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Existence and Stability of Standing Waves with Prescribed L^2-Norm for a Class of Schrödinger-Bopp-Podolsky System

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Abstract

In this paper, we look for solutions to the following Schrödinger-Bopp-Podolsky system with prescribed L^2-norm constraint $\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \, \mathrm{d}x = \rho^2, \end{cases}$, where $q \neq 0$, $a, \rho > 0$ are constants. At first, by the classical minimizing argument, we obtain a ground state solution to the above problem for sufficiently small ρ when $p \in \left(2, \frac{8}{3}\right]$. Secondly, in the case $p = 6$, we show the nonexistence of positive solutions by using a Liouville-type result. Finally, we argue by contradiction to investigate the orbital stability of standing waves for $p \in \left(2, \frac{8}{3}\right]$.

Keywords

Schrödinger-Bopp-Podolsky System, Standing Waves, Normalized Solution, Orbital Stability

1. Introduction

When one looks for standing waves of the Schrödinger equation coupled with the Bopp-Podolsky theory of the electromagnetic field in the purely electrostatic situation, it is equivalent to consider the existence of solutions for the following Schrödinger-Bopp-Podolsky system

$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u,\phi:\mathbb{R}^3\to\mathbb{R}$, $\omega,a>0$, $q\neq 0$, and $p\in(2,6)$. From the physical standpoint, u represents the modulus of the wave function and ϕ is the electrostatic potential, the parameter q has the meaning of the electric charge and a is the parameter of the Bopp-Podolsky term [1]. As is known to all, the Bopp-Podolsky theory, a second-order gauge theory of the electromagnetic field, was developed by Bopp [2] and then independented by Podolsky [3]. According to Mie theory [4] and its generalizations in [5] [6] [7] [8], the Bopp-Podolsky theory was introduced to solve the alleged infinity problem in classical Maxwell theory.

As far as system (1.1) is concerned, there are very few papers related to the existence of solutions. Indeed, to the best of our knowledge, Siciliano and d'Avenia in [9] for the first time showed that system (1.1) possesses nontrivial solutions by means of splitting lemma and the monotonicity trick, when p and q belong to different scope. Meanwhile, they also demonstrated that in the radial case, as $a\to 0$, the solutions they found tend to solutions of the classical Schrödinger-Poisson system. At the same time, Silva and Siciliano [1] proved by the fibering approach that system (1.1) has no solutions at all for large values of q and has two radial solutions for small $q's$, when $p\in(2,3]$. In addition, if system (1.1) is dependent on potentials, that is, non-autonomous, or the corresponding nonlinearity is of more general case, the authors in [10] [11] considered the existence of nontrivial solutions, the main results obtained in [12]-[16] are related to the existence of ground state solutions for system (1.1) with critical growth.

To deal with system (1.1), in the light of its variational structure, it can be reduced to search for nontrivial critical points of the associated energy functional. Actually, define the Hilbert space

$$\mathcal{D}:=\left\{\phi\in\mathcal{D}^{1,2}\left(\mathbb{R}^3\right):\Delta\phi\in L^2\left(\mathbb{R}^3\right)\right\}$$

normed by

$$\|\phi\|_{\mathcal{D}}^2=a^2\|\Delta\phi\|_2^2+\|\nabla\phi\|_2^2,$$

then, according to the Gauss law, it can be proved that for every $u\in H^1\left(\mathbb{R}^3\right)$ there is a unique solution $\phi_u\in\mathcal{D}$ of the second equation in system (1.1). Here, $H^1\left(\mathbb{R}^3\right)$ is the usual Sobolev space with standard norm

$$\|u\|=\left(\|\nabla u\|_2^2+\|u\|_2^2\right)^{\frac{1}{2}}.$$

In other words, a unique element $\phi_u\in\mathcal{D}$ satisfies

$$-\Delta\phi+a^2\Delta^2\phi=4\pi u^2$$

in the weak sense. Moreover, it turns out to be that

$$\phi_u:=\frac{1-\mathrm{e}^{-\frac{|\cdot|}{a}}}{|\cdot|}*u^2\triangleq\mathcal{K}*u^2, \tag{1.2}$$

where $\mathcal{K}:\mathbb{R}^3\setminus\{0\}\to(0,1/a)$.

In view of the solvability of ϕ_u, that is (1.2), system (1.1) can be naturally reduced to the following single equation

$$-\Delta u+\omega u+q^2\phi_u u=|u|^{p-2}u. \tag{1.3}$$

As for (1.3), the corresponding energy functional $J:H^1(\mathbb{R}^3)\to\mathbb{R}$ is defined by

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\omega|u|^2\,\mathrm{d}x+\frac{q^2}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x.$$

Based on the above arguments, if $u\in H^1(\mathbb{R}^3)$ is a critical point of *J*, we call that the pair (u,ϕ_u) is a weak solution of system (1.1). For the simplicity of the notations, throughout this paper we just say $u\in H^1(\mathbb{R}^3)$, instead of $(u,\phi_u)\in H^1(\mathbb{R}^3)\times\mathcal{D}$, is a solution of system (1.1).

Just as mentioned above, all the existing results involving system (1.1) focus on the case that ω is a fixed and assigned parameter. Nevertheless, as a model coupling the Schrödinger field and the electromagnetic field, the physicists are more interested in the existence of "normalized solutions", that is, solutions with prescribed L^2-norm. To this subject, we have not found any references dealing with system (1.1), except for the recent work [17], in which the authors investigated the normalized solutions to a Schrödinger-Bopp-Podolsky system defined on a connected, bounded, smooth open set under Neumann boundary conditions. However, it is must be pointed out that, although there are no results about the normalized solutions of system (1.1), as far as we know, many results concerning the existence or non-existence of normalized solutions to the elliptic problems have been extensively established, see [18]-[22] and the references therein.

Motivated by the above references, especially [1] [9] [18] [19], the purpose of this paper is to handle with the existence of normalized solutions for system (1.1) when p belongs to the scope $\left(2,\frac{8}{3}\right]\cup\{6\}$. As usual, for any given $\rho>0$, searching solution of (1.3) with $\|u\|_2=\rho$ (normalized solution) is equivalent to consider nontrivial solution of following constraint problem

$$\begin{cases}-\Delta u+\omega u+q^2\phi_u u=|u|^{p-2}u & \text{in } \mathbb{R}^3,\\ \int_{\mathbb{R}^3}|u|^2\,\mathrm{d}x=\rho^2.\end{cases} \tag{1.4}$$

It is worth mentioning that in this situation the parameter ω arises as a Lagrange multiplier, depending on the solution and is not a priori given. To solve the problem (1.4), it can be obtained as a critical point of the following C^1 functional

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x+\frac{q^2}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x \tag{1.5}$$

constrained on the L^2-spheres in $H^1(\mathbb{R}^3)$

$$B_\rho:=\left\{u\in H^1(\mathbb{R}^3):\|u\|_2=\rho\right\}. \tag{1.6}$$

A direct strategy to deal with (1.5)-(1.6) is to consider the constraint minimizing problem

$$I_{\rho^2} := \inf_{u \in B_\rho} I(u) \tag{1.7}$$

and verify that the minimizers are critical points of $I(u)\big|_{B_\rho}$.

Up to now, we can state our first result as follows.

Theorem 1.1. *For* $p \in \left(2, \frac{8}{3}\right]$, *there exists* $\rho_1 > 0$ (*depending on p*) *such that all the minimizing sequences for* (7) *are precompact in* $H^1(\mathbb{R}^3)$ *up to translations provided that*

$$0 < \rho < \rho_1.$$

That is, there exists a couple of $(u_\rho, \omega_\rho) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ *being solution of* (1.4).

We take advantage of the techniques used in [18] [19] to finish the proof of Theorem 1.1. In fact, for any minimizing sequence $\{u_n\}$ of (1.7), due to vanishing $u_n \rightharpoonup 0$ (or the dichotomy situation $u_n \rightharpoonup \bar{u} \neq 0$ and $0 < \|\bar{u}\|_2 < \rho$) may occur, which leads to the main difficulty, that is, the (bounded) minimizing sequence $\{u_n\} \subset B_\rho$ is lack of compactness. To avoid the above two cases, the effective procedure is to verify that any minimizing sequence weakly converges, up to translation, to a function $\bar{u}$ which is different from zero, excluding the vanishing case; and then, to show that $\|\bar{u}\|_2 = \rho$, which illustrates that the dichotomy property does not occur. On account of the above discussions, we firstly check that the energy functional I defined in (1.5) for problem (1.4) satisfies the hypothesis of Lemma 2.1, which guarantees that the condition (MD) as introduced in Remark 2.2 can be recovered. Furthermore, with the help of Proposition 2.5, we can examine that $I(\bar{u}) = I_{\rho^2}$.

Here, it should be noted that, different from [18] [19], to find the existence of constraint minimizers for problem (1.7), the main difficulty is caused by the inhomogeneity of $\mathcal{K}$ defined in (1.2), which makes the calculation involving the energy functional I more complicated and leads to more difficult to prove strong subadditivity with the current method.

In addition, we prove the nonexistence of the solution when $p = 6$.

Theorem 1.2. *If* $p = 6$, *for any* $\rho > 0$, (1.4) *has no positive solution in* $H^1(\mathbb{R}^3)$.

As previously said, our Theorem 1.1 is the first attempt to consider the existence of normalized solutions for system (1.1). Notice that, if $a = 0$, system (1.1) reduces to the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2} u, \\ -\Delta \phi = 4\pi u^2, \end{cases} \tag{1.8}$$

which has been widely studied in recent years, see [23] [24] and the references therein. It is well known that system (1.8) is equivalent to

$$-\Delta u + \omega u + q^2 \phi_u^0 u = |u|^{p-2} u, \tag{1.9}$$

where now

$$\phi_u^0 := \frac{1}{|\cdot|} * u^2.$$

Evidently, if $u(x)$ satisfies Equation (1.9), we are readily going to obtain standing wave solutions being of the form $\psi(x,t) = \mathrm{e}^{-i\omega t} u(x)$ corresponding to the following problem dependent on t

$$i\psi_t + \Delta\psi - q^2\left(|x|^{-1} * |\psi|^2\right)\psi + |\psi|^{p-2}\psi = 0.$$

In addition, there are also some papers dealing with the existence of normalized solutions of Equation (1.9). For the case that $2 < p < \frac{10}{3}$, normalized solutions can be found by considering the minimization problem, since the functional I^0 is bounded from below and coercive on B_ρ. Bellazzini and Siciliano in [18] [19] proved that $I^0_{\rho^2}$ is achieved when $\rho > 0$ is small for $2 < p < 3$ and when $\rho > 0$ is large for $3 < p < \frac{10}{3}$, respectively. Subsequently, for the range $2 \le p \le \frac{10}{3}$, Jeanjean and Luo in [25] explicated a threshold value of ρ separating the existence and nonexistence of minimizers of $I^0_{\rho^2}$. Catto and Lions in [26] showed that minimizers of $I^0_{\rho^2}$ exist for $p = \frac{8}{3}$ provided that $\rho \in (0, \hat{\rho})$ for some suitable $\hat{\rho} > 0$ small enough. When p is L^2-supercritical and Sobolev subcritical, that is, $\frac{10}{3} < p < 6$, the existence of normalized solutions can be generalized to the minimization problem in [27].

Since the normalized solution $u(x)$ obtained in Theorem 1.1 corresponds to the standing wave $\psi(x,t) = \mathrm{e}^{-i\omega t} u(x)$ of the following evolution equation

$$i\psi_t + \Delta\psi - q^2\left(\mathcal{K} * |\psi|^2\right)\psi + |\psi|^{p-2}\psi = 0.$$

Therefore, the stability of standing wave is the second concerned problem in present paper. Explicitly, we will discuss the orbital stability of standing waves with L^2-norm for the following initial problem

$$\begin{cases} i\psi_t + \Delta\psi - q^2\left(\mathcal{K} * |\psi|^2\right)\psi + |\psi|^{p-2}\psi = 0, \\ \psi(x,0) = \psi_0(x) \in H^1(\mathbb{R}^3). \end{cases} \tag{1.10}$$

Definition 1.3. *Define*

$$S_\rho = \left\{ \mathrm{e}^{i\theta} u(x) : \theta \in [0, 2\pi), \|u\|_2 = \rho, I(u) = I_{\rho^2} \right\}.$$

Then, S_ρ *is orbitally stable if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $\psi_0 \in H^1(\mathbb{R}^3)$ *with* $\inf_{v \in S_\rho} \|v - \psi_0\|_{H^1(\mathbb{R}^3)} < \delta$ *we have*

$$\forall\, t > 0, \quad \inf_{v \in S_\rho} \|\psi(\cdot,t) - v\|_{H^1(\mathbb{R}^3)} < \varepsilon.$$

Here, $\psi(\cdot,t)$ *is the solution of initial problem* (1.10).

We obtain the strong stability of standing waves for (1.10), which is shown in the following theorem.

Theorem 1.4. *Let* $p \in \left(2, \frac{8}{3}\right]$. *Then the set*

$$S_\rho = \left\{ e^{i\theta} u(x) : \theta \in [0, 2\pi), \|u\|_2 = \rho, I(u) = I_{\rho^2} \right\}$$

is orbitally stable for ρ *determined in Theorem* 1.1.

This paper is organized as follows. In Section 2, various preliminary results are presented to be used in the sequel. In Section 3, we focus our attention on the proofs of Theorem 1.1 and Theorem 1.2. Finally, the orbital stability obtained in Theorem 1.4 is established in Section 4.

2. Preliminaries

To prove our main results, some preliminaries are in order during this section. We first recall an abstract framework introduced in [18], however we could not narrate it again in order to avoid the repetition. For the simplicity, we directly apply it to our variational framework. Explicitly, for our constrained minimization problem (1.7), we rewrite it as follows

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + T(u),$$

where

$$T(u) := \frac{q^2}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx. \tag{2.1}$$

Under suitable assumption on T defined in (2.1), we have the strong convergence of the weakly convergent minimizing sequence.

Lemma 2.1. [18] *Let* $T \in C^1\left(H^1(\mathbb{R}^3), \mathbb{R}\right)$. *Let* $\rho > 0$ *and* $\{u_n\}$ *be a minimizing sequence for* I_{ρ^2} *weakly convergent, up to translations, to a nonzero function* $\bar{u}$. *Assume that the following inequality is satisfied*

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2 - \mu^2} \quad \text{for all } 0 < \mu < \rho, \tag{2.2}$$

and that

$$T(u_n - u) + T(\bar{u}) = T(u_n) + o(1), \tag{2.3}$$

$$T\left(\alpha_n (u_n - u)\right) - T(u_n - \bar{u}) = o(1), \text{ where } \alpha_n = \frac{\rho^2 - \|\bar{u}\|_2^2}{\|u_n - \bar{u}\|_2^2}, \tag{2.4}$$

then $\bar{u} \in B_\rho$.

Remark 2.2. In the above lemma, (2.2) is usually called strong subadditivity inequality. In order to ensure that any minimizing sequence on B_ρ is relatively compact, (2.2) is the necessary and sufficient condition and it is a stronger version of the following inequality

$$I_{\rho^2} \le I_{\mu^2} + I_{\rho^2 - \mu^2} \quad \text{for all } 0 < \mu < \rho, \tag{2.5}$$

which is referred as the weak subadditivity inequality. It is worth mentioning that in [18] [19], checking (2.2) for the Schrödinger-Poisson system is the essential step to solve problem (1.7). As a matter of fact, this is also very important for us. However, the inhomogeneity of $\mathcal{K}$ makes this more difficult than $a \neq 0$. To prove (2.2), we adopt the mediate approach which ensures that

(MD) the function $s \mapsto \frac{I_{s^2}}{s^2}$ is monotone decreasing.

Indeed, assuming that (MD) holds for $\mu \in (0, \rho)$, we get

$$\frac{\mu^2}{\rho^2} I_{\rho^2} < I_{\mu^2} \quad \text{and} \quad \frac{\rho^2 - \mu^2}{\rho^2} I_{\rho^2} < I_{\rho^2 - \mu^2}.$$

Therefore, one has

$$I_{\rho^2} = \frac{\mu^2}{\rho^2} I_{\rho^2} + \frac{\rho^2 - \mu^2}{\rho^2} I_{\rho^2} < I_{\mu^2} + I_{\rho^2 - \mu^2} \text{ for all } 0 < \mu < \rho.$$

In other words, verifying (MD) helps us to obtain (2.2), but it is not an easy work. In order to overcome this difficulty, the following Proposition 2.5 provides one criterion for (MD).

Before presenting it, we give some necessary definitions needed in the subsequence.

Definition 2.3. Let $u \in H^1(\mathbb{R}^3)$ with $u \neq 0$. A continuous path $g_u : \theta \in \mathbb{R}^+ \mapsto g_u(\theta) \in H^1(\mathbb{R}^3)$ such that $g_u(1) = u$ is said to be a scaling path of u if

$$\Theta_{g_u}(\theta) := \|g_u(\theta)\|_2^2 \|u\|_2^{-2} \text{ is differentiable and } \Theta'_{g_u}(1) \neq 0, \tag{2.6}$$

where the prime denotes the derivative. Furthermore, $\mathcal{G}_u$ is the set of scaling paths of u.

Definition 2.4. Let $u \neq 0$ be fixed and $g_u \in \mathcal{G}_u$. We say that the scaling path g_u is admissible for the functional (1.5) if

$$h_{g_u}(\theta) := I(g_u(\theta)) - \Theta_{g_u}(\theta) I(u), \ \theta \geq 0$$

is a differentiable function.

Proposition 2.5. *Let* $T \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ *satisfy the set of assumptions* (2.3) and (2.4). *Assume that for every* $\rho > 0$. *All the minimizing sequences* $\{u_n\}$ *for* I_{ρ^2} *have a weak limit up to translations different from zero. Assume finally* (2.5) *and the following conditions*

$$-\infty < I_{s^2} < 0 \text{ for all } s > 0 \ (I(0) = 0), \tag{2.7}$$

$$s \mapsto I_{s^2} \text{ is continuous}, \tag{2.8}$$

$$\lim_{s \to 0} \frac{I_{s^2}}{s^2} = 0. \tag{2.9}$$

Then, for every $\rho > 0$, *the set*

$$M(\rho) = \bigcup_{\mu \in (0, \rho]} \left\{ u \in B_\mu : I(u) = I_{\mu^2} \right\}$$

is nonempty. If, in addition,

$$\forall u \in M(\rho), \exists\, g_u \in \mathcal{G}_u \text{ is admissible such that } \left. \frac{\mathrm{d}}{\mathrm{d}\theta} h_{g_u}(\theta) \right|_{\theta=1} \neq 0, \tag{2.10}$$

then (MD) *holds. Moreover, if* $\{u_n\}$ *is a minimizing sequence for* I_{ρ^2} *weakly convergent, up to translations, to a nonzero function* $\bar{u}$,

$$\langle T'(u_n), u_n \rangle = O(1), \tag{2.11}$$

$$\langle T'(u_n) - T'(u_m), u_n - u_m \rangle = o(1) \text{ as } n, m \to +\infty, \tag{2.12}$$

then $\|u_n - \bar{u}\|_{H^1(\mathbb{R}^3)} \to 0$. *In particular, it follows that* $I(\bar{u}) = I_{\rho^2}$.

Next, we need to consider the local well posedness for the Cauchy problem (1.10). The framework established in [28] helps us to achieve this fact. To make our problem better correspond to the abstract results in [28], we will give the following details. In fact, the local well posedness considered in the following is applicable to more general nonlinearity

$$\begin{cases} i\psi_t + \Delta\psi + g(\psi) = 0, \\ \psi(0) = \psi_0. \end{cases} \tag{2.13}$$

For the nonlinearity being of the form $g = g_1 + g_2$, we assume that there exist $G_j \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ ($j = 1, 2$) such that

$$g_j = G_j'. \tag{2.14}$$

In addition, $g_j \in C(H^1(\mathbb{R}^3), H^{-1}(\mathbb{R}^3))$, there exist some $r_j, \rho_j \in [2, 6)$ such that

$$g_j \in C(H^1(\mathbb{R}^3), L^{\rho_j'}(\mathbb{R}^3)), \tag{2.15}$$

and for every $M < \infty$ there exists $C(M) < \infty$ such that

$$\|g_j(\varphi) - g_j(\psi)\|_{\rho_j'} \le C(M)\|\varphi - \psi\|_{r_j}, \forall \psi, \varphi \in H^1(\mathbb{R}^3, \mathbb{C}) \tag{2.16}$$

with $\|\psi\| + \|\varphi\| \le M$, where r_j', ρ_j' represent the conjugate exponent of r_j, ρ_j. Finally, we assume that for every $\psi \in H^1(\mathbb{R}^3, \mathbb{C})$

$$\operatorname{Im}(g_j(\psi)\bar{\psi}) = 0 \quad \text{a.e. in } \mathbb{R}^3. \tag{2.17}$$

Let $G = G_1 + G_2$ and define the energy E by

$$E(\psi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\psi|^2 \, dx - G(\psi)$$

for $\psi \in H^1(\mathbb{R}^3, \mathbb{C})$. Then, according to ([28], Theorem 4.3.1), the following proposition is a direct consequence.

Proposition 2.6. [28] *If g is defined as above. the initial-value problem* (2.13) *is locally well posed in* $H^1(\mathbb{R}^3)$. *Furthermore. there is conservation of charge and energy i.e.*

$$\|\psi(t)\|_2 = \|\psi_0\|_2, \quad E(\psi(t)) = E(\psi_0)$$

for all $t \in (-T_{\min}, T_{\max})$, *where* $\psi(t)$ *is the solution of* (2.13).

3. Proofs of Theorem 1.1 and Theorem 1.2

Firstly, we focus our attention on verifying the hypotheses of Proposition 2.5 to finish the proof of Theorem 1.1. We start this section to give some properties of

ϕ_u (see [9], Lemma 3.4), which will be used frequently in the later.

Lemma 3.1. ([9], Lemma 3.4]) *For any* $u \in H^1(\mathbb{R}^3)$. ϕ_u *has following properties*:

1) $\phi_u \ge 0$, $\phi_{tu} = t^2\phi_u$;

2) $\|\phi_u\|_6 \le C\|u\|^2$;

3) if $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, then $\phi_{v_n} \rightharpoonup \phi_v$ in $\mathcal{D}$;

4) $\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \le C\|u\|_{12/5}^4$.

Hereafter, we use $C, C_1, \cdots$ to denote suitable positive constants whose value may also change from line to line. The following lemma shows that problem (1.7) is well-defined for $p \in \left(2, \frac{8}{3}\right]$.

Lemma 3.2. *For every* $\rho > 0$ *and* $p \in \left(2, \frac{8}{3}\right]$. *the functional I as shown in* (1.5) *is bounded from below and coercive on* B_ρ.

Proof. In view of Lemma 3.1 and using the Gagliardo-Nirenberg inequality (see [29], Proposition 1.16), we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \nabla u^2 \mathrm{d}x + \frac{q^2}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x \\ &\ge \frac{1}{2}\|\nabla u\|_2^2 - C\|u\|_2^{p(1-\gamma_p)}\|\nabla u\|_2^{p\gamma_p} \\ &= \frac{1}{2}\|\nabla u\|_2^2 - C\rho^{p(1-\gamma_p)}\|\nabla u\|_2^{p\gamma_p}, \end{aligned} \tag{3.1}$$

where $\gamma_p = \frac{3(p-2)}{2p}$. Since $p \in \left(2, \frac{8}{3}\right]$, it results $p\gamma_p < 1$, which concludes the proof. □

Remark 3.3. For $0 < p\gamma_p < 2$, observing the above inequality (3.1) yields that the functional *I* is bounded from below and coercive on B_ρ. That is, Lemma 3.2 is effective for $p \in \left(2, \frac{10}{3}\right)$. In addition, as a consequence of this lemma, whenever ρ is fixed and $\{u_n\}$ is a minimizing sequence for I_{ρ^2}, we obtain that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and exists a weakly convergent subsequence.

In order to verify all the hypothesis of Proposition 2.5, we begin with the weak subadditivity inequality (2.5).

Lemma 3.4. *For the functional I defined in* (1.5). *The weak subadditivity inequality* (2.5) *is satisfied.*

Proof. According to the definition of infimum, for any $\varepsilon > 0$, there are $u_\varepsilon, v_\varepsilon$ such that

$$\begin{cases} I_{\mu^2} \le I(u_\varepsilon) \le I_{\mu^2} + \varepsilon, \ \|u_\varepsilon\|_2^2 = \mu^2, \\ I_{\rho^2-\mu^2} \le I(v_\varepsilon) \le I_{\rho^2-\mu^2} + \varepsilon, \ \|v_\varepsilon\|_2^2 = \rho^2 - \mu^2, \end{cases} \tag{3.2}$$

where $\rho > \mu > 0$. Denote $v_\varepsilon^n = v_\varepsilon(\cdot + n\chi)$, where χ is some given unit vector in $\mathbb{R}^3$. By ([9], Lemma B.5), we get $v_\varepsilon(\cdot + n\chi) \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and $v_\varepsilon(\cdot + n\chi) \to 0$ a.e. in $\mathbb{R}^3$, up to a subsequence if necessary. And, according to Brézis-Lieb Lemma (see [30], Lemma 1.32), we have

$$\|u_\varepsilon + v_\varepsilon^n\|_2^2 - \left(\|v_\varepsilon^n\|_2^2 + \|u_\varepsilon\|_2^2\right) \to 0;$$

by ([9], Lemma B.2), we derive that

$$I(u_\varepsilon + v_\varepsilon^n) - \left(I(u_\varepsilon) + I(v_\varepsilon^n)\right) \to 0.$$

Moreover, since $\|u\|_2^2$ and $I(u)$ are translation-invariant, we can infer from (3.2) that

$$\begin{aligned} I_{\mu^2} + I_{\rho^2-\mu^2} &\le I(u_\varepsilon) + I(v_\varepsilon) = \lim_{n\to\infty}\left(I(u_\varepsilon) + I(v_\varepsilon^n)\right) \\ &= I(u_\varepsilon + v_\varepsilon^n) \le I_{\mu^2} + I_{\rho^2-\mu^2} + 2\varepsilon \end{aligned}$$

and

$$\|u_\varepsilon + v_\varepsilon^n\|_2^2 = \lim_{n\to\infty}\left\{\|u_\varepsilon\|_2^2 + \|v_\varepsilon^n\|_2^2\right\} = \|u_\varepsilon\|_2^2 + \|v_\varepsilon\|_2^2 = \mu^2 + \rho^2 - \mu^2 = \rho^2.$$

As a result, according to the definition of infimum for I_{ρ^2}, it is achieved that

$$I_{\rho^2} \le I_{\mu^2} + I_{\rho^2-\mu^2}.$$

□

Lemma 3.5. *The functional T defined in* (2.1) *satisfies* (2.3) and (2.4).

Proof. For the convenience of notation, we redefine (2.1)

$$T(u) = \frac{q^2}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p \mathrm{d}x := N(u) - M(u).$$

It is obvious that $T \in C^1(\mathbb{R}^3, \mathbb{R})$. Therefore, we only need to verify that both *M* and *N* hold true for the relationships (2.3) and (2.4). By Lemma 3.2, for $\{u_n\}$ being an arbitrary minimization sequence for $I(u)$, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Due to the Sobolev embedding theorem, $\{u_n\}$ is also bounded in L^s-norm for $s \in [2, 2^*]$ and there is $\bar{u} \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup \bar{u} \in H^1(\mathbb{R}^3)$.

Note that, *M* and *N* satisfy the condition (2.3) which were proved by ([9], Lemma B.2). Therefore, it is sufficient to verify that the condition (2.4) is satisfied for *M* and *N*. Actually, by Hölder inequality and Lemma 3.1, we have

$$N(u_n) = \frac{q^2}{4}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^2 \mathrm{d}x \le C_1\|\phi_{u_n}\|_6 \|u_n\|_{\frac{12}{5}}^2 \le C\|u_n\|^2 \|u_n\|_{\frac{12}{5}}^2.$$

Then, $N(u_n)$ is bounded. In addition, once $\alpha_n = \frac{\rho^2 - \|\bar{u}\|_2^2}{\|u_n - \bar{u}\|_2^2} \to 1$, we conclude the proof from

$$N(\alpha_n(u_n - \bar{u})) - N(u_n - u) = (\alpha_n^4 - 1)N(u_n - \bar{u})$$

and

$$M(\alpha_n(u_n - \bar{u})) - M(u_n - u) = (\alpha_n^p - 1)M(u_n - \bar{u}).$$

Indeed, since $u_n - \bar{u} \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, up to a subsequence if necessary, by Brézis-Lieb Lemma we get

$$\|u_n - \bar{u}\|_2^2 + \|\bar{u}\|_2^2 = \|u_n\|_2^2 + o(1).$$

Hence,

$$\alpha_n = \frac{\rho^2 - \|\bar{u}\|_2^2}{\|u_n - \bar{u}\|_2^2} \to 1,$$

which implies that $M(\alpha_n(u_n - u)) - M(u_n - \bar{u}) = o(1)$ and $N(\alpha_n(u_n - u)) - N(u_n - \bar{u}) = o(1)$. So we deduce immediately that $T(\alpha_n(u_n - u)) - T(u_n - \bar{u}) = o(1)$ and complete the proof. □

We are now concentrating on testing that conditions (2.7)-(2.9) are achievable.

Lemma 3.6. *If* $2 < p \le 8/3$ *. then condition* (2.7) *is satisfied.*

Proof. By Lemma 3.2, we have $I_{s^2} > -\infty$ for all $s > 0$. Hence, it only needs to prove that $I_{s^2} < 0$ for every $s > 0$. Let $u \in H^1(\mathbb{R}^3)$ and choose the family of scaling paths of u parameterized with $\beta \in \mathbb{R}$ given by

$$\mathcal{G}_u^\beta = \left\{ g_u(\theta) = \theta^{1-\frac{3}{2}\beta} u(x/\theta^\beta) \right\} \subset \mathcal{G}_u,$$

such that $\Theta_{g_u}(\theta) = \theta^2$ and $\|g_u(\theta)\|_2 = \theta$, where $\theta \in \mathbb{R}^+$ and $\Theta_{g_u}(\theta)$ is given in Definition 2.3. For the simplicity of notations, we introduce the following quantities

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x, \; B(u) := \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x, \; C(u) := \int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x,$$

which gives that

$$I(u) := \frac{1}{2} A(u) + \frac{q^2}{4} B(u) - \frac{1}{p} C(u).$$

Meanwhile, some direct calculations bring the equalities as follows

$$A(g_u(\theta)) = \theta^{2-2\beta} A(u),$$

$$B(g_u(\theta)) = \theta^{4-\beta} \iint_{\mathbb{R}^3} \frac{1 - \mathrm{e}^{-\frac{|x-y|}{a\theta^{-\beta}}}}{|x-y|} u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y,$$

$$C(g_u(\theta)) = \theta^{\left(1-\frac{3}{2}\beta\right)p + 3\beta} C(u).$$

Taking $\beta = -2$, we readily see that

$$I(g_u(\theta)) = \frac{\theta^6}{2} A(u) + \frac{\theta^6 q^2}{4} \iint_{\mathbb{R}^3} \frac{1 - \mathrm{e}^{-\frac{|x-y|}{a\theta^2}}}{|x-y|} u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y - \frac{\theta^{4p-6}}{p} C(u) \to 0^-$$

as $\theta \to 0$, since $4p - 6 < 6$ and $C(u) > 0$. This signifies that there is a small θ_0 such that

$$I_{s^2} < 0, \quad \forall s \in (0, \theta_0].$$

Letting $\theta \in \left(\theta_0, \sqrt{2}\theta_0\right]$, then for every $s \in \left(\theta_0, \theta\right]$, we derive from Lemma 3.4 that

$$I_{s^2} \le I_{\theta_0^2} + I_{s^2-\theta_0^2} < 0,$$

since $s^2 - \theta_0^2 < \theta_0^2$. That is to say, $I_{s^2} < 0$ for s in the larger interval $\left(0, \theta\right]$. Iterating this procedure gives that $I_{s^2} < 0$ for every $s > 0$ and finishes the proof of this lemma. □

Lemma 3.7. *If* $2 < p \le 8/3$, *then the function* $s \mapsto I_{s^2}$ *satisfies* (2.8) *and* (2.9).

Proof. Firstly, we consider (2.8). Assume that $\rho_n \to \rho$ as $n \to \infty$, it is equivalent to show $\lim_{n\to\infty} I_{\rho_n^2} = I_{\rho^2}$. For every $n \in \mathbb{N}$, let $\omega_n \in B_{\rho_n}$ such that

$$I\left(\omega_n\right) \le I_{\rho_n^2} + \frac{1}{n} < \frac{1}{n}. \tag{3.3}$$

By the Gagliardo-Nirenberg inequality (see [29], Proposition 1.16), we have

$$\frac{1}{2}\left\|\nabla\omega_n\right\|_2^2 - C\rho_n^{\frac{6-p}{2}}\left\|\nabla\omega_n\right\|_2^{\frac{3(p-2)}{2}} \le \frac{1}{2}\left\|\nabla\omega_n\right\|_2^2 - \frac{1}{p}\left\|\omega_n\right\|_p^p \le I\left(\omega_n\right) < \frac{1}{n}.$$

Since $\frac{3\left(p-2\right)}{2} < 1$ and $\left\{\rho_n\right\}$ is bounded, we see that $\left\{\omega_n\right\}$ is bounded in $H^1\left(\mathbb{R}^3\right)$. Thus, $A\left(\omega_n\right)$, $B\left(\omega_n\right)$ and $C\left(\omega_n\right)$ are bounded sequences. So, by Lemma 3.1 and (3.3), using the fact that $\rho_n \to \rho$ as $n \to \infty$, it leads to

$$\begin{aligned} I_{\rho^2} &\le I\left(\frac{\rho}{\rho_n}\omega_n\right) = \frac{1}{2}\left(\frac{\rho}{\rho_n}\right)^2 A\left(\omega_n\right) + \frac{q^2}{4}\left(\frac{\rho}{\rho_n}\right)^4 B\left(\omega_n\right) - \frac{1}{p}\left(\frac{\rho}{\rho_n}\right)^p C\left(\omega_n\right) \\ &= I\left(\omega_n\right) + o\left(1\right) \le I_{\rho_n^2} + o\left(1\right). \end{aligned} \tag{3.4}$$

On the other hand, given a minimizing sequence $\left\{v_n\right\} \subset B_\rho$ for I_{ρ^2}, the following inequality holds

$$I_{\rho_n^2} \le I\left(\frac{\rho_n}{\rho}v_n\right) = I\left(v_n\right) + o\left(1\right) = I_{\rho^2} + o\left(1\right). \tag{3.5}$$

Combining (3.4) and (3.5), one has $\lim_{n\to\infty} I_{\rho_n^2} = I_{\rho^2}$, namely, condition (2.8) is established.

Next, we deal with $\lim_{\rho\to 0}\frac{I_{\rho^2}}{\rho^2} = 0$. Note that (2.7) implies that

$$\frac{G_{\rho^2}}{\rho^2} \le \frac{I_{\rho^2}}{\rho^2} < 0,$$

where

$$G_{\rho^2} = \inf_{B_\rho} G\left(u\right) \quad \text{and} \quad G\left(u\right) = \frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla u\right|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3}\left|u\right|^p \mathrm{d}x.$$

Therefore, it is sufficient to verify that $G_{\rho^2}/\rho^2 \to 0$ as $\rho \to 0$. Indeed, $G\left(u\right)$ is the functional associated to the following pure Schrödinger equation

$$-\Delta u - \left|u\right|^{p-2} u = \omega u,$$

with prescribed L^2-norm $\left\|u\right\|_2 = \rho$. It is known that, if $2 < p \le 8/3$, then for every $\rho > 0$, there exists $u_\rho \in B_\rho$ such that $G_{\rho^2} = G(u_\rho) < 0$. For the details, we refer the reader to [31] [32].

For the minimizer u_ρ, by the Gagliardo-Nirenberg inequality, there holds

$$0 > G(u_\rho) \geq \frac{1}{2}\|\nabla u_\rho\|_2^2 - C\rho^{\frac{6-p}{2}}\|\nabla u_\rho\|_2^{\frac{3(p-2)}{2}}, \tag{3.6}$$

which implies that the sequence $\{u_\rho\}_{\rho>0}$ is bounded in $D^{1,2}(\mathbb{R}^3)$ for $\rho \to 0$. On the other hand, since the minimizer u_ρ for G_{ρ^2} satisfies the following equation in weakly sense

$$-\Delta u_\rho - |u_\rho|^{p-2} u_\rho = \omega_\rho u_\rho, \tag{3.7}$$

we infer form (3.6) that

$$\begin{aligned} \frac{\omega_\rho}{2} &= \frac{\int_{\mathbb{R}^3}|\nabla u_\rho|^2 \mathrm{d}x - \int_{\mathbb{R}^3}|u_\rho|^p \mathrm{d}x}{2\int_{\mathbb{R}^3}|u_\rho|^2 \mathrm{d}x} \\ &\leq \frac{\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_\rho|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3}|u_\rho|^p \mathrm{d}x}{\int_{\mathbb{R}^3}|u_\rho|^2 \mathrm{d}x} = \frac{G(u_\rho)}{\rho^2} < 0, \end{aligned} \tag{3.8}$$

where ω_ρ is the Lagrange multiplier associated to the minimizer u_ρ. Observe (3.8), it reduces to prove that $\lim_{\rho\to 0}\omega_\rho = 0$.

We argue by contradiction assuming that there exists a sequence $\rho_n \to 0$ such that $\omega_{\rho_n} < -c$ for some $c \in (0,1)$. Since the minimizers $u_n := u_{\rho_n}$ satisfy (3.7), we are led to

$$\begin{aligned} c\|u_n\|^2 &\leq \int_{\mathbb{R}^3}|\nabla u_n|^2 \mathrm{d}x + c\int_{\mathbb{R}^3} u_n^2 \mathrm{d}x \\ &< \int_{\mathbb{R}^3}|\nabla u_n|^2 \mathrm{d}x - \omega_{\rho_n}\int_{\mathbb{R}^3} u_n^2 \mathrm{d}x \\ &= \int_{\mathbb{R}^3}|u_n|^p \mathrm{d}x \leq C\|u_n\|^p, \end{aligned}$$

which yields that there exists $c' > 0$ such that $\|\nabla u_n\|_2 > c' > 0$ due to $p > 2$. However, in view of (3.6), it holds that

$$0 > G(u_n) \geq \frac{1}{2}c' - o(1),$$

which is meaningless. Thus, we completed the proof. □

Based on Lemma 3.4-Lemma 3.7, we have shown that $M(\rho) \neq \varnothing$.

Lemma 3.8. *For every* $\rho > 0$. *all the minimizing sequences* $\{v_n\}$ *for* I_{ρ^2} *have a weak limit. up to translations. different from zero. Furthermore. the weak limit defined in Proposition* 2.5 *is contained in* $M(\rho)$.

Proof. Let $\{v_n\}$ be a minimizing sequence in B_ρ for I_{ρ^2}. Notice that for any sequence $\{y_n\} \subset \mathbb{R}^3$, the translation invariance guarantees that $\{v_n(\cdot + y_n)\}$ is still a minimizing sequence for I_{ρ^2}. Thus, we only need to prove the existence of one sequence $\{y_n\} \subset \mathbb{R}^3$ such that the weak limit of $\{v_n(\cdot + y_n)\}$ is different from zero. By the Lions' lemma (see [30], Lemma 1.21), it follows that if

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}\left|v_n\right|^2 \mathrm{d}x\right)=0,$$

then $v_n \to 0 \in L^q\left(\mathbb{R}^3\right)$ for any $q\in\left(2,2^*\right)$, where $B_r\left(a\right)=\left\{x\subset\mathbb{R}^3:\left|x-a\right|\le r\right\}$. So $C\left(v_n\right)\to 0$ as $n\to\infty$, and then $\lim_{n\to\infty} I\left(v_n\right)\ge 0$, which contradicts to Lemma 3.6. Therefore, we must have

$$\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}\left|v_n\right|^2 \mathrm{d}x\ge\delta>0.$$

In this case we can choose $\left\{y_n\right\}\subset\mathbb{R}^3$ such that

$$\int_{B_1(0)}\left|v_n\left(\cdot+y_n\right)\right|^2 \mathrm{d}x\ge\frac{\delta}{2}>0.$$

Due to the compactness of the embedding $H^1\left(B_1\left(0\right)\right)\hookrightarrow L^2\left(B_1\left(0\right)\right)$, we deduce that the weak limit of the sequence $\left\{v_n\left(\cdot+y_n\right)\right\}$, let us call it v, is nontrivial.

In the next moment, we verify that $v\in M\left(\rho\right)$. Indeed, if $\|v\|_2=\rho$, it is trivial. Thus, we only discuss the case of $\|v\|_2<\rho$. If $\mu_0:=\|v\|_2<\rho$, then using ([19], Proposition 3.1), we have $I_{\rho^2}=I_{\mu_0^2}+I_{\rho^2-\mu_0^2}$ and $I\left(v\right)=I_{\mu_0^2}$, which indicates that $v\in M\left(\rho\right)\neq\varnothing$. □

As previously stated, the strong subadditivity inequality (2.2) is only used to ensure that (MD) holds, which is the purpose of the following lemma.

Lemma 3.9. *For small* ρ. *the function* $h_{g_u}\left(\theta\right)$ *defined in Definition* 2.4 *satisfies* (2.10).

Proof. For $u\in M\left(\rho\right)$, that is $\|u\|_2=\mu\in\left(0,\rho\right]$ and $I\left(u\right)=I_{\mu^2}$, we set $v\left(\theta,u\right)=\theta^{-\frac{3}{2}}u\left(\frac{x}{\theta}\right)$ for $\theta>0$. It is obvious that $\|v\left(\theta,u\right)\|_2=\|u\|_2=\mu$.

Furthermore, a simple calculation gives that

$$A\left(v\left(\theta,u\right)\right)=\theta^{-2}A\left(u\right),$$

$$B\left(v\left(\theta,u\right)\right)=\theta^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1-\mathrm{e}^{-\frac{|x-y|}{a\theta^{-1}}}}{\left|x-y\right|}u^2\left(x\right)u^2\left(y\right)\mathrm{d}x\mathrm{d}y,$$

$$C\left(v\left(\theta,u\right)\right)=\theta^{\frac{6-3p}{2}}C\left(u\right).$$

Since the map $\theta\mapsto I\left(v\left(\theta,u\right)\right)$ is differentiable and u is the minimizer of $I\left(u\right)$ on B_μ, we infer that

$$\left.\frac{\mathrm{d}I\left(v\left(\theta,u\right)\right)}{\mathrm{d}\theta}\right|_{\theta=1}=0,$$

which means that

$$-A\left(u\right)-\frac{q^2}{4}B\left(u\right)+\frac{q^2}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\frac{|x-y|}{a}}}{a}u^2\left(x\right)u^2\left(y\right)\mathrm{d}x\mathrm{d}y-\frac{6-3p}{2p}C\left(u\right)=0. \tag{3.9}$$

Next, for $u\neq 0$ we compute explicitly $h_{g_u}\left(\theta\right)$ by choosing the family of scaling paths of u parameterized with $\beta\in\mathbb{R}$ given by

$$\mathcal{G}_u^\beta = \left\{ g_u(\theta) = \theta^{1-\frac{3}{2}\beta} u\left(x/\theta^\beta\right) \right\} \subset \mathcal{G}_u.$$

Evidently, all the paths of this family have the associated function $\Theta_{g_u}(\theta) = \theta^2$ where Θ_{g_u} is defined in (2.6). Denote by

$$\tilde{B}(\theta,u) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-\mathrm{e}^{-\frac{|x-y|}{a\theta^{-\beta}}}}{|x-y|} u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,$$

$h_{g_u}(\theta)$ can be rewritten as follows

$$\begin{aligned} h_{g_u}(\theta) = {} & \frac{1}{2}\left(\theta^{2-2\beta} - \theta^2\right)A(u) + \frac{q^2}{4}\theta^{4-\beta}\tilde{B}(\theta,u) - \frac{q^2}{4}\theta^2 B(u) \\ & -\frac{1}{p}\left(\theta^{\left(1-\frac{3}{2}\beta\right)p+3\beta} - \theta^2\right)C(u). \end{aligned} \tag{3.10}$$

Obviously, $h_{g_u}(\theta)$ is differentiable for every $g_u \in \mathcal{G}_u^\beta$, *i.e.*, the paths in $\mathcal{G}_u^\beta$ are admissible.

Meanwhile, for $g_u \in \mathcal{G}_u^\beta$

$$\begin{aligned} h'_{g_u}(1) = {} & -\beta A(u) + \frac{(2-\beta)q^2}{4}B(u) - \frac{\left(1-\frac{3}{2}\beta\right)p + 3\beta - 2}{p}C(u) \\ & + \frac{q^2\beta}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\frac{|x-y|}{a}}}{a}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Hence, it remains to demonstrate that the admissible scaling path satisfies $h'_{g_u}(1) \neq 0$, which can be chosen in $\mathcal{G}_u^\beta$. To prove this point, we argue by contradiction. Assume that there exists a sequence $\{u_n\} \subset M(\rho)$ with $\rho \geq \|u_n\|_2 = \rho_n \to 0$ such that for all $\beta \in \mathbb{R}$

$$\begin{aligned} h'_{g_{u_n}}(1) = {} & -\beta A(u_n) + \frac{(2-\beta)q^2}{4}B(u_n) - \frac{\left(1-\frac{3}{2}\beta\right)p + 3\beta - 2}{p}C(u_n) \\ & + \frac{q^2\beta}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\frac{|x-y|}{a}}}{a}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y \\ = {} & 0. \end{aligned}$$

Then, combining just the above with (3.9), we deduce that

$$\frac{q^2}{2}B(u_n) + \frac{2-p}{p}C(u_n) = 0. \tag{3.11}$$

As a result, it gives that

$$B(u_n) = \frac{2}{q^2}A(u_n) - \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\frac{|x-y|}{a}}}{a}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y,$$

$$C(u_n) = \frac{p}{p-2}A(u_n) - \frac{pq^2}{4(p-2)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\frac{|x-y|}{a}}}{a}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y,$$

$$
\begin{aligned}
I(u_n) &= \frac{1}{2}A(u_n)+\frac{q^2}{4}B(u_n)-\frac{1}{p}C(u_n) \\
&= \frac{p-3}{p-2}A(u_n)+\frac{q^2(4-p)}{8(p-2)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\frac{|x-y|}{a}}}{a}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.12}
$$

In the sequel, we derive the contradiction to four different situations by showing that the relationships (3.12) are impossible for $p\in\left(2,\frac{8}{3}\right]$ and ρ is small. Actually, by continuity, we know that

$$
\begin{cases} I(u_n)=I_{\rho_n^2}\to 0, \\ A(u_n),B(u_n),C(u_n)\to 0. \end{cases} \tag{3.13}
$$

Case 1: $2<p<12/5$.

By the Hardy-Littlewood-Sobolev inequality (see [33], Theorem 4.3) and the interpolation inequality (see [34], Lemma 6.32), we have

$$
\begin{aligned}
B(u) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1-\mathrm{e}^{-\frac{|x-y|}{a}}}{|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_n^2(x)u_n^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \\
&\le C\|u_n\|_{\frac{12}{5}}^4 \\
&\le C\|u_n\|_p^{4\alpha}\|u_n\|_6^{4(1-\alpha)},
\end{aligned}
$$

where $\alpha=\frac{3p}{2(6-p)}$. Then, according to Sobolev embedding theorem and (3.11), one has

$$
B(u)\le CB(u_n)^{\frac{4\alpha}{p}}A(u_n)^{\frac{4(1-\alpha)}{2}}.
$$

Since $p<\frac{12}{5}$, it results in $\frac{4\alpha}{p}>1$. Therefore, we are able to deduce that

$$
1\le CB(u_n)^{\frac{4\alpha}{p}-1}A(u_n)^{\frac{4(1-\alpha)}{2}},
$$

which is a contradiction with (3.13).

Case 2: $p=12/5$.

Due to (3.11) and Lemma (3.1), we obtain that

$$
\|u_n\|_{12/5}^{12/5}=3q^2B(u_n)\le C\|u_n\|_{12/5}^4,
$$

which is impossible, since $B(u_n)\to 0$.

Case 3: $12/5<p<8/3$.

Using the interpolation inequality, it follows that

$$
\|u_n\|_p^p=\frac{pq^2}{2(p-2)}B(u_n)\le C\|u_n\|_{12/5}^4\le C\|u_n\|_2^{4\alpha}\|u_n\|_p^{4(1-\alpha)}=C\rho_n^{4\alpha}\|u_n\|_p^{4(1-\alpha)},
$$

where $\alpha = \dfrac{5p-12}{6(p-2)}$. Since $p < 4(1-\alpha)$, we infer that $1 \le \rho_n^{4\alpha} \|u_n\|_p^{4(1-\alpha)-p}$, which contradicts (3.13).

Case 4: $p = 8/3$.

To this situation, one has

$$B(u_n) \le C\|u_n\|_{12/5}^4 \le C\|u_n\|_2^{4\alpha}\|u_n\|_{8/3}^{4(1-\alpha)} \le C\rho_n^{4/3}\|u_n\|_{8/3}^{8/3},$$

where $\alpha = 1/3$. Noting that $B(u_n) = \dfrac{1}{2q^2}\|u_n\|_{8/3}^{8/3}$, we also get a contradiction evidently. □

Remark 3.10. It is worth mentioning that in the above lemmas, except for Lemma 3.9, all the conclusions are effective for $2 < p < 10/3$. Unfortunately, we could not say anything more when $p \in (8/3, 6)$ as did in Lemma 3.9. In fact, when $8/3 < p < 10/3$, as usual, we want to establish the strong subadditivity (2.2). However, the appearance of $\tilde{B}$ in h_{g_u}, see (3.10), makes it impossible for us. In addition, since the functional $I(u)$ is unbounded from below on B_ρ if $10/3 < p < 6$, the above minimizing method is invalid any more.

Lemma 3.11. *The functional T defined in* (2.1) *satisfies* (2.11) *and* (2.12).

Proof. Based on Lemma 3.2, any minimizing sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Hence $\{u_n\}$ is bounded in all L^s norms for $s \in [2, 2^*]$ and up to a subsequence, by Lemma 2.1, there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup \bar{u} \in B_\rho$. According to the Gagliardo-Nirenberg inequality, using Lemma 2.1 and Lemma 3.5, we have

$$\begin{aligned}
\|u_n - u_m\|_p &\le \|u_n - u_m\|_2^\alpha \|\nabla(u_n - u_m)\|_{2^*}^{1-\alpha} \\
&\le \left(\|u_n - \bar{u}\|_2 + \|u_m - \bar{u}\|_2\right)^\alpha \|\nabla(u_n - u_m)\|_{2^*}^{1-\alpha} \\
&= o(1),
\end{aligned}$$

where $\dfrac{\alpha}{2} + \dfrac{1-\alpha}{2^*} = \dfrac{1}{p}$ and $2 < p < \dfrac{10}{3}$ discussed in Remark 3.10. As a result, using the Hölder inequality, we obtain that

$$\int_{\mathbb{R}^3} |u_n|^{p-1}|u_n - u_m| \mathrm{d}x \le \left(\int_{\mathbb{R}^3} |u_n|^p \mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^3} |u_n - u_m|^p \mathrm{d}x\right)^{\frac{1}{p}} = o(1),$$

and then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \left(|u_n|^{p-1} - |u_m|^{p-1}\right)(u_n - u_m)\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^3} |u_n|^{p-1}(u_n - u_m) - |u_m|^{p-1}(u_n - u_m)\mathrm{d}x\right| \\
&= o(1).
\end{aligned} \tag{3.14}$$

Additionally, by the Hölder inequality and Lemma 3.1, it holds that

$$\begin{aligned}
\int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u_m)\mathrm{d}x &\le \|\phi_{u_n}\|_6 \|u_n\|_2 \|u_n - u_m\|_3 \\
&\le C\|u_n\|^2 \|u_n\|_2 \|u_n - u_m\|_3 \\
&= o(1).
\end{aligned} \tag{3.15}$$

On the basis of (3.14), (3.15) and

$$\begin{aligned}&\left\langle T'(u_n)-T'(u_m),u_n-u_m\right\rangle\\&=\int_{\mathbb{R}^3}\left(\phi_{u_n}u_n-\phi_{u_m}u_m\right)(u_n-u_m)\mathrm{d}x-\int_{\mathbb{R}^3}\left(|u_n|^{p-1}-|u_m|^{p-1}\right)(u_n-u_m)\mathrm{d}x,\end{aligned}$$

(2.12) is a direct consequence. Moreover, the boundedness of $\{u_n\}$ in L^s-norm for $s\in\left[2,2^*\right]$ brings that

$$\begin{aligned}\left|\left\langle T'(u_n),u_n\right\rangle\right|&=\left|q^2\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x-\int_{\mathbb{R}^3}|u_n|^p\,\mathrm{d}x\right|\\&\le q^2\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x+\left|\int_{\mathbb{R}^3}|u_n|^p\,\mathrm{d}x\right|\\&\le C\|u_n\|_{\frac{12}{5}}^4+C_1\\&=O(1).\end{aligned}$$

Thus, (2.11) is achieved. □

Proof of Theorem 1.1.

Proof. Summing up, we have verified all the hypotheses of Lemma 2.1 and Proposition 2.5. Therefore, the limit $\bar{u}$ of the minimizing sequence $\{u_n\}$ makes problem (1.7) solved. In other words, $u_\rho=\bar{u}$ and the corresponding Lagrange multiplier ω_ρ is a couple of solution for problem (1.4). □

Next, to prove Theorem 1.2, we first recall a Liouville-type result, see ([35], Theorem 2.1).

Proposition 3.12. *Assume that* $N\ge 3$ *and the nonlinearity* $f:(0,\infty)\mapsto(0,\infty)$ *is continuous and satisfies*

$$\liminf_{s\to 0} s^{-\frac{N}{N-2}}f(s)>0.$$

Then the differential inequality $-\Delta u\ge f(u)$ *has no positive solution in any exterior domain of* $\mathbb{R}^N$.

Proof of Theorem 1.2. First of all, we assert that if $u\in B_\rho$ is a minimizer to problem (1.7) for $p=6$, then the associated Lagrange multiplier ω is negative. Indeed, firstly we have the following Pohozaev type identity ([9], A.3)

$$\begin{aligned}&\frac{1}{2}\|\nabla u\|_2^2+\frac{3}{2}\omega\|u\|_2^2+\frac{5q^2}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1-\mathrm{e}^{-\frac{|x-y|}{a}}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y\\&+\frac{q^2}{4a}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{e}^{-\frac{|x-y|}{a}}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y-\frac{3}{p}\|u\|_p^p=0,\end{aligned}$$

which can be rewritten as

$$\frac{1}{2}A(u)+\frac{3\omega}{2}\rho^2+\frac{5q^2}{4}B(u)-\frac{3}{p}C(u)+\frac{q^2}{4a}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{e}^{-\frac{|x-y|}{a}}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y=0. \quad (3.16)$$

In addition, it is easy to see that

$$A(u)+\omega\rho^2+q^2B(u)-C(u)=0. \quad (3.17)$$

Thus, (3.16) together with (3.17) gives that

$$A(u)+\frac{q^2}{4}B(u)-\frac{3p-6}{2p}C(u)-\frac{q^2}{4a}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{e}^{-\frac{|x-y|}{a}}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y=0. \quad (3.18)$$

As a result, substituting (3.17) into (3.18), it derives that

$$\begin{aligned}&2(6-p)A(u)-(5p-12)q^2B(u)-\frac{pq^2}{a}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{e}^{-\frac{|x-y|}{a}}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y\\&=2(3p-6)\omega\rho^2.\end{aligned} \quad (3.19)$$

For $p=6$, obviously one has

$$2(6-p)=0,\ -(5p-12)q^2<0,\ -\frac{6q^2}{a}<0,\ 2(3p-6)=24>0,$$

which, combining with (3.19), brings that $\omega<0$.

By Lemma 3.1, we know that

$$\begin{aligned}0\le\phi_u=\mathcal{K}*u^2&=\int_{\mathbb{R}^3}\mathcal{K}(x)u^2(y-x)\mathrm{d}x\\&=\int_{\mathbb{R}^3}\frac{1-\mathrm{e}^{-\frac{|x|}{a}}}{|x|}u^2(y-x)\mathrm{d}x\\&\le\int_{\mathbb{R}^3}\frac{1}{|x|}u^2(y-x)\mathrm{d}x=\frac{1}{|x|}*u^2.\end{aligned} \quad (3.20)$$

Moreover, note that ([36], Lemma 2.3) gives that

$$\lim_{|x|\to\infty}\left(|x|^{-1}*u^2\right)(x)=0. \quad (3.21)$$

Thus, together with (3.20) and (3.21) means that

$$\lim_{|x|\to\infty}\left(\mathcal{K}*u^2\right)(x)=0. \quad (3.22)$$

Assume that $u\in H^1(\mathbb{R}^3)$ is positive satisfying problem (1.7). In view of (3.22) and $\omega<0$, there exists $R_0>0$ large enough such that

$$\left(\mathcal{K}*u^2\right)(x)\le-\frac{\omega}{2q^2}\ \text{ for } |x|>R_0.$$

Therefore, we infer that

$$\begin{aligned}-\Delta u(x)&=\left(-\omega-q^2\phi_u+|u(x)|^4\right)u(x)\\&\ge\left(-\omega+\frac{\omega}{2q^2}\cdot q^2\right)u(x)=-\frac{\omega}{2}u(x)\quad\text{for } |x|>R_0.\end{aligned}$$

By applying Proposition 3.12 with $f(s)=-\frac{\omega}{2}s$, we have $-\Delta u\ge-\frac{\omega}{2}u$ and reach to a contradiction. □

4. Proof of Theorem 1.4

In this section, we consider Cauchy problem (1.10). To do this, the first task is to establish the local well posedness with aid of Proposition 2.6. In addition, to discuss the orbital stability, the solution with given initial value should exist globally.

According to the framework discussed in Section 2, for problem (1.10) our nonlinearity is of the following form

$$g(\psi)=g_1(\psi)+g_2(\psi):=-q^2\left(\mathcal{K}*\psi^2\right)\psi+|\psi|^{p-2}\psi.$$

Observe that $\mathcal{K}$ defined in (1.2) satisfies the following conditions: $\mathcal{K}$ is an even real-valued potential and $\mathcal{K}\in L^2\left(\mathbb{R}^3\right)$. Then, by ([28], Proposition 3.2.9), g_1 holds (2.14)-(2.17). Moreover, according to the discussion in ([28], Remark 3.2.6), we see that g_2 satisfies (2.14)-(2.17) evidently. So, g fulfils the hypotheses of Proposition 2.6, which means that the local well posedness is established.

Lemma 4.1. $T_{\max}=\infty$. *i.e. the solution of problem* (1.10) *is global.*

Proof. Let $\psi(x,t)$ be the solution of (1.10) and $T_{\max}\in(0,\infty]$ is its maximal time of existence. Then we either have

$$T_{\max}=\infty$$

or

$$T_{\max}<\infty \quad \text{and} \quad \lim_{t\to T_{\max}}\left\|\nabla\psi(x,t)\right\|_2^2=\infty.$$

If $T_{\max}<\infty$, due to

$$\begin{aligned}
I(\psi(x,t))&=\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla\psi(x,t)\right|^2\mathrm{d}x+\frac{q^2}{4}\int_{\mathbb{R}^3}\phi_\psi(x,t)\left|\psi(x,t)\right|^2\mathrm{d}x\\
&\quad-\frac{1}{p}\int_{\mathbb{R}^3}\left|\psi(x,t)\right|^p\mathrm{d}x\\
&\geq\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla\psi(x,t)\right|^2\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^3}\left|\psi(x,t)\right|^p\mathrm{d}x\\
&\geq\frac{1}{2}\left\|\nabla\psi(x,t)\right\|_2^2-C\left\|\psi(x,t)\right\|_2^{p(1-\gamma_p)}\left\|\nabla\psi(x,t)\right\|_2^{p\gamma_p}\\
&=\frac{1}{2}\left\|\nabla\psi(x,t)\right\|_2^2-C\rho^{p(1-\gamma_p)}\left\|\nabla\psi(x,t)\right\|_2^{p\gamma_p},
\end{aligned}$$

where $\gamma_p=\dfrac{3(p-2)}{2p}$ and $p\gamma_p<2$, we infer that $I(\psi(x,t))\to\infty$ as $t\to T_{\max}$. However, this contradicts the conservation of energy $I(\psi(x,t))=I(\psi_0(x))$, $\forall t\in(0,T_{\max})$. Hence, $T_{\max}=\infty$. □

Proof of Theorem 1.4. Observe that S_ρ is invariant by translation, namely, if $v\in S_\rho$ then also $v(\cdot-y)\in S_\rho$ for any $y\in\mathbb{R}^3$. Assume that for some $\rho>0$ small enough S_ρ is orbitally unstable, that is, there exist $\varepsilon>0$, a sequence of initial value $\{\psi_{n,0}\}\subset H^1\left(\mathbb{R}^3\right)$ and $\{t_n\}\subset\mathbb{R}$ such that the solution ψ_n, which is global and $\psi_n(\cdot,0)=\psi_{n,0}$, satisfying

$$\lim_{n\to\infty}\inf_{v\in S_\rho}\left\|\psi_{n,0}-v\right\|_{H^1(\mathbb{R}^3)}=0 \quad\text{and}\quad \inf_{v\in S_\rho}\left\|\psi_n(\cdot,t_n)-v\right\|_{H^1(\mathbb{R}^3)}\geq\varepsilon. \tag{4.1}$$

Consequently, there exists $u_\rho\in H^1\left(\mathbb{R}^3\right)$ minimizer of I_{ρ^2} and $\theta\in\mathbb{R}$ such that $v=\mathrm{e}^{i\theta}u_\rho$,

$$\left\|\psi_{n,0}\right\|_2\to\|v\|_2=\rho \quad\text{and}\quad I(\psi_{n,0})\to I(v)=I_{\rho^2}.$$

Indeed, we can assume that $\psi_{n,0} \in B_\rho$, because there exists $\alpha_n = \rho / \|\psi_{n,0}\|_2 \to 1$ such that $\alpha_n \psi_{n,0} \in B_\rho$ and $I(\alpha_n \psi_{n,0}) \to I_{\rho^2}$. In other words, $\psi_{n,0}$ can be replaced by $\alpha_n \psi_{n,0}$. Thus, $\{\psi_{n,0}\}$ is a minimizing sequence for I_{ρ^2}. Since $I(\psi_n(\cdot,t_n)) = I(\psi_{n,0})$, we know that $\{\psi_n(\cdot,t_n)\}$ is also a minimizing sequence for I_{ρ^2}. However, we have proved that every minimizing sequence has a subsequence converging (up to translation) in H^1-norm to a minimizer on the sphere B_ρ, which leads to a contradiction with (4.1). □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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