various Discoveries Regarding e—1748

Leonhard Euler had made several discoveries of the properties of e in the years that followed and in the year 1748 he published his research *Introductio in Analysin infinitorum* and gave detailed explanations presented formally of the unique properties and ideas relating to e. In particular, Euler proved that

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

and also showed Jacob Bernoulli's work showing es first major approximation

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n .$$

In this piece of work, Euler stated an approximation for number e to 18 decimal places

$$\mathrm{e} = 2.718281828459045235.$$

It remains a mystery of how he went about to derive e to 18 decimal places. He listed it without showing any calculations on how it was done. The most applicable explanation is taking about 20 terms of

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

This gives the answer for e with the same degree of accuracy which Euler had stated.

Another fascinating feature of e that Euler had discovered was the connection between the complex exponential function and the sine and cosine functions. Euler successfully deduced this using De Moivre's formula.

Interestingly, Euler also gave the continued fraction expansion of e and showed that there was a pattern that followed as the calculation propagated. In particular he showed that

$$\frac{\mathrm{e}-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cdots}}}}}$$

Where it can also be observed that

$$e-1=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{6+\cdots}}}}}}}}$$

Euler provide in his work evidence for the patterns he spotted continues. Despite this, he knew that if some form of calculation was found to indicate that the calculations he showed were infinite, it would act as evidence that e was an irrational number.

This is clear to see because the propagated calculations of the fraction (e − 1)/2 follows the pattern like observed for example $6,10,14,\cdots$. Obviously e could not be rational. This simple calculation leads to the conclusion that this may have been the first attempt by a mathematician to prove the irrationality of e [9].

Shank and Glaisher work calculate the value of e to 205 decimal places-1854 The same interest and drive that mathematicians had of calculating more and more decimals for the number e was not as powerful and inspiring as the passion they had at that time for π [2].

There were a few who did calculate its decimal expansion. In the year 1854, Shanks gave a large decimal expansion of e. Interestingly, Shanks was an even more enthusiastic about the calculation of π to a large number of decimal places though. Glaisher showed in his calculations that the first 137 places of Shanks calculations for e were correct but found an error on the 138^{th} value which, after correction by Shanks, gave e to 205 decimal places. A mathematician attempting to solve the problem at that time would need about 120 terms of

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

to achieve an accuracy of such degree and a huge amount of patience[1]. Benjamin Peirce relates the constants e, π and imaginary number i-1864 Benjamin Peirce (see **Figure 15**) during one of his lectures he stated to his students:

Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.

On Euler's identity,

$$e^{i\pi}+1=0$$

Euler's identity can be seen as one equation that represents deep mathematical beauty. There are five of the mathematical constants that are linked by the three basic arithmetic operations-addition, exponentiation and multiplication. And what's more fascinating is that they all occur only once. Here are the five mathematical constants used:

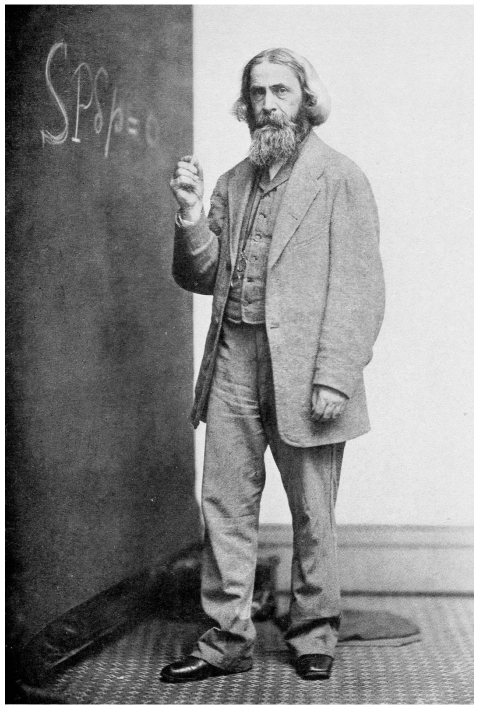

Figure 15. Benjamin Peirce (1809-1880).

- The number 0 which is the identity for addition;
- The number 1 which is the identity for multiplication;
- The number π which is the fundamental circle constant;
- The number e, which is a widely occurring constant in growth and decay analysis;
- The number i which is used in complex number system calculations representing the imaginary unit [7].

12. Hermite Officially Proves That e Is an Irrational Number—1873

Charles Hermite (see **Figure 16**) was a French mathematician who did research concerning number theorems, quadratic forms and complex geometry.

Leonhard Euler is widely regarded as the first mathematician to prove the irrationality of the number e (as show previously). However, in the year 1873 Hermite successfully proved the irrationality of the number e by showing that it wasn't an algebraic number. It would be absurd for any mathematician to seriously believe that e would be algebraic [3].

13. Boorman Calculates e to 346 Decimal Places—1884

In the years that followed other mathematicians worked on the decimal expansion of e. In 1884 Boorman calculated e successfully to 346 decimal places. In his calculations he proved that Shanks calculations were accurate to only 187 decimal places.

To present day, using computer programs the value of e has been given to billions of decimal places. Much like π, the irrational number e still fascinates mathematicians worldwide. It is used in business, engineering and geography. The irrational constant is truly, a wonder.

Here as a clear depiction of the beauty of e. If the equation is graphed

$$y = e^x$$

It can be find that:

1) The slope at any given point on the curve is also equal to e raised to *x*;

2) The area formed under the curve from negative infinity up to *x* is also e raised to *x*.

The function of $y = e^x$ is the only constant in all the fields of science and mathematics studied to date for which the two points above hold true. It brings into perception how intimately e is related the idea of growth-exponential growth (see Figure 17).

Figure 16. Charles Hermite circa (1822-1901).

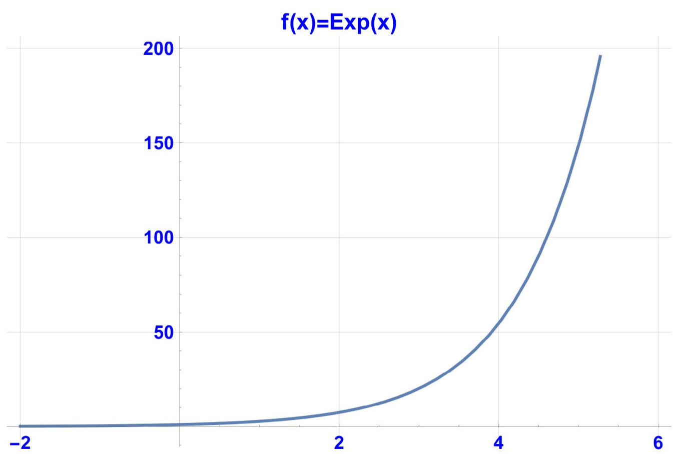

Figure 17. A graph of $y = \exp(x)$ sketched using a graphing software.

14. Conclusion

The universe that encapsulates all, and what human beings can perceive and understand is infinite. There is no saying what other mysterious and beautiful irrational constants lay out there waiting to be discovered. The will of the human spirit is unyielding, and hence if the search to understand better the universe and all structurally-critical constants such as e are what is desired, it can be and will be discovered and understood.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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