Journal of Applied Mathematics and Physics, 2022, 10, 2074-2085
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

A Family of Inertial Manifolds of Coupled Kirchhoff Equations

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How to cite this paper: Lin, G.G. and Chen, F.M. (2022) A Family of Inertial Manifolds of Coupled Kirchhoff Equations. *Journal of Applied Mathematics and Physics*, **10**, 2074-2085.
https://doi.org/10.4236/jamp.2022.106141

Received: May 7, 2022
Accepted: June 27, 2022
Published: June 30, 2022

Abstract

In this paper, we study the long-time behavior of the solution of the initial boundary value problem of the coupled Kirchhoff equations. Based on the relevant assumptions, the equivalent norm on E_k is obtained by using the Hadamard graph transformation method, and the Lipschitz constant l_F of F is further estimated. Finally, a family of inertial manifolds satisfying the spectral interval condition is obtained.

Keywords

Kirchhoff Equation, the Family of Inertial Manifolds, Hadamard Graph Transformation, Spectral Interval Condition

1. Introduction

This paper mainly studies the initial boundary value problem of the coupled Kirchhoff equations:

$$\begin{cases} u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g_1(u_t, v_t) = f_1(x), & (1) \\ v_{tt} + M\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} v + \beta(-\Delta)^{2m} v_t + g_2(u_t, v_t) = f_2(x), & (2) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, & (3) \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in \Omega, & (4) \\ \dfrac{\partial^i u}{\partial n^i} = 0, \dfrac{\partial^i v}{\partial n^i} = 0, (i = 0,1,2,\cdots,2m-1), x \in \partial\Omega. & (5) \end{cases}$$

where $\Omega \subseteq R^n (n \ge 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $g_1(u_t, v_t), g_2(u_t, v_t)$ are nonlinear terms, $M\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} u$,

$M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} v$ are the rigid terms which $M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)$ is real function, $f_1(x), f_2(x)$ are the external force terms, and $\beta(-\Delta)^{2m} u_t$, $\beta(-\Delta)^{2m} v_t (\beta \geq 0)$ are strong dissipative terms. This paper mainly studies the long-time behavior of the solution of the initial boundary value problem. Based on the relevant assumptions, the family of inertial manifolds satisfying the spectral interval condition is obtained by using the Hadamard graph transformation method.

As we all know, an inertial manifold is a Lipschitz manifold that contains a global attractor and attracts all solution orbits at an exponential rate, and it is finite-dimensional and positively invariant. The inertial manifold is of great significance to study the long-term behavior of infinite dimensional dynamical systems. Because it transforms infinite dimensional problems into finite-dimensional problems, and an inertial manifold is of great significance to the development of nonlinear science.

In 1989, Constantin, Foias, Nicolaenko, *et al.* [1] tried to refine the spectral separation conditions by using the concept of spectral barrier in Hilbert space. In 1991, Eugene Fabes, Mitchell Luskin and George R Sell [2] used the elliptic regularization method to construct the inertial manifold. Two famous methods used to prove the existence of inertial manifold are the Lyapunov Perron method and the Hadamard graph transformation method.

Based on the above references, Guoguang Lin and Lingjuan Hu [3] studied the inertial manifold for nonlinear higher-order coupled Kirchhoff equations with strong linear damping

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g_1(u,v)=f_1(x), \\ v_{tt}+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v)=f_2(x), \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \\ v(x,0)=v_0(x), v_t(x,0)=v_1(x), x\in\Omega, \\ \dfrac{\partial^i u}{\partial n^i}=0, \dfrac{\partial^i v}{\partial n^i}=0, (i=0,1,2,\cdots,m-1), x\in\partial\Omega. \end{cases}$$

where $\Omega\subseteq R^n (n\geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $g_1(u,v), g_2(u,v)$ are nonlinear source terms, $f_1(x), f_2(x)$ are the external force terms, $M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m u$, $M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m v$ are rigid terms which $M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)$ is real function, $\beta(-\Delta)^m u_t$, $\beta(-\Delta)^m v_t (\beta\geq 0)$ are strong dissipative terms. Using the Hadamard graph transformation method, they obtain the existence of the inertial manifold while such equations satisfy the spectrum interval condition.

Guoguang Lin and Lujiao Yang in [4] first studied the family of inertial manifolds and exponential attractors for the Kirchhoff equations.

$$\begin{cases} u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u) = f(x), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, t > 0, \\ u(x,t) = 0, \dfrac{\partial^i u}{\partial v^i} = 0, (i = 1, 2, \cdots, 2m-1), x \in \partial\Omega. \end{cases}$$

where $m > 1$, $p \ge 2$, $\Omega \in R^n (n \ge 1)$ is the bounded domain with smooth boundary $\partial\Omega$, $\beta > 0$ is the dissipative coefficient, $\beta(-\Delta)^{2m} u_t$ is the strong dissipative term, $g(u)$ is the nonlinear term among, and $f(x)$ is the external force term, $M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u$ is the rigid term which $M\left(\left\|\nabla^m u\right\|_p^p\right) \in C^2\left([0,+\infty); R^+\right)$ is real function. After making appropriate assumptions, the existence of exponential attractor is obtained by proving the discrete squeezing property of the equation. Then according to Hadamard's graph transformation method, the spectral interval condition is proved to be true, therefore, the existence of a family of the inertial manifolds for the equation is obtained.

Because an inertial manifold plays a very important role in describing the long-time behavior of solutions, it is of great significance to the development of nonlinear science. The relevant research theoretical results are shown in references [5]-[19].

On the basis of previous studies, this paper further improves the order of the strong dissipative term and the rigid term mentioned by Guoguang Lin and Lingjuan Hu [3], where the coefficient of the rigid term is extended from $M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^m v\right\|^2\right)$ to $M\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)$, and $g_1(u_t, v_t), g_2(u_t, v_t)$ are new nonlinear terms. When constructing the equivalent norm in E_k space, through reasonable assumptions and combined with the Lipschitz property of the nonlinear term, the family of inertial manifolds satisfying the spectral interval condition is obtained.

2. Preliminaries

The following symbols and assumptions are introduced for the convenience of statement:

$$V_0 = L^2(\Omega), V_{2m} = H^{2m}(\Omega) \cap H_0^1(\Omega), V_{2m+k} = H^{2m+k}(\Omega) \cap H_0^1(\Omega),$$
$$V_k = H^k(\Omega) \cap H_0^1(\Omega), E_0 = V_{2m} \times V_0 \times V_{2m} \times V_0,$$
$$E_k = V_{2m+k} \times V_k \times V_{2m+k} \times V_k, (k = 0, 1, 2, \cdots, 2m)$$

The inner product of the $L^2(\Omega)$ space is $(u,v) = \int_\Omega u(x) v(x) \mathrm{d}x$ and the norm is $\|u\| = \|u\|_{L^2} = \left(\int_\Omega |u(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}$.

The norm of $L^p(\Omega)$ space is called $\|u\|_p = \|u\|_{L^p(\Omega)}$.

l_F is the Lipschitz constant of $F \in C_b(E_k, E_k)$, l is the Lipschitz constant of $g_i(u_t, v_t)(i = 1, 2)$.

Relevant assumptions:

(H1) $\dfrac{\beta^2}{2}\left(1 - \mu_1^{-k}\right) \ge 0$;

(H$_2$) $\frac{1}{2}\left[\left(\mu_{N+1}-\mu_N\right)\left(\beta-\sqrt{\beta^2\mu_1-4M(s)}\right)\right]\geq\frac{16l}{\beta^2\mu_1-4M(s)}+1$.

Definition 2.1 [5] Assuming $S=\left(S(t)\right)_{t\geq 0}$ is a solution semigroup on Banach space $E_k=V_{2m+k}\times V_k\times V_{2m+k}\times V_k$, subset $\mu_k\subset E_k$ is said to be a family of inertial manifolds, if they satisfy the following three properties:

1) μ_k are a finite-dimensional Lipschitz manifold;

2) μ_k is positively invariant, *i.e.*, $S(t)\mu_k\subseteq\mu_k, t\geq 0$;

3) μ_k attracts exponentially all orbits of solution , that is, for any $x\in E_k$, there are constants $\eta>0, C>0$ such that

$$dist\left(S(t)x,\mu_k\right)\leq Ce^{-\eta t}, t\geq 0.$$

Definition 2.2 [5] Assuming the operator $A: X\to X$ have countable positive real part eigenvalues and $F\in C_b(X,X)$ satisfies the Lipschitz condition:

$$\left\|F(u)-F(v)\right\|_X\leq l_F\left\|u-v\right\|_X,\quad u,v\in X.$$

If the point spectrum of the operator A can be divided into the following two parts σ_1 and σ_2, where σ_1 is finite

$$\Lambda_1=\sup\left\{\operatorname{Re}\lambda\middle|\lambda\in\sigma_1\right\},\Lambda_2=\inf\left\{\operatorname{Re}\lambda\middle|\lambda\in\sigma_2\right\},$$

$$X_i=span\left\{w_j\middle|\lambda_j\in\sigma_i\right\},(i=1,2),$$

and satisfy

$$\Lambda_2-\Lambda_1>4l_F, \tag{6}$$

$$X=X_1\oplus X_2, \tag{7}$$

where $P_1: E_k\to E_{k1}$ and $P_2: E_k\to E_{k2}$ are orthogonal projection. So the operator A is said to satisfy the spectral interval condition.

Lemma 2.1 [5] Assuming eigenvalue $\left\{\lambda_k^-\right\}_{k\geq 1}$ is non-decreasing sequence, there is $N_1\in N^+$, for any $N\geq N_1$, such that μ_N^- and μ_{N+1}^- are continuous adjacent values.

3. The Family of Inertial Manifolds

From the above preparation knowledge, Equations (1)-(5) are equivalent to

$$U_t+AU=F(U),\quad U\in E_k, \tag{8}$$

where $U=(u,w,v,q)$, $w=u_t$, $q=v_t$,

$$A=\begin{pmatrix}0 & -I & 0 & 0\\ M(s)(-\Delta)^{2m} & \beta(-\Delta)^{2m} & 0 & 0\\ 0 & 0 & 0 & -I\\ 0 & 0 & M(s)(-\Delta)^{2m} & \beta(-\Delta)^{2m}\end{pmatrix},$$

$$F(U)=\begin{pmatrix}0\\ f_1(x)-g_1(u_t,v_t)\\ 0\\ f_2(x)-g_2(u_t,v_t)\end{pmatrix},$$

$$D(A)=\left\{(u,v)\in V_{2m+k}\times V_{2m+k}\left|(u,v)\in V_0\times V_0,\left(\nabla^{2m+k}u,\nabla^{2m+k}v\right)\in V_0\times V_0\right.\right\}$$
$$\times V_k\times V_k,$$

$$E_k=V_{2m+k}\times V_k\times V_{2m+k}\times V_k.$$

In order to determine the eigenvalue of operator A, we must first consider the graph norm generated by the inner product in E_k

$$\begin{aligned}(U,V)_{E_k}&=\left(M(s)\cdot\nabla^{2m+k}u,\nabla^{2m+k}\overline{a}\right)+\left(\nabla^k w,\nabla^k\overline{b}\right)\\&\quad+\left(M(s)\cdot\nabla^{2m+k}v,\nabla^{2m+k}\overline{c}\right)+\left(\nabla^k q,\nabla^k\overline{d}\right),\end{aligned}\tag{9}$$

where $U=(u,w,v,q)$, $V=(a,b,c,d)$, $s=\|\nabla^m u\|_p^p+\|\nabla^m v\|^2$, $\overline{a},\overline{b},\overline{c},\overline{d}$ represent the conjugation of a,b,c,d respectively.

Indeed, for $U\in D(A)$,

$$\begin{aligned}(AU,U)_{E_k}&=-\left(M(s)\nabla^{2m+k}w,\nabla^{2m+k}\overline{u}\right)+\left(M(s)\nabla^{2m+k}u,\nabla^{2m+k}\overline{w}\right)\\&\quad+\beta\left(\nabla^{2m+k}w,\nabla^{2m+k}\overline{w}\right)\\&=-\left(M(s)\nabla^{2m+k}q,\nabla^{2m+k}\overline{v}\right)+\left(M(s)\nabla^{2m+k}v,\nabla^{2m+k}\overline{q}\right)\\&\quad+\beta\left(\nabla^{2m+k}q,\nabla^{2m+k}\overline{q}\right)\\&=\beta\left(\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\right)\geq 0,\end{aligned}$$

therefore, $(AU,U)_{E_k}$ is a nonnegative real number.

To further determine the eigenvalues of A, we consider the following characteristic equation

$$AU=\lambda U,U=(u,w,v,q)\in E_k\tag{10}$$

then

$$\begin{cases}-w=\lambda u,\\M(s)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}w=\lambda w,\\-q=\lambda v,\\M(s)(-\Delta)^{2m}v+\beta(-\Delta)^{2m}q=\lambda q.\end{cases}\tag{11}$$

Substitute the first equation into the second equation, and substitute the third equation into the fourth equation in (11), thus

$$\begin{cases}\lambda^2u-\lambda\beta(-\Delta)^{2m}u+M(s)(-\Delta)^{2m}u=0,\\\lambda^2u-\lambda\beta(-\Delta)^{2m}u+M(s)(-\Delta)^{2m}u=0,\\\left.\dfrac{\partial^i u}{\partial n^i}\right|_{\partial\Omega}=\left.\dfrac{\partial^i v}{\partial n^i}\right|_{\partial\Omega}=0,i=0,1,2,\cdots,2m-1.\end{cases}\tag{12}$$

Take $(-\Delta)^k u,(-\Delta)^k v$ inner product with the first equation and the second equation of the above Equations (12) respectively to obtain

$$\begin{cases}\lambda^2\left\|\nabla^k u\right\|^2-\lambda\beta\left\|\nabla^{2m+k}u\right\|^2+M(s)\left\|\nabla^{2m+k}u\right\|^2=0,\\\lambda^2\left\|\nabla^k v\right\|^2-\lambda\beta\left\|\nabla^{2m+k}v\right\|^2+M(s)\left\|\nabla^{2m+k}v\right\|^2=0.\end{cases}\tag{13}$$

The above Equation (13) are sorted out

$$\begin{aligned}&\lambda^2\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k v\right\|^2\right)-\lambda\beta\left(\left\|\nabla^{2m+k} u\right\|^2+\left\|\nabla^{2m+k} v\right\|^2\right)\\&+M(s)\left(\left\|\nabla^{2m+k} u\right\|^2+\left\|\nabla^{2m+k} v\right\|^2\right)=0.\end{aligned} \tag{14}$$

The above Equation (14) is a quadratic equation of one variable about λ, and u_k, v_k are used to replace u, v in the above equation. For any positive integer *k*, the above Equation (10) has paired eigenvalues

$$\lambda_k^{\pm}=\frac{\beta\mu_k\pm\sqrt{\beta^2\mu_k^2-4M(s)\cdot\mu_k}}{2},$$

where μ_k is the eigenvalue of $\begin{pmatrix}(-\Delta)^{2m} & 0\\ 0 & (-\Delta)^{2m}\end{pmatrix}$ in $V_{2m}\times V_{2m}$, taking $\mu_k=\lambda_1 k^{\frac{2m}{n}}$.

If $\mu_k\ge\frac{4}{\beta^2}M(s)$, that is $\mu_k\ge\frac{4}{\beta^2}m^*$, then all eigenvalues of operator A are real numbers, and the corresponding characteristic function have the following forms

$$U_k^{\pm}=\left(u_k,-\lambda_k^{\pm}u_k,v_k,-\lambda_k^{\pm}v_k\right).$$

For the convenience of the following description, for any positive integer *k*,

$$\begin{aligned}&\left\|\nabla^k u_k\right\|^2+\left\|\nabla^k v_k\right\|^2=1,\left\|\nabla^{2m+k} u_k\right\|^2+\left\|\nabla^{2m+k} v_k\right\|^2=\mu_k,\\&\left\|\nabla^{-2m-k} u_k\right\|^2+\left\|\nabla^{-2m-k} v_k\right\|^2=\frac{1}{\mu_k}.\end{aligned} \tag{15}$$

Lemma 3.1 $g_i(u_t,v_t):V_k\times V_k\to V_k\times V_k$, $i=1,2$ is uniformly bounded and globally Lipschitz continuous.

Proof. for arbitrary $(u_t,v_t),(\overline{u_t},\overline{v_t})\in V_k\times V_k$, $w=u_t,\overline{w}=\overline{u_t},q=v_t,\overline{q}=\overline{v_t}$,

$$\begin{aligned}&\left\|g_i\left(\overline{u_t},\overline{v_t}\right)-g_i\left(u_t,v_t\right)\right\|_{V_k\times V_k}\\&=\Big\|g_{iu_t}\left(\overline{u_t}+\theta_1\left(\overline{u_t}-u_t\right),\overline{v_t}+\theta_1\left(\overline{v_t}-v_t\right)\right)\left(\overline{u_t}-u_t\right)\\&\quad+g_{iv_t}\left(\overline{u_t}+\theta_1\left(\overline{u_t}-u_t\right),\overline{v_t}+\theta_1\left(\overline{v_t}-v_t\right)\right)\left(\overline{v_t}-v_t\right)\Big\|_{V_k\times V_k}\\&\le l\left(\left\|\overline{w}-w\right\|_{V_k}+\left\|\overline{q}-q\right\|_{V_k}\right),\end{aligned}$$

where $\theta_1\in(0,1)$, l is the Lipschitz constant.

Lemma 3.1 is proved.

Theorem 3.1 When $\mu_k\ge\frac{4}{\beta^2}m^*$, there is a large enough $N_1\in N^*$ so that $\forall N\ge N_1$ has

$$\frac{1}{2}\left[\left(\mu_{N+1}-\mu_N\right)\left(\beta-\sqrt{\beta^2\mu_1-4M(s)}\right)\right]\ge\frac{16l}{\beta^2\mu_1-4M(s)}+1, \tag{16}$$

where l is the Lipschitz constant of $g_i(u_t,v_t)(i=1,2)$, then operator A satisfies the spectral interval condition of Definition 2.2.

Proof. When $\mu_k \ge \frac{4}{\beta^2} m^*$, all eigenvalues of A are positive real numbers, and the sequences $\{\lambda_k^-\}_{k\ge1}$ and $\{\lambda_k^+\}_{k\ge1}$ are monotonic.

The following is divided into four steps to prove this theorem.

Step 1 Since $\{\lambda_k^-\}_{k\ge1}$ is a non-decreasing sequence, according to Lemma 2.1, there is $N_1 \in N^+$, for any $N \ge N_1$, such that μ_N^- and μ_{N+1}^- are continuous adjacent values.

Step 2 The existence of N makes μ_N^- and μ_{N+1}^- continuous adjacent values, so the eigenvalues of A can be decomposed into

$$\sigma_1 = \left\{\lambda_k^- \middle| 1 \le k \le N\right\},$$

$$\sigma_2 = \left\{\lambda_j^\pm, \lambda_k^+ \middle| 1 \le k \le N \le j\right\}.$$

The corresponding E_k can be decomposed into

$$E_{k1} = Span\left\{U_k^- \middle| \lambda_k^- \in \sigma_1\right\}, \tag{17}$$

$$E_{k2} = Span\left\{U_j^\pm, U_k^+ \middle| \lambda_j^\pm, \lambda_k^+ \in \sigma_2\right\}. \tag{18}$$

In order to prove the spectral interval condition, we will find out the orthogonality of subspaces E_{k1} and E_{k2}.

Further decompose subspace $E_{k2} = E_C \oplus E_R$, where

$$E_C = Span\left\{U_k^+ \middle| \lambda_k^+ \in \sigma_2\right\},$$

$$E_R = Span\left\{U_j^\pm \middle| \lambda_j^\pm \in \sigma_2\right\}.$$

Note that E_{k1} and E_C are finite-dimensional subspaces, $U_N^- \in E_{k1}$, $U_{N+1}^- \in E_R$. Because E_{k1} and E_R are orthogonal and E_{k1} and E_C are not orthogonal, E_{k1} and E_{k2} are not orthogonal.

Next, the equivalent norm of eigenvalues on E_k is specified so that E_{k1} and E_{k2} are orthogonal.

Under the new graph norm, let $E_N = E_{k1} \oplus E_c$.

Let the functions $\Phi: E_N \to R$ and $\Psi: E_R \to R$,

$$\begin{aligned}
\Phi(U,V) = & -2M(s)\left(\nabla^k u, \nabla^k \bar{a}\right) + 2\beta\left(\nabla^{2m+k}\bar{u}, \nabla^{2m+k} a\right) + \beta\left(\nabla^{2m} u, \nabla^{-2m-k}\bar{b}\right) \\
& + \beta\left(\nabla^{-2m-k}\bar{w}, \nabla^{2m} a\right) + 2\left(\nabla^{-2m-k} w, \nabla^{-2m-k}\bar{b}\right) \\
& + \left(\beta^2 - 2\beta\right)\left(\nabla^{2m+k} u, \nabla^{2m+k}\bar{a}\right) - 2M(s)\left(\nabla^k v, \nabla^k \bar{c}\right) \\
& + 2\beta\left(\nabla^{2m+k}\bar{v}, \nabla^{2m+k} c\right) + \beta\left(\nabla^{2m} v, \nabla^{-2m-k}\bar{d}\right) + \beta\left(\nabla^{-2m-k}\bar{q}, \nabla^{2m} c\right) \\
& + 2\left(\nabla^{-2m-k} q, \nabla^{-2m-k}\bar{d}\right) + \left(\beta^2 - 2\beta\right)\left(\nabla^{2m+k} v, \nabla^{2m+k}\bar{c}\right),
\end{aligned}$$

$$\begin{aligned}
\Psi(U,V) = & \beta^2\left(\nabla^{2m+k} u, \nabla^{2m+k}\bar{a}\right) + 2\beta\left(\nabla^{2m+k} u, \nabla^k \bar{b}\right) + 2\beta\left(\nabla^k \bar{w}, \nabla^{2m+k} a\right) \\
& + \left(8 + \frac{\beta^2 \mu_1}{2}\right)\left(\nabla^k w, \nabla^k \bar{b}\right) + \beta^2\left(\nabla^{2m+k} v, \nabla^{2m+k}\bar{c}\right) \\
& + 2\beta\left(\nabla^{2m+k} v, \nabla^k \bar{d}\right) + 2\beta\left(\nabla^k \bar{q}, \nabla^{2m+k} c\right) + \left(8 + \frac{\beta^2 \mu_1}{2}\right)\left(\nabla^k q, \nabla^k \bar{d}\right),
\end{aligned}$$

where $U=(u,w,v,q),V=(a,b,c,d)\in E_N$ or E_R.

For $U=(u,w,v,q)\in E_N$, then

$$
\begin{aligned}
&\Phi(U,U)\\
&=-2M(s)\left(\nabla^k u,\nabla^k \bar{u}\right)+2\beta\left(\nabla^{2m+k}\bar{u},\nabla^{2m+k}u\right)+\beta\left(\nabla^{2m}u,\nabla^{-2m-k}\bar{w}\right)\\
&\quad+\beta\left(\nabla^{-2m-k}\bar{w},\nabla^{2m}u\right)+2\left(\nabla^{-2m-k}w,\nabla^{-2m-k}\bar{w}\right)+\left(\beta^2-2\beta\right)\left(\nabla^{2m+k}u,\nabla^{2m+k}\bar{u}\right)\\
&\quad-2M(s)\left(\nabla^k v,\nabla^k \bar{v}\right)+2\beta\left(\nabla^{2m+k}\bar{v},\nabla^{2m+k}v\right)+\beta\left(\nabla^{2m}v,\nabla^{-2m-k}\bar{q}\right)\\
&\quad+\beta\left(\nabla^{-2m-k}\bar{q},\nabla^{2m}v\right)+2\left(\nabla^{-2m-k}q,\nabla^{-2m-k}\bar{q}\right)+\left(\beta^2-2\beta\right)\left(\nabla^{2m+k}v,\nabla^{2m+k}\bar{v}\right)\\
&\ge-2M(s)\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k v\right\|^2\right)+\beta^2\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)\\
&\quad+2\left(\left\|\nabla^{-2m-k}w\right\|^2+\left\|\nabla^{-2m-k}q\right\|^2\right)\\
&\quad-2\beta\left(\mu_1^{-\frac{k}{2}}\left\|\nabla^{2m+k}u\right\|\left\|\nabla^{-2m-k}w\right\|+\mu_1^{-\frac{k}{2}}\left\|\nabla^{2m+k}v\right\|\left\|\nabla^{-2m-k}q\right\|\right)\\
&\ge-2M(s)\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k v\right\|^2\right)+\left(\frac{\beta^2}{2}+\left(\frac{\beta^2}{2}-\frac{\beta^2}{2}\mu_1^{-k}\right)\right)\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)\\
&\ge-2M(s)\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k v\right\|^2\right)+\frac{\beta^2}{2}\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)\\
&\ge\left(\frac{\beta^2}{2}\mu_1-2M(s)\right)\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k v\right\|^2\right),
\end{aligned}
$$

for any *k*, having $\frac{\beta^2}{2}\mu_k\ge 2M(s)$, then $\Phi(U,U)\ge 0$, $\forall U=(u,w,v,q)\in E_N$, that is, Φ is positive definite.

Similarly, $U=(u,w,v,q)\in E_R$,

$$
\begin{aligned}
\Psi(U,U)&=\beta^2\left(\nabla^{2m+k}u,\nabla^{2m+k}\bar{u}\right)+2\beta\left(\nabla^{2m+k}u,\nabla^k\bar{w}\right)+2\beta\left(\nabla^k\bar{w},\nabla^{2m+k}u\right)\\
&\quad+\left(8+\frac{\beta^2\mu_1}{2}\right)\left(\nabla^k w,\nabla^k\bar{w}\right)+\beta^2\left(\nabla^{2m+k}v,\nabla^{2m+k}\bar{v}\right)\\
&\quad+2\beta\left(\nabla^{2m+k}v,\nabla^k\bar{q}\right)+2\beta\left(\nabla^k\bar{q},\nabla^{2m+k}v\right)+\left(8+\frac{\beta^2\mu_1}{2}\right)\left(\nabla^k q,\nabla^k\bar{q}\right)\\
&\ge\frac{\beta^2}{2}\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)+\frac{\beta^2}{2}\mu_1\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right)\\
&\ge\frac{\beta^2}{2}\mu_1\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k v\right\|^2+\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right),
\end{aligned}
$$

then $\Psi(U,U)\ge 0,\forall U=(u,w,v,q)\in E_R$, that is, Ψ is positive definite.

Then redefine the equivalent norm on E_k

$$\left\langle\left\langle U,V\right\rangle\right\rangle_{E_k}=\Phi\left(P_N U,P_N V\right)+\Psi\left(P_R U,P_R V\right), \tag{19}$$

where P_N and P_R are projections of $E_k\to E_N$ and $E_k\to E_R$ respectively. For convenience, Equation (19) is abbreviated as

$$\left\langle\left\langle U,V\right\rangle\right\rangle_{E_k}=\Phi(U,V)+\Psi(U,V).$$

Based on the redefinition of the equivalent norm in E_k, to prove that E_{k_1}

and E_{k_2} are orthogonal, we only need to prove that E_{k_1} and E_c are orthogonal.

Through Equation (19), the following equation holds

$$\left\langle\left\langle U_j^-,U_j^+\right\rangle\right\rangle_{E_k}=\Phi\left(U_j^-,U_j^+\right)=0,\left(U_j^-\in E_{k1},U_j^+\in E_c\right).$$

The main calculation process is as follows

$$\begin{aligned}\Phi\left(U_j^-,U_j^+\right)&=-2M(s)\left(\nabla^k u_j,\nabla^k\overline{u_j}\right)+2\beta\left(\nabla^{2m+k}\overline{u_j},\nabla^{2m+k}u_j\right)\\&\quad+\beta\left(\nabla^{2m}u_j,-\lambda_j^+\nabla^{-2m-k}\overline{u_j}\right)+\beta\left(-\lambda_j^-\nabla^{-2m-k}\overline{u_j},\nabla^{2m}u_j\right)\\&\quad+2\left(-\lambda_j^-\nabla^{-2m-k}u_j,-\lambda_j^+\nabla^{-2m-k}\overline{u_j}\right)+\left(\beta^2-2\beta\right)\left(\nabla^{2m+k}u_j,\nabla^{2m+k}\overline{u_j}\right)\\&\quad-2M(s)\left(\nabla^k v_j,\nabla^k\overline{v_j}\right)+2\beta\left(\nabla^{2m+k}\overline{v_j},\nabla^{2m+k}v_j\right)\\&\quad+\beta\left(\nabla^{2m}v_j,-\lambda_j^+\nabla^{-2m-k}\overline{v_j}\right)+\beta\left(-\lambda_j^-\nabla^{-2m-k}\overline{v_j},\nabla^{2m}v_j\right)\\&\quad+2\left(-\lambda_j^-\nabla^{-2m-k}v_j,-\lambda_j^+\nabla^{-2m-k}\overline{v_j}\right)+\left(\beta^2-2\beta\right)\left(\nabla^{2m+k}v_j,\nabla^{2m+k}\overline{v_j}\right)\\&=-2M(s)\left(\left\|\nabla^k u_j\right\|^2+\left\|\nabla^k v_j\right\|^2\right)+\beta^2\left(\left\|\nabla^{2m+k}u_j\right\|^2+\left\|\nabla^{2m+k}v_j\right\|^2\right)\\&\quad-\beta\left(\lambda_j^++\lambda_j^-\right)\left(\left\|\nabla^{-k}u_j\right\|^2+\left\|\nabla^{-k}v_j\right\|^2\right)\\&\quad+2\left(\lambda_j^+\cdot\lambda_j^-\right)\left(\left\|\nabla^{-2m-k}u_j\right\|^2+\left\|\nabla^{-2m-k}v_j\right\|^2\right)\\&=-2M(s)+\beta^2\mu_k-\beta\left(\lambda_j^++\lambda_j^-\right)+2\left(\lambda_j^+\cdot\lambda_j^-\right)\frac{1}{\mu_k},\end{aligned}\tag{20}$$

where $U_j^-\in E_{k1},U_j^+\in E_c$.

Through Equation (14), we can get

$$\lambda_j^++\lambda_j^-=\beta\mu_k,\lambda_j^+\cdot\lambda_j^-=M(s)\mu_k.$$

So

$$\Phi\left(U_j^-,U_j^+\right)=-2M(s)+\beta^2\mu_k-\beta\left(\lambda_j^++\lambda_j^-\right)+2\left(\lambda_j^+\cdot\lambda_j^-\right)\frac{1}{\mu_k}=0.$$

That is, E_{k_1} and E_c are orthogonal, further E_{k_1} and E_{k_2} are orthogonal.

Step 3 After the Step 2, $E_k=E_{k1}\oplus E_{k2}$ has been established. Now we estimate the Lipschitz constant l_F of F. By lemma 3.1, $g_i(u_t,v_t):V_k\times V_k\to V_k\times V_k,(i=1,2)$ is uniformly bounded and globally Lipschitz continuous, and $F(U)=\left(0,f_1(x)-g_1(u_t,v_t),0,f_2(x)-g_2(u_t,v_t)\right)^{\mathrm{T}}$.

Let $P_1:E_k\to E_{k1}$ and $P_2:E_k\to E_{k2}$ be orthogonal projections, if

$$U=(u,w,v,q)\in E_k,U_1=(u_1,w_1,v_1,q_1)\in P_1U,U_2=(u_2,w_2,v_2,q_2)\in P_2U,$$

then

$$P_1u=u_1,P_1w=w_1,P_1v=v_1,P_1q=q_1.$$

So

$$\begin{aligned}\|U\|_{E_k}^2&=\Phi(P_1U,P_1U)+\Psi(P_2U,P_2U)\\&\ge\left(\frac{\beta^2}{2}\mu_1-2M(s)\right)\left(\left\|\nabla^kP_1u\right\|^2+\left\|\nabla^kP_1v\right\|^2\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta^2}{2}\mu_1\left(\left\|\nabla^k P_2 u\right\|^2+\left\|\nabla^k P_2 v\right\|^2+\left\|\nabla^k P_2 w\right\|^2+\left\|\nabla^k P_2 q\right\|^2\right)\\
&\geq\left(\frac{\beta^2}{2}\mu_1-2M(s)\right)\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k v\right\|^2\right)\\
&+\left(\frac{\beta^2}{2}\mu_1-2M(s)\right)\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right)\\
&\geq\left(\frac{\beta^2}{2}\mu_1-2M(s)\right)\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right).
\end{aligned}
$$

Given $U=(u,w,v,q),W=(\bar{u},\bar{w},\bar{v},\bar{q})\in E_k$, from lemma 3.1

$$
\begin{aligned}
&\left\|F(U)-F(W)\right\|_{E_k}\\
&=\left\|g_1(u_t,v_t)-g_1(\overline{u_t},\overline{v_t})\right\|_{V_k\times V_k}+\left\|g_2(u_t,v_t)-g_2(\overline{u_t},\overline{v_t})\right\|_{V_k\times V_k}\\
&\leq 2l\left(\left\|w-\bar{w}\right\|_{V_k}+\left\|q-\bar{q}\right\|_{V_k}\right)\\
&\leq\frac{2l}{\sqrt{\frac{\beta^2}{2}\mu_1-2M(s)}}\left\|U-W\right\|_{E_k},
\end{aligned}
\tag{21}
$$

so

$$
l_F\leq\frac{2l}{\sqrt{\frac{\beta^2}{2}\mu_1-2M(s)}}=\frac{4l}{\sqrt{\beta^2\mu_1-4M(s)}}. \tag{22}
$$

Step 4 Prove Equation (6) holds in Definition 2.2.

According to the eigenvalue of A decomposition, letting $\Lambda_1=\lambda_N^-,\Lambda_2=\lambda_{N+1}^-$, then

$$
\Lambda_2-\Lambda_1=\lambda_{N+1}^--\lambda_N^-=\frac{\beta}{2}\left(\mu_{N+1}^--\mu_N^-\right)+\frac{1}{2}\left(\sqrt{R(N)}-\sqrt{R(N+1)}\right), \tag{23}
$$

where $\sqrt{R(N)}=\beta^2\mu_N^2-4M(s)\mu_N$.

There is $N_1\in N^+$, for any $N\geq N_1$, having

$$
\begin{aligned}
&\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_1-4M(s)}\left(\mu_{N+1}^--\mu_N^-\right)\\
&=\sqrt{\beta^2\mu_1-4M(s)}\left(\left(\sqrt{\frac{R(N)}{\beta^2\mu_1-4M(s)}}-\mu_N\right)-\left(\sqrt{\frac{R(N+1)}{\beta^2\mu_1-4M(s)}}-\mu_{N+1}\right)\right)\\
&=\sqrt{\beta^2\mu_1-4M(s)}\left(R_1(N)-R_1(N+1)\right),
\end{aligned}
\tag{24}
$$

letting $R_1(N)=\sqrt{\frac{R(N)}{\beta^2\mu_1-4M(s)}}-\mu_N$.

Because $\lim\limits_{N\to+\infty}R_1(N)=\lim\limits_{N\to+\infty}\sqrt{\frac{R(N)}{\beta^2\mu_1-4M(s)}}-\mu_N=0$,

$$
\lim_{N\to+\infty}\left(\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_1-4M(s)}\left(\mu_{N+1}^--\mu_N^-\right)\right)=0. \tag{25}
$$

According to the assumptions (16) of Theorem 3.1 and (22)-(25), we can get

$$\Lambda_2 - \Lambda_1 \geq \frac{1}{2}\left[\left(\mu_{N+1} - \mu_N\right)\left(\beta - \sqrt{\beta^2 \mu_1 - 4M(s)}\right)\right] - 1 \geq \frac{16l}{\beta^2 \mu_1 - 4M(s)} \geq 4l_F, \quad (26)$$

so operator A satisfies the spectral interval condition.

Theorem 3.1 is proved.

The equivalent norm on E_k is obtained by using the Hadamard graph transformation method. On this basis, it is proved that $E_k = E_{k1} \oplus E_{k2}$, where $P_1 : E_k \to E_{k1}$ and $P_2 : E_k \to E_{k2}$ are orthogonal projections. Since $g_i(u_t, v_t) : V_k \times V_k \to V_k \times V_k, (i = 1, 2)$ is uniformly bounded and globally Lipschitz continuous, the Lipschitz constant l_F of F can be further estimated. Finally, Formula (26) holds and then operator A satisfies the spectral interval condition. Next, we will further obtain that the initial boundary value problems (1)-(5) have a family of inertial manifolds.

Theorem 3.2 If $F \in C_b(E_k, E_k)$ satisfies Lipschitz condition and operator A satisfies spectral interval condition , then the initial boundary value problem (1)-(5) has a family of inertial manifolds $\mu_k \in E_k$,

$$\mu_k = graph(\Gamma) \in E_k := \left\{\varsigma + \Gamma(\varsigma) : \varsigma \in E_{k_1}\right\},$$

where E_{k_1} and E_{k_2} are defined in (17)-(18), and $\Gamma : E_{k_1} \to E_{k_2}$ is Lipschitz continuous function.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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