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The Family of Global Attractors for Kirchhoff-Type Coupled Equations

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Abstract

This paper mainly studies the initial value problems of Kirchhoff-type coupled equations. Firstly, by giving the hypothesis of Kirchhoff stress term $M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)$, the Galerkin's method obtains the existence uniqueness of the overall solution of the above problem by using a priori estimates in the spaces of E_0 and E_k, and secondly, it proves that there is a family of global attractors for the above problem, and finally estimates the Hausdorff dimension and the Fractal dimension of the family of global attractors.

Keywords

Kirchhoff Equation, the Existence and Uniqueness of Global Solution, the Family of Global Attractors, Hausdorff Dimension and Fractal Dimension of Global Attractor

1. Introduction

This paper investigates the following primal value problems of a system of generalized Kirchhoff-type coupled equations:

$$\begin{cases} u_{tt}+M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)(-\Delta)^{2m} u+\beta(-\Delta)^{2m} u_t+g_1(u,v)=f_1(x), & (1)\\ v_{tt}+M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)(-\Delta)^{2m} v+\beta(-\Delta)^{2m} v_t+g_2(u,v)=f_2(x), & (2)\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, & (3)\\ v(x,0)=v_0(x), v_t(x,0)=v_1(x), x\in\Omega, & (4)\\ \dfrac{\partial^i u}{\partial n^i}=0, \dfrac{\partial^i v}{\partial n^i}=0\ (i=0,1,2,\cdots,2m). & (5)\end{cases}$$

where Ω is a bounded region with a smooth boundary in R^n, $\partial\Omega$ represents the boundary of Ω, $u_0(x), u_1(x)$ and $v_0(x), v_1(x)$ are known functions, where $g_j(u,v), f_j(u,v)\,(j=1,2)$ are nonlinear terms and external interference terms, respectively, and are known functions on $\Omega\times(0,T)$, β is the normal number, $M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)$ is a non-negative first-order continuous derivative function, and $m>1$ is the normal number, $\left\|D^m u\right\|_p^p=\int_\Omega\left|D^m u\right|^p \mathrm{d}x$.

The innovation of this article is that the rigid term is changed from $M\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)$ to $M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)$, and we mainly make appropriate assumptions about this, and then use the Holder's inequality, the Young's inequality, the Poincare's inequality, interpolation inequality, and the Gronwall's inequality to obtain the required a priori estimate.

Recently, Yang Zhijian [1] studied the long-term behavior of kirchhoff-type equations with strong damping on R^n, demonstrating that the related continuous semigroup has a connected, fractal dimension and Hausdorff dimension of the global attractor in the equation

$$u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u-\Delta u_t+u+u_t+g(u,v)=f(x), \text{ in } R^N\times R^+.$$

At the same time, Yang Zhijian [2] processed a class of Kirchhoff-type global attractors and Hausdorff dimensions, and obtained the global attractors, regularities and Hausdorff-dimensional equations of the Kirchhoff type produced in a class of elastoplastic flows

$$u_{tt}-\operatorname{div}\left\{\sigma\left(|\nabla u|^2\right)\nabla u\right\}-\Delta u_t+\Delta^2 u+h(u_t)+g(u)=f(x), \text{ in } \Omega\times R^+.$$

In addition, Xiaoming Fan, and Shengfan Zhou [3] also demonstrated the presence of a tight-core section during the nonlinear vibration of the nonlinear elastic string in which the non-degraded Kirchhoff type strong damping wave equation simulates, and obtained an accurate estimate of the upper boundary of the Kirchhoff type of the kernel section in the equation

$$u_{tt}-\alpha\Delta u_t-\left(\beta+\gamma\left(\int_\Omega|\nabla u|^2\,\mathrm{d}x\right)^\rho\right)\Delta u+h(u_t)+f(u,t)=g(x,t), x\in\Omega, t>\tau.$$

In addition, Lin Guoguang and Gao Yunlong [4] studied the long-term behavior of a class of strongly damped high-order Kirchhoff-type equations for solving the initial edge value problem

$$u_{tt}+(-\Delta)^m u_t+\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q(-\Delta)^m u+g(u)=f(x),(x,t)\in\Omega\times[0,+\infty).$$

They used the Galerkin method to obtain the understanding of the uniqueness of existence, and based on the attractor theorem to obtain the existence of the global attractor at $H_0^m(\Omega)\times L^2(\Omega)$, and established an estimate of the Hausdorff dimension of the attractor.

In paper [5], Lin Guoguang and Zhou Chunmeng studied a class of high-order strong-damped Kirchhoff equations on the initial edge value problems

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+|u|^\rho u=f(x),\alpha>0, \\ u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega, \\ u(x,t)=\dfrac{\partial^j u}{\partial v^j},j=1,2,\cdots,2m-1,x\in\partial\Omega. \end{cases}$$

Wherein $m>0$ $p\geq 2$, $\Omega\subset R^n$ is a bounded region with a smooth boundary $\partial\Omega$, $\beta>0$ is a dissipation coefficient, $\beta(-\Delta)^{2m}u_t$ is a strong dissipation term, $|u|^\rho u$ is a nonlinear term, and $\rho\geq -1$, $f(x)$ is an external force interference term, when studying rigid term $M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)$, the method in the literature is referred to.

Guoguang Lin and Lingjuan Hu [6] studied a class of nonlinearly coupled Kirchhoff equations with strong damping

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g_1(u,v)=f_1(x), \\ v_{tt}+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v)=f_2(x), \\ u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega, \\ v(x,0)=v_0(x),v_t(x,0)=v_1(x),x\in\Omega, \\ \dfrac{\partial^i u}{\partial n^i}=0,\dfrac{\partial^i v}{\partial n^i}=0 \quad (i=0,1,2,\cdots,2m-1) \quad x\in\partial\Omega. \end{cases}$$

where Ω is a bounded region with a smooth boundary in R^n, $\partial\Omega$ represents the boundary of Ω, $g_j(u,v)\,(j=1,2)$ is a nonlinear source term, $f_1(x), f_2(x)$ is an external force interference term, and $\beta(-\Delta)^m u$, $\beta(-\Delta)^m v\,(\beta\geq 0)$ is a strong dissipation terms.

More research on the Kirchhoff equations see [7]-[13].

Using the Rellich-Kondrachov compact embedding theorem, it is obtained that the solution semigroup $S(t)$ generated by the Kirchhoff equation has a family of global attractors in space $E_k=V_{2m+k}\times V_k\times V_{2m+k}\times V_k\,(k=0,1,\cdots,2m)$; then proves that the solution semigroup $S(t)$ has Fréchet differentiability on space E_k; Dimensional estimation of the family of global attractors yields that both the Hausdorff and Fractal dimensions are finite, and that the Fractal dimension does not exceed twice the Hausdorff dimension.

2. The Existence and Uniqueness of Global Solution

For narrative convenience, we introduce the following symbols and assumptions:

Set $\nabla=D$. Consider the Hilbert space $V_\alpha=D\left((-\Delta)^{\alpha/2}\right),\alpha\in R$, whose inner product and norm are $(\bullet,\bullet)_{V_\alpha}=\left((-\Delta)^{\alpha/2},(-\Delta)^{\alpha/2}\right)$ and $\|\bullet\|_{V_\alpha}=\left\|(-\Delta)^{\alpha/2}\right\|$, respectively. Apparently

$$V_0=L^2(\Omega),V_{2m}=H^{2m}(\Omega)\cap H_0^1(\Omega),V_{2m+k}=H^{2m+k}(\Omega)\cap H_0^1(\Omega),$$
$$V_k=H^k(\Omega)\cap H_0^1(\Omega),E_0=V_{2m}\times V_0\times V_{2m}\times V_0,$$
$$E_k=V_{2m+k}\times V_k\times V_{2m+k}\times V_k,(k=1,2,\cdots,2m).$$

The assumption is as follows:

(H1) Let $M(s)$ be a continuous function on interval $D_1(D_1 \in \Omega)$, and $M(s) \in C^1(R^+)$:

1) $1 \le \mu_0 \le M(s) \le \mu_1$, set $M(s) = M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)$.

2)

$$\begin{cases} \text{When } \frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2 \ge 0, \frac{1}{2}M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2 \ge \frac{1}{2}\mu_0\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2, \\ \text{When } \frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2 < 0, \frac{1}{2}M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2 \ge \frac{1}{2}\mu_1\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2. \end{cases}$$

3)

$$\begin{cases} \text{When } \frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^{2m}u\|^2 \ge 0, \frac{1}{2}M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^{2m}u\|^2 \ge \frac{1}{2}\mu_0\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^{2m}u\|^2, \\ \text{When } \frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^{2m}u\|^2 < 0, \frac{1}{2}M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^{2m}u\|^2 \ge \frac{1}{2}\mu_1\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^{2m}u\|^2. \end{cases}$$

(H2) For any $u, v \in V$, $J(u,v) = \int_\Omega \left[G_1(u,v) + G_2(u,v)\right]\mathrm{d}x$,

$G_1(u,v) = \int_0^u g_1(\varsigma, v)\mathrm{d}\varsigma$, $G_2(u,v) = \int_0^v g_2(u,\eta)\mathrm{d}\eta$.

Then for any $\mu \ge 0$, the existence of $c > 0$, $c_\mu \ge 0$, $c'_\mu \ge 0$, makes

$$\left(g_1(u,v),u\right) + \left(g_2(u,v),v\right) - cJ(u,v) + \mu\left(\|D^m u\|^2 + \|D^m v\|^2\right) \ge -c_\mu;$$

$$2J(u,v) + \varepsilon^2\left(\|u\|^2 + \|v\|^2\right) \ge -2c'_\mu.$$

(H3) $g_j(u,v)(j=1,2) \in C^1(R)$ is a differentiable non-subtractive function that makes

$$|g_j(u,v)| \le c\left(1 + |u|^{r_j} + |v|^{r_j}\right),$$
$$|g_{ju}(u,v)| \le c\left(1 + |u|^{r_j - 1} + |v|^{r_j}\right),$$
$$|g_{jv}(u,v)| \le c\left(1 + |u|^{r_j} + |v|^{r_j - 1}\right).$$

(H4) Let $0 < \kappa_1, \kappa_2 < 1$, for E_k, there are constants $l_k = l_k(E_k)$, $l'_k = l'_k(E_k)$, such that

$$\|g_{1i}(\tilde{u},\tilde{v}) - g_{1i}(u,v)\| \le l_k \|(\tilde{u},\tilde{v}) - (u,v)\|_{V_{2m+k}\times V_{2m+k}}^{\kappa_1},$$
$$\|g_{2i}(\tilde{u},\tilde{v}) - g_{2i}(u,v)\| \le l'_k \|(\tilde{u},\tilde{v}) - (u,v)\|_{V_{2m+k}\times V_{2m+k}}^{\kappa_2}, \ (i = u, v),$$
$$\forall (\tilde{u},\tilde{v}),(u,v) \in V_{2m+k}\times V_{2m+k}, \ \|(\tilde{u},\tilde{v})\|_{V_{2m+k}\times V_{2m+k}} \le c(E_k),$$
$$\|(u,v)\|_{V_{2m+k}\times V_{2m+k}} \le c(E_k).$$

where $\|(\tilde{u},\tilde{v}) - (u,v)\|_{V_{2m+k}\times V_{2m+k}}^l = \|D^{2m+k}(\tilde{u} - u)\|^l + \|D^{2m+k}(\tilde{v} - v)\|^l$.

Make a priori estimates as following:

Lemma 1 Assumes that (H1) - (H2) holds, and $(u_0, z_0, v_0, q_0) \in E_0$, $f_1(x), f_2(x) \in L^2(\Omega)$, then Equations (1)-(5) have solutions (u,z,v,q) and have the following properties

1) $(u,z,v,q) \in L^\infty\left((0,+\infty); E_0\right)$;

2) $E_0(t) \le \frac{1}{k_1}\left(\frac{2C_\mu}{C} + \frac{C}{2\varepsilon_0^2}\left(\|f_1\|^2 + \|f_2\|^2\right)\right)$,

where $E_0(t) = \left\|D^{2m}u\right\|^2 + \|z\|^2 + \left\|D^{2m}v\right\|^2 + \|q\|^2 \le c(R_0)$.

3) There are positive constants $c(R_0)$ and $t_0 = t_0(\Omega) > 0$, such that

$$\|(u,z,v,q)\|_{E_0}^2 = \left\|D^{2m}u\right\|^2 + \|z\|^2 + \left\|D^{2m}v\right\|^2 + \|q\|^2 \le c(R_0).$$

Proof: $z = u_t + \varepsilon u$ and the Equation (1) as the inner product, that is

$$\left(u_{tt} + M\left(\left\|D^m u\right\|_p^p + \left\|D^m v\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g_1(u,v), z\right) = \left(f_1(x), z\right).$$

where $(u_{tt}, z) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 - \varepsilon\|z\|^2 + \varepsilon^3\|u\|^2 + \frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2$;

$$\begin{aligned}
&\left(M\left(\left\|D^m u\right\|_p^p + \left\|D^m v\right\|_p^p\right)(-\Delta)^{2m} u, z\right) \\
&= M\left(\left\|D^m u\right\|_p^p + \left\|D^m v\right\|_p^p\right)\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2 + \varepsilon\left\|D^{2m}u\right\|^2\right) \\
&\ge \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2 + \varepsilon\mu_0\left\|D^{2m}u\right\|^2;
\end{aligned}$$

$$\begin{aligned}
\left(\beta(-\Delta)^{2m} u_t, z\right) &= \beta\left(D^{2m}u_t, D^{2m}z\right) \\
&= \beta\left\|D^{2m}z\right\|^2 - \varepsilon\beta\left(D^{2m}u, D^{2m}z\right) \\
&\ge \frac{\beta}{2}\left\|D^{2m}z\right\|^2 - \frac{\beta\varepsilon^2}{2}\left\|D^{2m}u\right\|^2;
\end{aligned}$$

$$\left(g_1(u,v), z\right) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^u g_1(\varsigma, v)\,\mathrm{d}\varsigma\,\mathrm{d}x + \varepsilon\left(g_1(u,v), u\right).$$

Similarly, $q = v_t + \varepsilon v$ and the Equation (2) as the inner product, that is

$$\left(v_{tt} + M\left(\left\|D^m u\right\|_p^p + \left\|D^m v\right\|_p^p\right)(-\Delta)^{2m} v + \beta(-\Delta)^{2m} v_t + g_2(u,v), q\right) = \left(f_2(x), q\right).$$

where,

$$(v_{tt}, q) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 - \varepsilon\|q\|^2 + \varepsilon^3\|v\|^2 + \frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2;$$

$$\begin{aligned}
&\left(M\left(\left\|D^m u\right\|_p^p + \left\|D^m v\right\|_p^p\right)(-\Delta)^{2m} v, q\right) \\
&= M\left(\left\|D^m u\right\|_p^p + \left\|D^m v\right\|_p^p\right)\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}v\right\|^2 + \varepsilon\left\|D^{2m}v\right\|^2\right) \\
&\ge \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}v\right\|^2 + \varepsilon\mu_0\left\|D^{2m}v\right\|^2;
\end{aligned}$$

$$\begin{aligned}
\left(\beta(-\Delta)^{2m} v_t, q\right) &= \beta\left(D^{2m}v_t, D^{2m}q\right) \\
&= \beta\left\|D^{2m}q\right\|^2 - \varepsilon\beta\left(D^{2m}v, D^{2m}q\right) \\
&\ge \frac{\beta}{2}\left\|D^{2m}q\right\|^2 - \frac{\beta\varepsilon^2}{2}\left\|D^{2m}v\right\|^2;
\end{aligned}$$

$$\left(g_2(u,v), q\right) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^v g_2(u, \eta)\,\mathrm{d}\eta\,\mathrm{d}x + \varepsilon\left(g_2(u,v), v\right).$$

The above result and other product terms are collated, using the Holder's inequality, the Young's inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2-\varepsilon\|z\|^2+\varepsilon^3\|u\|^2+\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2-\varepsilon\|q\|^2+\varepsilon^3\|v\|^2+\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2$$
$$+\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2+\varepsilon\mu_0\left\|D^{2m}u\right\|^2+\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}v\right\|^2+\varepsilon\mu_0\left\|D^{2m}v\right\|^2$$
$$+\frac{\beta}{2}\left\|D^{2m}z\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|D^{2m}u\right\|^2+\frac{\beta}{2}\left\|D^{2m}q\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|D^{2m}v\right\|^2$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^u g_1(\varsigma,v)\,\mathrm{d}\varsigma\mathrm{d}x+\varepsilon\left(g_1(u,v),u\right)+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^v g_2(u,\eta)\,\mathrm{d}\eta\mathrm{d}x+\varepsilon\left(g_2(u,v),v\right)$$
$$\le(f_1,z)+(f_2,q)\le\frac{1}{4\varepsilon}\left(\|f_1\|^2+\|f_2\|^2\right)+\varepsilon\left(\|z\|^2+\|q\|^2\right).$$

Again by hypothesis (H2), obtained

$$-\varepsilon\left(g_1(u,v),u\right)-\varepsilon\left(g_2(u,v),v\right)\le-\varepsilon cJ(u,v)+\varepsilon\mu\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)+\varepsilon c_\mu$$
$$\le-\varepsilon cJ(u,v)+\varepsilon\mu c_*^2\left(\left\|D^{2m}u\right\|^2+\left\|D^{2m}v\right\|^2\right)+\varepsilon c_\mu.$$

Reuse the Poincare's inequality to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\left(\left\|D^{2m}u\right\|^2+\left\|D^{2m}v\right\|^2\right)+\|z\|^2+\|q\|^2+2J(u,v)+\varepsilon^2\left(\|u\|^2+\|v\|^2\right)\right)$$
$$+2\varepsilon\left(\mu_0-\frac{\beta\varepsilon}{2}-\mu c_*^2\right)\left(\left\|D^{2m}u\right\|^2+\left\|D^{2m}v\right\|^2\right)+\left(\frac{\beta}{c_*^2}-4\varepsilon\right)\left(\|z\|^2+\|q\|^2\right)$$
$$+2\varepsilon cJ(u,v)+2\varepsilon^3\left(\|u\|^2+\|v\|^2\right)$$
$$\le 2\varepsilon c_\mu+\frac{1}{2\varepsilon}\left(\|f_1\|^2+\|f_2\|^2\right).$$

Again by hypothesis (H2), obtained $2J(u,v)+\varepsilon^2\left(\|u\|^2+\|v\|^2\right)+2c'_\mu\ge 0$,
At this point, Order

$$\bar{y}(t)=\mu\left(\left\|D^{2m}u\right\|^2+\left\|D^{2m}v\right\|^2\right)+\|z\|^2+\|q\|^2+2J(u,v)+\varepsilon^2\left(\|u\|^2+\|v\|^2\right).$$

Then there is a constant of $k_1=\min\{1,\mu\}$, such that $\bar{y}(t)+2c'_\mu\ge k_1E_0(t)\ge 0$, of which, $E_0(t)=\left\|D^{2m}u\right\|^2+\|z\|^2+\left\|D^{2m}v\right\|^2+\|q\|^2$. Since $c>0$,

Then Order $\varepsilon_0=\min\left\{\frac{2\varepsilon}{\mu}\left(\mu_0-\frac{\beta\varepsilon}{2}-\mu c_*^2\right),\frac{\beta}{c_*^2}-4\varepsilon,c\varepsilon,2\varepsilon\right\}$.

Here to denote $y(t)=\bar{y}(t)+2c'_\mu$, using the Gronwall's inequality, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+\varepsilon_0 y(t)\le 2\varepsilon c_\mu+\frac{1}{2\varepsilon}\left(\|f_1\|^2+\|f_2\|^2\right),$$
$$y(t)\le y(0)\mathrm{e}^{-\varepsilon_0 t}+\frac{2c_\mu}{c}+\frac{c}{2\varepsilon_0^2}\left(\|f_1\|^2+\|f_2\|^2\right).$$

Therefore,

$$E_0(t)\le\frac{1}{k_1}\left(\bar{y}(t)+2c'_\mu\right)=\frac{y(t)}{k_1}=\frac{1}{k_1}y(0)\mathrm{e}^{-\varepsilon_0 t}+\frac{1}{k_1}\left(\frac{2c_\mu}{c}+\frac{c}{2\varepsilon_0^2}\left(\|f_1\|^2+\|f_2\|^2\right)\right).$$

Well, $E_0(t) \le \frac{1}{k_1}\left(\frac{2c_\mu}{c} + \frac{c}{2\varepsilon_0^2}\left(\|f_1\|^2 + \|f_2\|^2\right)\right)$, that is, there are positive constants $c(R_0)$ and $t_0 = t_0(\Omega) > 0$, such that for $t > t_0$,
$\|(u,z,v,q)\|_{E_0}^2 = \|D^{2m}u\|^2 + \|z\|^2 + \|D^{2m}v\|^2 + \|q\|^2 \le c(R_0)$.

Lemma 1 is proved.

Lemma 2 Assumes that (H1) - (H3) holds, and $f_1(x), f_2(x) \in H^k(\Omega)$, $|(f_1,u) + (f_2,v)| \le 2c$, then Equations (1)-(5) have solutions (u,z,v,q) and have the following properties

1) $(u,z,v,q) \in L^\infty((0,+\infty); E_k)$;

2) $E_k(t) \le y(0)\mathrm{e}^{-k_1 t} + \frac{c_3}{k_1}$;

3) There are positive constants $c(E_k)$ and $t_k = t_k(\Omega) > 0$, such that

$$\|(u,z,v,q)\|_{E_k}^2 = \|D^{2m+k}u\|^2 + \|D^k z\|^2 + \|D^{2m+k}v\|^2 + \|D^k q\|^2 \le c(E_k).$$

Prove: $(-\Delta)^k z$ and the Equation (1) as the inner product, that is

$$\left(u_{tt} + M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g_1(u,v), (-\Delta)^k z\right)$$
$$= \left(f_1(x), (-\Delta)^k z\right).$$

where, $\left(u_{tt}, (-\Delta)^k z\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^k z\|^2 - \varepsilon\|D^k z\|^2 + \varepsilon^3\|D^k u\|^2 + \frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^k u\|^2$;

$$\left(M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)(-\Delta)^{2m} u, (-\Delta)^k z\right)$$
$$= M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m+k}u\|^2 + \varepsilon\|D^{2m+k}u\|^2\right)$$
$$\ge \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m+k}u\|^2 + \varepsilon\mu_0\|D^{2m+k}u\|^2;$$

$$\left(\beta(-\Delta)^{2m} u_t, (-\Delta)^k z\right) = \beta\left(\Delta^{2m} u_t, \Delta^k z\right)$$
$$= \beta\|D^{2m+k}z\|^2 - \varepsilon\beta\left((-\Delta)^{2m} u, (-\Delta)^k z\right)$$
$$\ge \frac{\beta}{2}\|D^{2m+k}z\|^2 - \frac{\beta\varepsilon^2}{2}\|D^{2m+k}u\|^2.$$

Similarly, $(-\Delta)^k q$ and the Equation (2) as the inner product, that is

$$\left(v_{tt} + M\left(\|D^m u\|_p^p + \|D^m v\|_p^p\right)(-\Delta)^{2m} v + \beta(-\Delta)^{2m} v_t + g_2(u,v), (-\Delta)^k q\right)$$
$$= \left(f_2(x), (-\Delta)^k q\right).$$

where,

$$\left(v_{tt}, (-\Delta)^k q\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^k q\|^2 - \varepsilon\|D^k q\|^2 + \varepsilon^3\|D^k v\|^2 + \frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^k v\|^2;$$

$$\begin{aligned}&\left(M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)(-\Delta)^{2m} v,(-\Delta)^k q\right)\\&=M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}v\right\|^2+\varepsilon\left\|D^{2m+k}v\right\|^2\right)\\&\geq\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}v\right\|^2+\varepsilon\mu_0\left\|D^{2m+k}v\right\|^2;\end{aligned}$$

$$\begin{aligned}\left(\beta(-\Delta)^{2m} v_t,(-\Delta)^k q\right)&=\beta\left(\Delta^{2m}v_t,\Delta^k q\right)\\&=\beta\left\|D^{2m+k}q\right\|^2-\varepsilon\beta\left((-\Delta)^{2m} v,(-\Delta)^k q\right)\\&\geq\frac{\beta}{2}\left\|D^{2m+k}q\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|D^{2m+k}v\right\|^2.\end{aligned}$$

The above result and other internal product terms are sorted out

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k z\right\|^2-\varepsilon\left\|D^k z\right\|^2+\varepsilon^3\left\|D^k u\right\|^2+\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k u\right\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k q\right\|^2-\varepsilon\left\|D^k q\right\|^2\\&+\varepsilon^3\left\|D^k v\right\|^2+\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k v\right\|^2+\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\varepsilon\mu_0\left\|D^{2m+k}u\right\|^2+\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}v\right\|^2\\&+\varepsilon\mu_0\left\|D^{2m+k}v\right\|^2+\frac{\beta}{2}\left\|D^{2m+k}z\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|D^{2m+k}u\right\|^2+\frac{\beta}{2}\left\|D^{2m+k}q\right\|^2\\&-\frac{\beta\varepsilon^2}{2}\left\|D^{2m+k}v\right\|^2+\left(g_1(u,v),(-\Delta)^k z\right)+\left(g_2(u,v),(-\Delta)^k q\right)\\&\leq\left(f_1(x),(-\Delta)^k z\right)+\left(f_2(x),(-\Delta)^k q\right)=\left(D^k f_1,D^k z\right)+\left(D^k f_2,D^k q\right).\end{aligned}$$

Again by hypothesis (H3), obtained

$$\begin{aligned}\left\|g_1(u,v)\right\|^2&=\int_\Omega\left|g_1(u,v)\right|^2\mathrm{d}x\leq\int_\Omega\left|c\left(1+|u|^{r_1}+|v|^{r_1}\right)\right|^2\mathrm{d}x\\&=c\int_\Omega\left|\left(1+2|u|^{r_1}+2|v|^{r_1}+|u|^{2r_1}+|v|^{2r_1}+2|u|^{r_1}|v|^{r_1}\right)\right|\mathrm{d}x\\&\leq c\int_\Omega\left(1+1+|u|^{2r_1}+1+|v|^{2r_1}+2|u|^{2r_1}+2|v|^{2r_1}\right)\mathrm{d}x\\&=3c\int_\Omega\left(1+|u|^{2r_1}+|v|^{2r_1}\right)\mathrm{d}x\leq3c\left(1+\|u\|_{2r_1}^{2r_1}+\|v\|_{2r_1}^{2r_1}\right).\end{aligned}$$

Similarly, $\left\|g_2(u,v)\right\|^2\leq3c\left(1+\|u\|_{2r_2}^{2r_2}+\|v\|_{2r_2}^{2r_2}\right)$.

By interpolation inequality, there is $\|u\|_{2r_j}\leq c\left\|D^{2m}u\right\|^{\frac{n(r_j-1)}{4mr_j}}\|u\|^{\frac{4mr_j-n(r_j-1)}{4mr_j}}$,

This can be concluded $\begin{cases}\|u\|_{2r_j}^{2r_j}\leq c\left\|D^{2m}u\right\|^{\frac{n(r_j-1)}{2m}}\|u\|^{\frac{4mr_j-n(r_j-1)}{2m}}\\\|v\|_{2r_j}^{2r_j}\leq c\left\|D^{2m}v\right\|^{\frac{n(r_j-1)}{2m}}\|v\|^{\frac{4mr_j-n(r_j-1)}{2m}}\end{cases}$

where, $0<r_j\leq2n/[n-4m]^+\left(n=4m,0<r_j<+\infty\right)$,

Therefore, $\left\|g_1(u,v)\right\|^2+\left\|g_2(u,v)\right\|^2\leq c_1$.

Reuse the Young's inequality and the Sobolev-Poincare's inequality, thus

$$\left(g_1(u,v),(-\Delta)^k z\right)+\left(g_2(u,v),(-\Delta)^k q\right)\geq-\frac{c_1}{2\varepsilon}-\frac{\varepsilon}{2}\left(\left\|(-\Delta)^k z\right\|^2+\left\|(-\Delta)^k q\right\|^2\right),$$

$$\frac{\beta-\varepsilon}{2}\left(\left\|(-\Delta)^{k} z\right\|^{2}+\left\|(-\Delta)^{k} q\right\|^{2}\right) \geq \frac{\beta-\varepsilon}{2 c_{*}^{2}}\left(\left\|D^{k} z\right\|^{2}+\left\|D^{k} q\right\|^{2}\right).$$

In summary, according to the Holder's inequality, the Young's inequality is obtained

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\left(\left\|D^{2m+k} u\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}\right)+\left\|D^{k} z\right\|^{2}+\left\|D^{k} q\right\|^{2}+\varepsilon^{2}\left(\left\|D^{k} u\right\|^{2}+\left\|D^{k} v\right\|^{2}\right)\right) \\
&+2 \varepsilon\left(\mu_{0}-\frac{\beta \varepsilon}{2}\right)\left(\left\|D^{2m+k} u\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}\right)+\left(\frac{\beta-\varepsilon}{c_{*}^{2}}\right)\left(\left\|D^{k} z\right\|^{2}+\left\|D^{k} q\right\|^{2}\right) \\
&+2 \varepsilon^{3}\left(\left\|D^{k} u\right\|^{2}+\left\|D^{k} v\right\|^{2}\right) \\
&\leq \frac{c_{1}}{\varepsilon}+2\left[\left(D^{k} f_{1}, D^{k} z\right)+\left(D^{k} f_{2}, D^{k} q\right)\right],
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\left(\left\|D^{2m+k} u\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}\right)+\left\|D^{k} z\right\|^{2}+\left\|D^{k} q\right\|^{2}+\varepsilon^{2}\left(\left\|D^{k} u\right\|^{2}+\left\|D^{k} v\right\|^{2}\right)\right) \\
&+2 \varepsilon\left(\mu_{0}-\frac{\beta \varepsilon}{2}\right)\left(\left\|D^{2m+k} u\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}\right)+\left(\frac{\beta-\varepsilon}{c_{*}^{2}}-4 \varepsilon\right)\left(\left\|D^{k} z\right\|^{2}+\left\|D^{k} q\right\|^{2}\right) \\
&+2 \varepsilon^{3}\left(\left\|D^{k} u\right\|^{2}+\left\|D^{k} v\right\|^{2}\right) \\
&\leq \frac{c_{1}}{\varepsilon}+\frac{1}{2 \varepsilon}\left(\left\|D^{k} f_{1}\right\|^{2}+\left\|D^{k} f_{2}\right\|^{2}\right)
\end{aligned}$$

At this point, Order

$$y'(t)=\mu\left(\left\|D^{2m+k} u\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}\right)+\left\|D^{k} z\right\|^{2}+\left\|D^{k} q\right\|^{2}+\varepsilon^{2}\left(\left\|D^{k} u\right\|^{2}+\left\|D^{k} v\right\|^{2}\right),$$

So there are $\frac{\mathrm{d}}{\mathrm{d}t} y'(t)+k_{1} y'(t) \leq c_{3}$, $k_{1}=\min \left\{\frac{2 \varepsilon}{\mu}\left(\mu_{0}-\frac{\beta \varepsilon}{2}\right), \frac{\beta-\varepsilon}{c_{*}^{2}}-4 \varepsilon, 2 \varepsilon\right\}$,

Reuse the Gronwall's inequality,

$$y'(t) \leq y'(0) \mathrm{e}^{-k_{1} t}+\frac{c_{3}}{k_{1}}, t \geq 0.$$

$$\begin{aligned}
E_{k}(t)&=\left\|D^{2m+k} u\right\|^{2}+\left\|D^{k} z\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}+\left\|D^{k} q\right\|^{2} \\
&\leq \mu\left(\left\|D^{2m+k} u\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}\right)+\left\|D^{k} z\right\|^{2}+\left\|D^{k} q\right\|^{2} \\
&\leq \mu\left(\left\|D^{2m+k} u\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}\right)+\left\|D^{k} z\right\|^{2}+\left\|D^{k} q\right\|^{2}+\varepsilon^{2}\left(\left\|D^{k} u\right\|^{2}+\left\|D^{k} v\right\|^{2}\right) \\
&=y'(t) \leq y'(0) \mathrm{e}^{-k_{1} t}+\frac{c_{3}}{k_{1}}.
\end{aligned}$$

that is $E_{k}(t) \leq \frac{c_{3}}{k_{1}}$.

Thus, there is a positive constant, there is a positive constant $c(E_k)$ and $t_k = t_k(\Omega) > 0$, so that for $t > t_k$, there are

$$\|(u, z, v, q)\|_{E_{k}}^{2}=\left\|D^{2m+k} u\right\|^{2}+\left\|D^{k} z\right\|^{2}+\left\|D^{2m+k} v\right\|^{2}+\left\|D^{k} q\right\|^{2} \leq c\left(E_{k}\right).$$

Lemma 2 is proved.

Theorem 1 (The existence and uniqueness of solution) Suppose (H1) -

(H3) holds, and $(u_0, z_0, v_0, q_0) \in E_k$, $f_1(x), f_2(x) \in H^k(\Omega)$, then Equations (1) - (5) have an unique solution

$$(u(x,t), z(x,t), v(x,t), q(x,t)) \in L^\infty((0,+\infty); E_k).$$

Proof: Using the Galerkin method, combining lemmas 1 and 2, where the first a priori estimation has been proved; the second step: approximate solution.

We can take sequences $w_1, w_2, \cdots, w_r, \cdots, w_i \in H_0^{2m+k}(\Omega) \cap L^2(\Omega)(\forall i)$, $\forall r$, $w_1, w_2, \cdots, w_r$, $w_i = (u_i, v_i)$, where the linear combination of w_i is dense in $H_0^{2m+k}(\Omega) \cap L^2(\Omega)$, so w_j represents the eigenvalue function corresponding to the eigenvalue, and $w_1, w_2, \cdots, w_r$ is the standard orthogonal basis that constitutes H^{2m+k}; and λ_j is the eigenvalue of $(-\Delta)$ with a homogeneous Dirichlet boundary condition on Ω, then there is

$(-\Delta)^{2m+k} w_j = \lambda_j^{2m+k} w_j (k = 0,1,2,\cdots,2m)$.

Set the approximate solution of the initial edge value problem (1)-(5), $u_r = u_r(t) = \sum_{j=1}^{r} g_{jr}(t) w_j$, $v_r = v_r(t) = \sum_{j=1}^{r} f_{jr}(t) w_j$.

Easy (u_r, v_r) is dense in $H^{2m+k} \times H^{2m+k}$ and satisfies the following conditions

$$\begin{cases} (u_r''(t), u_j) + \left(M\left(\|D^m u_r(t)\|_p^p + \|D^m v_r(t)\|_p^p\right)(-\Delta)^{2m} u_r(t), u_j\right) \\ +\left(\beta(-\Delta)^{2m} u_r'(t), u_j\right) + \left(g(u_r'(t), v_r'(t)), u_j\right) = (f_1(x), u_j), 1 \le j \le r, \\ (v_r''(t), v_j) + \left(M\left(\|D^m u_r(t)\|_p^p + \|D^m v_r(t)\|_p^p\right)(-\Delta))^{2m} v_r(t), v_j\right) \\ +\left(\beta(-\Delta)^{2m} v_r'(t), v_j\right) + \left(g(u_r'(t), v_r'(t)), v_j\right) = (f_2(x), v_j), 1 \le j \le r. \end{cases} \tag{6}$$

And the above nonlinear system of ordinary differential Equations (6) satisfies the initial conditions:

$$u_r(0) = u_{0r} = \sum_{j=1}^{r} (u_0, w_j) w_j \to u_0, (r \to \infty), \text{ in } H_0^{2m+k}(\Omega) \cap L^2(\Omega) \tag{7}$$

$$u_r'(0) = u_{1r} = \sum_{j=1}^{r} (u_1, w_j) w_j \to u_1, (r \to \infty), \text{ in } H_0^{k}(\Omega) \cap L^2(\Omega) \tag{8}$$

$$v_r(0) = v_{0r} = \sum_{j=1}^{r} (v_0, w_j) w_j \to v_0, (r \to \infty), \text{ in } H_0^{2m+k}(\Omega) \cap L^2(\Omega) \tag{9}$$

$$v_r'(0) = v_{1r} = \sum_{j=1}^{r} (v_1, w_j) w_j \to v_1, (r \to \infty) \text{ in } H_0^{k}(\Omega) \cap L^2(\Omega) \tag{10}$$

The general conclusion of the system of nonlinear ordinary differential equations is easy to know, which ensures that the approximate solution of the problem (6)-(10) exists on the interval $[0, t_r]$.

Known $z = u_t + \varepsilon u$, $q = v_t + \varepsilon v$, binding lemma 1, lemma 2, in space E_k, and we can pick subsequence $\{u_s\}$ from sequence $\{u_h\}$ and subsequence $\{v_s\}$ from sequence $\{v_h\}$, such that $(u_s, z_s, v_s, q_s) \to (u, z, v, q)$ is weak * convergence in

$$L^{\infty}\left(\left[0,+\infty\right);E_k\right). \tag{11}$$

and z_r, q_r is bounded on $L^2\left(\left(0,T\right);E_0^{2m+k}\right)$.

By the Rellich-Kondrachov compact embedding theorem, E_k is compactly embedding in E_0, $\left(u_s,z_s,v_s,q_s\right)\to\left(u,z,v,q\right)$ is strong convergence almost everywhere.

This can be obtained from the above assumptions and lemmas

$$M\left(\left\|D^m u_m\left(t\right)\right\|_p^p+\left\|D^m v_m\left(t\right)\right\|_p^p\right)\left(-\Delta\right)^{2m}u_r\left(t\right) \to M\left(\left\|D^m u_m\left(t\right)\right\|_p^p+\left\|D^m v_m\left(t\right)\right\|_p^p\right)\left(-\Delta\right)^{2m}u\left(t\right)$$ weak converges in

$L^{\infty}\left(0,T;H_0^{2m+k}\left(\Omega\right)\right)$, and $\beta\left(-\Delta\right)^{2m}u_r'\left(t\right)\to\beta\left(-\Delta\right)^{2m}u'\left(t\right)$ weak converges in $L^{\infty}\left(0,T;H_0^{2m+k}\left(\Omega\right)\right)$, $g\left(u_r\left(t\right),v_r\left(t\right)\right)\to g\left(u\left(t\right),v\left(t\right)\right)$ weak converges in $L^{\infty}\left(0,T;H_0^{2m+k}\left(\Omega\right)\right)$.

Thus it is possible to take $r=\mu$ in (1), (2), and take the limit. To the fixed *j* and $\mu\geq j$, get

$$\left(u_\mu'',u_j\right)+\left(M\left(\left\|D^m u_r\left(t\right)\right\|_p^p+\left\|D^m v_r\left(t\right)\right\|_p^p\right)\left(-\Delta\right)^{2m}u_\mu,u_j\right)+\left(\beta\left(-\Delta\right)^{2m}u_\mu',u_j\right)$$
$$+\left(g\left(u\left(t\right),v\left(t\right)\right),u_j\right)=\left(f_1\left(x\right),u_j\right).$$

it satisfies all *j*, and thus for $\forall u\in L^{\infty}\left(0,T;H_0^{2m+k}\left(\Omega\right)\cap L^2\left(\Omega\right)\right)$,

$$\left(v_\mu'',v_j\right)+\left(M\left(\left\|D^m u_r\left(t\right)\right\|_p^p+\left\|D^m v_r\left(t\right)\right\|_p^p\right)\left(-\Delta\right)^{2m}v_\mu,v_j\right)+\left(\beta\left(-\Delta\right)^{2m}v_\mu',v_j\right)$$
$$+\left(g\left(u\left(t\right),v\left(t\right)\right),v_j\right)=\left(f_2\left(x\right),v_j\right).$$

it satisfies all *j*, and thus for $\forall u\in L^{\infty}\left(0,T;H_0^{2m+k}\left(\Omega\right)\cap L^2\left(\Omega\right)\right)$.

It is easy to obtain that the system of Equations (1)-(5) exists

$$\begin{cases} u_{tt}+M\left(s_1\right)\left(-\Delta\right)^{2m}u_1-M\left(s_2\right)\left(-\Delta\right)^{2m}u_2+\beta\left(-\Delta\right)^{2m}u_t+g_1\left(u_1,v_1\right)-g_1\left(u_2,v_2\right)=0 \\ v_{tt}+M\left(s_1\right)\left(-\Delta\right)^{2m}v_1-M\left(s_2\right)\left(-\Delta\right)^{2m}v_2+\beta\left(-\Delta\right)^{2m}v_t+g_2\left(u_1,v_1\right)-g_2\left(u_2,v_2\right)=0 \end{cases} \tag{12}$$

where,

$$M\left(s_1\right)=M\left(\left\|D^m u_1\right\|_p^p+\left\|D^m v_1\right\|_p^p\right),M\left(s_2\right)=M\left(\left\|D^m u_2\right\|_p^p+\left\|D^m v_2\right\|_p^p\right),$$
$$u=u_1-u_2,v=v_1-v_2.$$

Use u_t,v_t and Equation (12) as the inner product in turn, and get it

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|u_t\right\|^2+\left\|v_t\right\|^2\right)+\beta\left(\left\|D^{2m}u_t\right\|^2+\left\|D^{2m}v_t\right\|^2\right) \\ &+\left(M\left(s_1\right)\left(-\Delta\right)^{2m}u_1-M\left(s_2\right)\left(-\Delta\right)^{2m}u_2,u_t\right) \\ &+\left(M\left(s_1\right)\left(-\Delta\right)^{2m}v_1-M\left(s_2\right)\left(-\Delta\right)^{2m}v_2,v_t\right) \\ &+\left(g_1\left(u_1,v_1\right)-g_1\left(u_2,v_2\right),u_t\right)+\left(g_2\left(u_1,v_1\right)-g_2\left(u_2,v_2\right),v_t\right)=0. \end{aligned} \tag{13}$$

Using the Young's inequality and the Sobolev-Poincare's inequality, it is derived

$$\begin{aligned}&\left(M(s_1)(-\Delta)^{2m}u_1-M(s_2)(-\Delta)^{2m}u_2,u_t\right)\\&=\left(M(s_1)(-\Delta)^{2m}u+M(s_1)(-\Delta)^{2m}u_2-M(s_2)(-\Delta)^{2m}u_2,u_t\right)\\&\ge\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2+(\mu_0-\mu_1)\left((-\Delta)^{2m}u_2,u_t\right)\\&\ge\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2+c_4\|u_t\|.\end{aligned}\tag{14}$$

In summary

$$\left(M(s_1)(-\Delta)^{2m}v_1-M(s_2)(-\Delta)^{2m}v_2,v_t\right)\ge\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}v\right\|^2+c_4\|v_t\|\tag{15}$$

$$\begin{aligned}&\left|\left(g_1(u_1,v_1)-g_1(u_2,v_2),u_t\right)\right|\\&=\left|\left(g_1(u_1,v_1)-g_1(u_1,v_2)+g_1(u_1,v_2)-g_1(u_2,v_2),u_t\right)\right|\\&\le\left|\left(|v_1|^{r_1}-|v_2|^{r_1},u_t\right)\right|+\left|\left(|u_1|^{r_1}-|u_2|^{r_1},u_t\right)\right|\\&\le\left|\left(c\left(1+|v_1|^{r_1-1}+|v_2|^{r_1-1}\right)v,u_t\right)\right|+\left|\left(c\left(1+|u_1|^{r_1-1}+|u_2|^{r_1-1}\right)u,u_t\right)\right|\\&\le\left\|c\left(1+|v_1|^{r_1-1}+|v_2|^{r_1-1}\right)\right\|_\infty\|v\|\|u_t\|+\left\|c\left(1+|u_1|^{r_1-1}+|u_2|^{r_1-1}\right)\right\|_\infty\|u\|\|u_t\|\\&\le c_5\left(\|v\|^2+\|u\|^2+\frac{\|u_t\|^2}{2}\right)\le c_5\left(c_*^2\left\|D^{2m}v\right\|^2+c_*^2\left\|D^{2m}u\right\|^2+\frac{\|u_t\|^2}{2}\right).\end{aligned}\tag{16}$$

Similarly,

$$\left|\left(g_1(u_1,v_1)-g_1(u_2,v_2),u_t\right)\right|\le c_5\left(c_*^2\left\|D^{2m}v\right\|^2+c_*^2\left\|D^{2m}u\right\|^2+\frac{\|v_t\|^2}{2}\right)\tag{17}$$

Substituting (14)-(17) into Equation (13), combining lemmas 1 and 2, using the Poincare's inequality, to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2+\|v_t\|^2+\mu\left(\left\|D^{2m}u\right\|^2+\left\|D^{2m}v\right\|^2\right)\right)+\left(\frac{2\beta}{c_*^2}+2c_4+c_5\right)\left(\|u_t\|^2+\|v_t\|^2\right)\\&+4c_5c_*^2\left(\left\|D^{2m}u\right\|^2+\left\|D^{2m}v\right\|^2\right)\le 0.\end{aligned}$$

Order $y''(t)=\|u_t\|^2+\|v_t\|^2+\mu\left(\left\|D^{2m}u\right\|^2+\left\|D^{2m}v\right\|^2\right)$,

Then there are $\frac{\mathrm{d}}{\mathrm{d}t}y''(t)+k_4y''(t)\le 0$, $k_4=\min\left\{\frac{2\beta}{c_*^2}+2c_4+c_5,4c_5c_*^2\right\}$,

Using the Gronwall's inequality, get $y''(t)\le y''(0)\mathrm{e}^{-t}$, $t\ge 0$,

So $y''(t)=\|u_t\|^2+\|v_t\|^2+\mu\left(\left\|D^{2m}u\right\|^2+\left\|D^{2m}v\right\|^2\right)\equiv 0$.

That's $\|u_t\|^2=\|v_t\|^2=\left\|D^{2m}u\right\|^2=\left\|D^{2m}v\right\|^2=0$, hence $w(x,t)=(u(x,t),v(x,t))=(0,0)$.

Theorem 1 is proved.

3. The Family of Global Attractors and Dimension Estimation

Theorem 2 [7] Assume E is a Banach space, and $\{S(t)\}_{t\ge 0}$ is the operator semigroup on *E*,

$$S(t):E\to E\,,\ \ S(t+r)=S(t)+S(r)(\forall t,r>0)\,,\ \ S(0)=\boldsymbol{I}\,.$$

where $\boldsymbol{I}$ is the identity operator，if $S(t)$ satisfies

1) Semigroup $S(t)$ is uniformly bounded in *E*;

2) There exists a bounded absorbing set B_0 in *E*;

3) $\{S(t)\}_{t\ge 0}$ is completely continuous operator.

That is to say that operator semigroup $S(t)$ has compact global attractor *A*.

Where (1) means $\forall R>0$, exists a constant $C(R)$ such that when $\|u\|_E\le R$, there is $\|S(t)u\|_E\le c(R)$ $(\forall t\in[0,\infty))$, and (2) means for any bounded set $B\subset E$, there exists a constant $t_0>0$, such that $S(t)B\subset B_0$ $(\forall t\ge t_0)$. In theorem 2, if $S(t)$ is a solution semigroup generated by the initial boundary value problem (1)-(5), $(u(t),z(t),v(t),q(t))=S(t)(u_0,z_0,v_0,q_0)$, and Banach space E is changed into Hilbert space E_k, there will be family of global attractors.

Theorem 3 Let $S(t)$ is a solution semigroup generated by the initial boundary value problems (1)-(5), under the hypothesis of lemma 1 and lemma 2. Assuming that the existence and uniqueness of solution, then the equation has a global attraction subfamily. That is:

$$A_k\subset E_k\subset E_0\,(k=1,2,\cdots,2m)\text{, and }\ A_k=\omega(B_{0k})=\bigcap_{s\ge 0}\overline{\bigcup_{t\ge s}S(t)B_{0k}}\,.$$

where

$$B_{0k}=\left\{(u,z,v,q)\in E_k:\left\|D^{2m+k}u\right\|^2+\left\|D^kz\right\|^2+\left\|D^{2m+k}v\right\|^2+\left\|D^kq\right\|^2\le c(R_k)\right\},$$

1) Invariability: $S(t)A_k=A_k$;

2) Attractiveness: A_k attracts all bounded sets of E_k, that is, any bounded set $B_{0k}\subset E_k$, $dist(S(t)B_{0K},A_k)=\sup_{x\in B_{0K}}\inf_{y\in A_k}\|S(t)x-y\|_{E_k}\to 0\,(t\to 0)$.

Then compact set A_k is called family of global attractors of semigroup $S(t)$.

Proof: Verify theorem 2 to prove the existence of family of global attractors, under the condition of theorem 1, and the initial boundary value problems (1)-(5) generate solution semigroups $S(t):E_k\to E_k$.

1) So for any bounded set $B_{0k}\subset E_k$, having

$$\left\|S(t)(u_0,z_0,v_0,q_0)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^kz\right\|^2+\left\|D^{2m+k}v\right\|^2+\left\|D^kq\right\|^2\le C(R_k),$$

where $t\ge 0$ and $(u_0,z_0,v_0,q_0)\in B_k$, shows that $\{S(t)\}_{t\ge 0}$ is uniformly bounded in E_k;

2) $\forall(u_0,z_0,v_0,q_0)\in E_k$, when $t\ge\max\{t_0,t_{0k}\}$, there is $\left\|S(t)(u_0,z_0,v_0,q_0)\right\|_{E_k}^2\le C(R_k)$, thus B_k is a bounded absorption set of semigroup $S(t)$;

3) E_k is compactly embedded in E_0, *i.e.*, the bounded set in E_k is a compact set in E_0, so the operator semigroup $S(t)$ is completely continuous operator.

Theorem 2 is proved.

Since the solution semigroup $S(t)$ has a family of global attractors in space E_k, the dimensionality estimates of the global attractors subfamily are now made, and the resulting Hausdorff and Fractal dimensions are finite to prove. Linearize the problems (1)-(5) first, as follows:

$$\begin{cases} U_{tt}+M'\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)'\left(D^m U+D^m V\right)(-\Delta)^{2m}u \\ +M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)(-\Delta)^{2m}U+\beta(-\Delta)^{2m}U_t+g_{2u}(u,v)U+g_{2v}(u,v)V=0, \\ V_{tt}+M'\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)'\left(D^m U+D^m V\right)(-\Delta)^{2m}v \\ +M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)(-\Delta)^{2m}V+\beta(-\Delta)^{2m}V_t+g_{2u}(u,v)U+g_{2v}(u,v)V=0, \\ \left.\frac{\partial^i U}{\partial n^i}\right|_{\partial\Omega}=\left.\frac{\partial^i V}{\partial n^i}\right|_{\partial\Omega}=0, t\ge 0, \\ U(x,0)=\xi_1, U_t(x,0)=\xi_2, \\ V(x,0)=\eta_1, V_t(x,0)=\eta_2. \end{cases} \tag{18}$$

where $(\xi_1,\eta_1,\xi_2,\eta_2)\in E_k$, $(u_0,v_0,u_1,v_1)\in A_k$, $(u,v,u_t,v_t)=S(t)(u_0,v_0,u_1,v_1)$ is the solution to the problem (18) obtained by $(u_0,v_0,u_1,v_1)\in A_k$, Given $(u_0,v_0,u_1,v_1)\in A_k$, $S(t):E_k\to E_k$, it can be proved that for any $(\xi_1,\eta_1,\xi_2,\eta_2)\in E_k$, there is a unique solution to the linear initial edge value problem $\left(U(t),V(t),U_t(t),V_t(t)\right)\in L^\infty\left((0,+\infty);E_k\right)$.

Theorem 4 for the arbitrary $t>0$, $r>0$, map $S(t):E_k\to E_k$ is a Fractal differentiable. $\mathbf{\Phi}_0=(u_0,v_0,u_1,v_1)^{\mathrm{T}}$ the differential is a linear operator on $F:(\xi_1,\eta_1,\xi_2,\eta_2)^{\mathrm{T}}\to\left(U(t),V(t),U_t(t),V_t(t)\right)^{\mathrm{T}}$, where $\left(U(t),V(t),U_t(t),V_t(t)\right)$ is the solution to the problem (18).

Proof: Set $\mathbf{\Phi}_0=(u_0,v_0,u_1,v_1)^{\mathrm{T}}\in E_k$, there is $\bar{\mathbf{\Phi}}_0=(u_0+\xi_1,v_0+\eta_1,u_1+\xi_2,v_1+\eta_2)^{\mathrm{T}}\in E_k$, so $\|\mathbf{\Phi}_0\|_{E_k}\le r$, $\|\bar{\mathbf{\Phi}}_0\|_{E_k}\le r$.

Thus one obtains the Lipchitz property of $S(t)$ on bounded set E_k, hence

$$\left\|S(t)\mathbf{\Phi}_0-S(t)\bar{\mathbf{\Phi}}_0\right\|_{E_k}^2\le \mathrm{e}^{ct}\left\|(\xi_1,\eta_1,\xi_2,\eta_2)\right\|_{E_k}^2.$$

Let $\sigma=\bar{u}-u-U, H=\bar{v}-v-V$ be the solution to the problem, then

$$\begin{cases} \sigma_{tt}+M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)(-\Delta)^{2m}\sigma+\beta(-\Delta)^{2m}\sigma_t=h_1=h_{11}+h_{12} \\ H_{tt}+M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)(-\Delta)^{2m}H+\beta(-\Delta)^{2m}H_t=h_2=h_{21}+h_{22} \end{cases} \tag{19}$$

where

$$h_{11}=M'\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)'\left(D^m U+D^m V\right)(-\Delta)^{2m}u$$
$$+M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)(-\Delta)^{2m}\bar{u}+M\left(\|D^m\bar{u}\|_p^p+\|D^m\bar{v}\|_p^p\right)(-\Delta)^{2m}\bar{u},$$

$$h_{12}=-g_1(\bar{u},\bar{v})+g_1(u,v)+g_{1u}(u,v)U+g_{1v}(u,v)V,$$

$$h_{21}=M'\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)'\left(D^mU+D^mV\right)(-\Delta)^{2m}v$$
$$+M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)(-\Delta)^{2m}\bar{v}+M\left(\|D^m \bar{u}\|_p^p+\|D^m \bar{v}\|_p^p\right)(-\Delta)^{2m}\bar{v},$$

$$h_{22}=-g_2(\bar{u},\bar{v})+g_2(u,v)+g_{2u}(u,v)U+g_{2v}(u,v)V.$$

Take the inner product with the first equation of (19) and $(-\Delta)^k\sigma_t$, and get it

$$\left(\sigma_{tt},(-\Delta)^k\sigma_t\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^k\sigma_t\|^2,$$

$$\left(M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)(-\Delta)^{2m}\sigma,(-\Delta)^k\sigma_t\right)$$
$$=\frac{1}{2}M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m+k}\sigma_t\|^2,$$

$$\left(\beta(-\Delta)^{2m}\sigma_t,(-\Delta)^k\sigma_t\right)=\beta\|D^{2m+k}\sigma_t\|^2.$$

this is,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^k\sigma_t\|^2+\frac{1}{2}M\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m+k}\sigma_t\|^2+\beta\|D^{2m+k}\sigma_t\|^2$$
$$=\left(h_1,(-\Delta)^k\sigma_t\right).$$

Order

$$s=D^m u,w=D^m v,\bar{s}=D^m\bar{u},\bar{w}=D^m\bar{v},\tilde{u}=\bar{u}-u,\tilde{v}=\bar{v}-v,$$
$$\theta_1=(1-\alpha_1)\bar{s}+\alpha_1 s,\theta_3=(1-\alpha_3)\bar{w}+\alpha_3 w,$$
$$\theta_2=(1-\alpha_2)s+\alpha_2\theta_1,\theta_4=(1-\alpha_4)w+\alpha_4\theta_3.$$
$$N(\theta)=M'\left(\|s\|_p^p+\|w\|_p^p\right)\left(\|s\|_p^p+\|w\|_p^p\right)',\alpha_1,\alpha_2,\alpha_3,\alpha_4\in(0,1).$$

get

$$h_{11}=-N(\theta_1,\theta_3)\left(D^m\tilde{u}+D^m\tilde{v}\right)(-\Delta)^{2m}\bar{u}+N(\theta)D^m\tilde{u}(-\Delta)^{2m}u$$
$$-N(\theta)D^m\sigma(-\Delta)^{2m}u-N(\theta)D^m\tilde{v}(-\Delta)^{2m}u-N(\theta)D^mH(-\Delta)^{2m}u$$
$$=-N(\theta_1,\theta_3)D^m\tilde{u}(-\Delta)^{2m}\tilde{u}-N'(\theta_2,\theta_4)(1-\alpha_1)\left(D^m\tilde{u}\right)^2(-\Delta)^{2m}u$$
$$-N(\theta)D^m\sigma(-\Delta)^{2m}u-N(\theta_1,\theta_3)D^m\tilde{v}(-\Delta)^{2m}\tilde{u}$$
$$-N'(\theta_2,\theta_4)(1-\alpha_3)\left(D^m\tilde{v}\right)^2(-\Delta)^{2m}u-N(\theta)D^mH(-\Delta)^{2m}u.$$

So

$$\left|\left(h_{11},(-\Delta)^k\sigma_t\right)\right|$$
$$\le\|N(\theta_1,\theta_3)\|_\infty\|D^{2m+k}\tilde{u}\|\|\Delta^m\tilde{u}\|\|D^{m+k}\sigma_t\|$$
$$+c_1\|N'(\theta_2,\theta_4)\|_\infty\|D^{2m+k}\tilde{u}\|^2\left\|(-\Delta)^m\tilde{u}\right\|\|\sigma_t\|$$
$$+\|N(\theta)\|_\infty\left\|(-\Delta)^m u\right\|\|D^{m+k}\sigma\|\|D^{2m+k}\sigma_t\|$$
$$+\|N(\theta_1,\theta_3)\|_\infty\|D^{2m+k}\tilde{v}\|\|\Delta^m\tilde{u}\|\|D^{m+k}\sigma_t\|$$

$$
\begin{aligned}
&+c_1\left\|N'(\theta_2,\theta_4)\right\|_\infty\left\|D^{2m+k}\tilde{v}\right\|^2\left\|(-\Delta)^m u\right\|\left\|\sigma_t\right\| \\
&+\left\|N(\theta)\right\|_\infty\left\|(-\Delta)^m u\right\|\left\|D^{m+k}H\right\|\left\|D^{2m+k}\sigma_t\right\| \\
&\le \frac{c_3}{\lambda_1^k}\left\|D^k\sigma_t\right\|^2+\frac{c_4}{2\lambda_1^m}\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|\right)+c_5\left\|D^{2m+k}\sigma_t\right\|^2 \\
&+\left(\frac{c_6}{2}+\frac{c_7^2}{4c_5\lambda_1^{m+k}}\right)\left(\left\|D^{2m+k}\tilde{u}\right\|^2+\left\|D^{2m+k}\tilde{v}\right\|^2\right).
\end{aligned}
$$

In the same way, the second equation of (19) and $(-\Delta)^k H_t$ is used to take the internal product and organize it

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k\sigma_t\right\|^2+\left\|D^kH_t\right\|^2+M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|_p^p\right)\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|^2\right)\right. \\
&\left.+\beta\left(\left\|D^{2m+k}\sigma_t\right\|^2+\left\|D^{2m+k}H_t\right\|^2\right)\right) \\
&\le \frac{c_3}{\lambda_1^k}\left\|D^k\sigma_t\right\|^2+\frac{c_4}{2\lambda_1^m}\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|\right)+c_5\left\|D^{2m+k}\sigma_t\right\|^2 \\
&+\left(\frac{c_6}{2}+\frac{c_7^2}{4c_5\lambda_1^{m+k}}\right)\left(\left\|D^{2m+k}\tilde{u}\right\|^2+\left\|D^{2m+k}\tilde{v}\right\|^2\right)+h_{12}+h_{22}.
\end{aligned}
\tag{20}
$$

At the same time, when $U_t+\varepsilon U=P$, $V_t+\varepsilon V=P^*$, there is

$$h_{12}=-\left(g_1(\bar{u},\bar{v})-g_1(u,v)-g_{1u}(u,v)(\bar{u}-u)-g_{1v}(u,v)(\bar{v}-v)\right),$$

$$h_{22}=-\left(g_2(\bar{u},\bar{v})-g_2(u,v)-g_{2u}(u,v)(\bar{u}-u)-g_{2v}(u,v)(\bar{v}-v)\right).$$

From there, get

$$
\begin{aligned}
h_{12}=-\int_0^1\Big[&\left\{g_{1u}\left(u+P_1(\bar{u}-u),v+P_1(\bar{v}-v)\right)-g_{1u}(u,v)\right\}(\bar{u}-u) \\
&+\left\{g_{1v}\left(u+P_1(\bar{u}-u),v+P_1(\bar{v}-v)\right)-g_{1v}(u,v)\right\}(\bar{v}-v)\Big]\mathrm{d}P_1,
\end{aligned}
$$

$$
\begin{aligned}
h_{22}=-\int_0^1\Big[&\left\{g_{2u}\left(u+P_2(\bar{u}-u),v+P_2(\bar{v}-v)\right)-g_{2u}(u,v)\right\}(\bar{u}-u) \\
&+\left\{g_{2v}\left(u+P_2(\bar{u}-u),v+P_2(\bar{v}-v)\right)-g_{2v}(u,v)\right\}(\bar{v}-v)\Big]\mathrm{d}P_2,
\end{aligned}
$$

Get $\forall P_1,P_2\in[0,1]$,

$$\left\|g_{1u}\left(u+P_1(\bar{u}-u),v+P_1(\bar{v}-v)\right)-g_{1u}(u,v)\right\|\le l_kP_1^{\kappa_1}\left\|(\bar{u},\bar{v})-(u,v)\right\|_{V_{2m+k}\times V_{2m+k}}^{\kappa_1},$$

$$\left\|g_{1v}\left(u+P_1(\bar{u}-u),v+P_1(\bar{v}-v)\right)-g_{1v}(u,v)\right\|\le l_kP_1^{\kappa_1}\left\|(\bar{u},\bar{v})-(u,v)\right\|_{V_{2m+k}\times V_{2m+k}}^{\kappa_1},$$

$$\left\|g_{2u}\left(u+P_2(\bar{u}-u),v+P_2(\bar{v}-v)\right)-g_{2u}(u,v)\right\|\le l_k'P_2^{\kappa_1}\left\|(\bar{u},\bar{v})-(u,v)\right\|_{V_{2m+k}\times V_{2m+k}}^{\kappa_2},$$

$$\left\|g_{2v}\left(u+P_2(\bar{u}-u),v+P_2(\bar{v}-v)\right)-g_{2v}(u,v)\right\|\le l_k'P_2^{\kappa_1}\left\|(\bar{u},\bar{v})-(u,v)\right\|_{V_{2m+k}\times V_{2m+k}}^{\kappa_2}.$$

thereupon

$$\left\|h_{12}\right\|\le 2c_1\left\|(\bar{u},\bar{v})-(u,v)\right\|_{V_{2m+k}\times V_{2m+k}}^{\kappa_1+1},$$

Similarly,

$$\left\|h_{22}\right\|\le 2c_1'\left\|(\bar{u},\bar{v})-(u,v)\right\|_{V_{2m+k}\times V_{2m+k}}^{\kappa_2+1}.$$

By assumptions, the Formula (20) can be obtained

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k\sigma_t\right\|^2+\left\|D^kH_t\right\|^2+\mu_0\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|^2\right)\right)\\&\le\frac{2c_3'}{\lambda_1^k}\left(\left\|D^k\sigma_t\right\|^2+\left\|D^kH_t\right\|^2\right)+\frac{c_4'}{2\lambda_1^m}\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|^2\right)\\&\quad+2\left(c_5'-\beta\right)\left(\left\|D^{2m+k}\sigma_t\right\|^2+\left\|D^{2m+k}H_t\right\|^2\right)\\&\quad+\left(c_6'+\frac{c_7'^2}{2c_5'\lambda_1^{m+k}}\right)\left(\left\|D^{2m+k}\tilde{u}\right\|^4+\left\|D^{2m+k}\tilde{v}\right\|^4\right)\\&\quad+4c_1\left\|\left(\overline{u},\overline{v}\right)-\left(u,v\right)\right\|_{E_k}^{\kappa_1+1}+4c_1'\left\|\left(\overline{u},\overline{v}\right)-\left(u,v\right)\right\|_{E_k}^{\kappa_2+1}\end{aligned}$$

$$\begin{aligned}&\le\left(\frac{2c_3'}{\lambda_1^k}+\frac{2\left(c_5'-\beta\right)}{\lambda_1^{4m}}\right)\left(\left\|D^k\sigma_t\right\|^2+\left\|D^kH_t\right\|^2\right)+\frac{c_4'}{2\lambda_1^m}\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|^2\right)\\&\quad+\left(c_6'+\frac{c_7'^2}{2c_5'\lambda_1^{m+k}}+c_8\right)\left(\left\|D^{2m+k}\tilde{u}\right\|^4+\left\|D^{2m+k}\tilde{v}\right\|^4\right).\end{aligned}$$

Order $\alpha=\max\left\{\frac{4c_3'}{\lambda_1^k}+\frac{4\left(c_5'-\beta\right)}{\lambda_1^{4m}},\frac{c_4'}{\mu_0\lambda_1^m}\right\}$,

Then there is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k\sigma_t\right\|^2+\left\|D^kH_t\right\|^2+\mu_0\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|^2\right)\right)\\&\le\alpha\left(\left\|D^k\sigma_t\right\|^2+\left\|D^kH_t\right\|^2+\mu_0\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|^2\right)\right)\\&\quad+c_{10}\left(\left\|D^{2m+k}\tilde{u}\right\|^4+\left\|D^{2m+k}\tilde{v}\right\|^4\right).\end{aligned}$$

Using the Gronwall's inequality, there is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k\sigma_t\right\|^2+\left\|D^kH_t\right\|^2+\mu_0\left(\left\|D^{2m+k}\sigma\right\|^2+\left\|D^{2m+k}H\right\|^2\right)\right)\\&\le c_{11}\mathrm{e}^{c_{12}t}\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_k}^4.\end{aligned}$$

Then when $\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_k}^2\to0$,

$$\frac{\left\|S(t)\boldsymbol{\Phi}_0-S(t)\overline{\boldsymbol{\Phi}}_0-F\left(\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right)\right\|_{E_k}^2}{\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_k}^2}\le c_{11}\mathrm{e}^{c_{12}t}\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_k}^2\to0.$$

Theorem 4 is proved.

The Hausdorff and Fractal dimensions of the global attractors subfamily are estimated below.

The problem of linearization is reduced to $\boldsymbol{\Psi}'+\boldsymbol{P}(\boldsymbol{\varphi})\boldsymbol{\Psi}=\boldsymbol{\Gamma}_1(\boldsymbol{\varphi})\boldsymbol{\Psi}+\boldsymbol{\Gamma}_2(\boldsymbol{\varphi})\boldsymbol{\Psi}$.

At this point, $\boldsymbol{\Psi}=\left(U,P,V,P^*\right)^{\mathrm{T}}\in E_k$, $P=U_t+\varepsilon U$, $P^*=V_t+\varepsilon V$, $\boldsymbol{\varphi}=\left(u,z,v,q\right)^{\mathrm{T}}\in E_k$ are the solutions to Equation (21), $\boldsymbol{\Psi}(0)=\left\{\xi,\zeta,\eta,\sigma\right\}\in E_k$, $t>0$.

$$\boldsymbol{P}(\boldsymbol{\varphi})=\begin{pmatrix} \varepsilon I & -I & 0 & 0 \\ (1-\beta\varepsilon)(-\Delta)^{2m}+\varepsilon^2 I & \beta(-\Delta)^{2m}-\varepsilon I & 0 & 0 \\ 0 & 0 & \varepsilon I & -I \\ 0 & 0 & (1-\beta\varepsilon)(-\Delta)^{2m}+\varepsilon^2 I & \beta(-\Delta)^{2m}-\varepsilon I \end{pmatrix},$$

$$\boldsymbol{\Gamma}_1(\boldsymbol{\varphi})=\begin{pmatrix} 0 & 0 & 0 & 0 \\ -g_{1u}(u,v) & 0 & -g_{1v}(u,v) & 0 \\ 0 & 0 & 0 & 0 \\ -g_{2u}(u,v) & 0 & -g_{2v}(u,v) & 0 \end{pmatrix},$$

$$\boldsymbol{\Gamma}_2(\boldsymbol{\varphi})\boldsymbol{\Psi}=\begin{pmatrix} 0 \\ (1-M(s))(-\Delta)^{2m}U-2M'(s)s'(D^mU+D^mV)(-\Delta)^{2m}u \\ 0 \\ (1-M(s))(-\Delta)^{2m}V-2M'(s)s'(D^mU+D^mV)(-\Delta)^{2m}v \end{pmatrix}.$$

where $s=\|D^m u\|_p^p+\|D^m v\|_p^p$, Order

$$D_1=M'\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)' D^mU(-\Delta)^{2m}u,$$

$$D_2=M'\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)' D^mV(-\Delta)^{2m}u,$$

$$D_3=M'\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)' D^mU(-\Delta)^{2m}v,$$

$$D_4=M'\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)\left(\|D^m u\|_p^p+\|D^m v\|_p^p\right)' D^mV(-\Delta)^{2m}v.$$

Theorem 5 Under the conditions of theorem 4, problems (1)-(5) the global attraction subfamily A_k has the Hausdorff dimension and the Fractal dimension, and

$$d_H(A_k)\le\min\left\{N\mid N\in N_+,\frac{1}{N}\sum_j^N\lambda_j^{t_k-1}<\frac{\ell}{2\Upsilon}\right\},$$

$$d_F(A_k)\le 2N,(k=1,2,\cdots,2m)$$

Proof: Let $N\in N_+$, the N solutions of the problem (21) are $\chi_1,\chi_2,\cdots,\chi_N$, considering N of them, given time τ, there is

$$\left|\chi_1(s)\wedge\chi_2(s)\wedge\cdots\wedge\chi_N(s)\right|_{\wedge^N E_k}$$
$$=\left|\chi_1(0)\wedge\chi_2(0)\wedge\cdots\wedge\chi_N(0)\right|_{\wedge^N E_k}\exp\int_0^s TrF'(\boldsymbol{\varphi}(\tau))\cdot B_N(\tau)\mathrm{d}\tau.$$

where $\boldsymbol{\varphi}(\tau)=(u(\tau),p(\tau),v(\tau),q(\tau))$, $B_N(\tau)=B_N(\tau,\boldsymbol{\varphi}_0;\chi_1(0),\cdots,\chi_N(0))$ is an orthogonal projection from E_k to $span\{\chi_1(\tau),\cdots,\chi_N(\tau)\}$, and
$y_j(\tau)=\{\xi_j,\zeta_j,\eta_j,\sigma_j\}, j=1,\cdots,N$ is
$B_N(\tau)E_k=span\{\chi_1(\tau),\chi_2(\tau),\cdots,\chi_N(\tau)\}$ standard orthogonal radicals.

Set the corresponding inner product and norm,

$$(y_j,\bar{y}_j)_{E_k}=\left(D^{2m+k}\xi_j,D^{2m+k}\tilde{\xi}_j\right)+\left(D^k\zeta_j,D^k\tilde{\zeta}_j\right)$$
$$+\left(D^{2m+k}\eta_j,D^{2m+k}\tilde{\eta}_j\right)+\left(D^k\sigma_j,D^k\tilde{\sigma}_j\right),$$

$$\left\|y_j\right\|_{E_k}^2=\left(y_j,y_j\right)_{E_k}=\left\|D^{2m+k}\xi_j\right\|^2+\left\|D^k\zeta_j\right\|^2+\left\|D^{2m+k}\eta_j\right\|^2+\left\|D^k\sigma_j\right\|^2=1.$$

$$\begin{aligned}TrF'\left(\varphi(\tau)\right)\cdot B_N(\tau)&=\sum_{j=1}^{+\infty}\left(F'\left(\boldsymbol{\varphi}(\tau)\right)\cdot B_N(\tau)y_j(\tau),y_j(\tau)\right)_{E_k}\\&=\sum_{j=1}^{N}\left(F'\left(\boldsymbol{\varphi}(\tau)\right)y_j(\tau),y_j(\tau)\right)_{E_k}.\end{aligned}$$

so,

$$\begin{aligned}&-\left(P(\varphi)y_j,y_j\right)\\&=-\left(\varepsilon D^{2m+k}\xi_j-D^{2m+k}\zeta_j,D^{2m+k}\xi_j\right)-\left(\varepsilon D^{2m+k}\eta_j-D^{2m+k}\sigma_j,D^{2m+k}\eta_j\right)\\&\quad-\left((1-\beta\varepsilon)(-\Delta)^{2m}D^k\xi_j+\varepsilon^2D^k\xi_j,D^k\zeta_j\right)-\left(\beta(-\Delta)^{2m}D^k\zeta_j,D^k\zeta_j\right)\\&\quad-\left(\varepsilon D^k\zeta_j,D^k\zeta_j\right)-\left((1-\beta\varepsilon)(-\Delta)^{2m}D^k\eta_j+\varepsilon^2D^k\eta_j,D^k\sigma_j\right)\\&\quad-\left(\beta(-\Delta)^{2m}D^k\sigma_j,D^k\sigma_j\right)-\left(\varepsilon D^k\sigma_j,D^k\sigma_j\right)\\&\le\frac{\beta-2}{2}\varepsilon\left\|D^{2m+k}\xi_j\right\|^2+\frac{\beta\varepsilon c_{13}\lambda_1^{2m}}{2}\left\|D^k\zeta_j\right\|^2+\frac{\varepsilon^2}{2}\left\|D^k\xi_j\right\|^2\\&\quad+\frac{\beta-2}{2}\varepsilon\left\|D^{2m+k}\eta_j\right\|^2+\left(\frac{\varepsilon c_{14}-4}{2}\beta\lambda_1^{2m}+2\varepsilon^2\right)\left\|D^k\sigma_j\right\|^2+\frac{\varepsilon^2}{2}\left\|D^k\eta_j\right\|^2.\end{aligned}\tag{22}$$

$$\begin{aligned}&\left(\Gamma_1(\varphi)y_j,y_j\right)_{E_k}\\&=\left(-g_{1u}(u,v)D^k\xi_j,D^k\zeta_j\right)+\left(-g_{1v}(u,v)D^k\eta_j,D^k\zeta_j\right)\\&\quad+\left(-g_{2u}(u,v)D^k\xi_j,D^k\sigma_j\right)+\left(-g_{2v}(u,v)D^k\eta_j,D^k\sigma_j\right)\\&\le\left\|g_{1u}(u,v)\right\|_\infty\left\|D^k\xi_j\right\|\left\|D^k\zeta_j\right\|+\left\|g_{1v}(u,v)\right\|_\infty\left\|D^k\eta_j\right\|\left\|D^k\zeta_j\right\|\\&\quad+\left\|g_{2u}(u,v)\right\|_\infty\left\|D^k\xi_j\right\|\left\|D^k\sigma_j\right\|+\left\|g_{2v}(u,v)\right\|_\infty\left\|D^k\eta_j\right\|\left\|D^k\sigma_j\right\|\\&\le\frac{c_{15}+c_{17}}{2}\left\|D^k\xi_j\right\|^2+\frac{c_{15}+c_{16}}{2}\left\|D^k\zeta_j\right\|^2+\frac{c_{16}+c_{18}}{2}\left\|D^k\eta_j\right\|^2+\frac{c_{17}+c_{18}}{2}\left\|D^k\sigma_j\right\|^2.\end{aligned}\tag{23}$$

$$\begin{aligned}&\left(\Gamma_2(\varphi)y_j,y_j\right)_{E_k}\\&=\left(1-M(s)\right)\left((-\Delta)^{2m}D^k\xi_j,D^k\zeta_j\right)-2\left(D^kD_1\xi_j,D^k\zeta_j\right)-2\left(D^kD_2\zeta_j,D^k\zeta_j\right)\\&\quad+\left(1-M(s)\right)\left((-\Delta)^{2m}D^k\eta_j,D^k\sigma_j\right)-2\left(D^kD_3\xi_j,D^k\sigma_j\right)-2\left(D^kD_4\zeta_j,D^k\sigma_j\right)\\&=\left(1-M(s)\right)\left(D^{2m+k}\xi_j,D^{2m+k}\zeta_j\right)-2\left(D^kD_1\xi_j,D^k\zeta_j\right)-2\left(D^kD_2\zeta_j,D^k\zeta_j\right)\\&\quad+\left(1-M(s)\right)\left(D^{2m+k}\eta_j,D^{2m+k}\sigma_j\right)-2\left(D^kD_3\xi_j,D^k\sigma_j\right)-2\left(D^kD_4\zeta_j,D^k\sigma_j\right)\\&\le\frac{1-\mu_0}{2}\left(\left\|D^{2m+k}\xi_j\right\|^2+\left\|D^{2m+k}\zeta_j\right\|^2\right)+c_{19}\left\|D^k\xi_j\right\|^2+\left(c_{19}+2c_{20}+c_{22}\right)\left\|D^k\zeta_j\right\|^2\\&\quad+\frac{1-\mu_0}{2}\left(\left\|D^{2m+k}\eta_j\right\|^2+\left\|D^{2m+k}\sigma_j\right\|^2\right)+c_{21}\left\|D^k\xi_j\right\|^2+c_{21}\left\|D^k\sigma_j\right\|^2\\&\le\frac{1-\mu_0}{2}\left\|D^{2m+k}\xi_j\right\|^2+\frac{1-\mu_0}{2}\left\|D^{2m+k}\eta_j\right\|^2\\&\quad+\left(\frac{c_{23}(1-\mu_0)\lambda_1^{2m}}{2}+c_{19}+2c_{20}+c_{22}\right)\left\|D^k\zeta_j\right\|^2\\&\quad+\left(\frac{c_{24}(1-\mu_0)\lambda_1^{2m}}{2}+c_{21}\right)\left\|D^k\sigma_j\right\|^2+\left(c_{19}+c_{21}\right)\left\|D^k\xi_j\right\|^2.\end{aligned}\tag{24}$$

According to Formulas (22), (23) and (24), there is

$$\begin{aligned}&\left(F'(\varphi)y_j,y_j\right)_{E_k}=\left(\left(-P(\varphi)+\Gamma_1(\varphi)+\Gamma_2(\varphi)\right)h_j,h_j\right)\\&\le\left(\frac{\beta-2}{2}\varepsilon+\frac{1-\mu_0}{2}\right)\left\|D^{2m+k}\xi_j\right\|^2+\left(\frac{\beta\varepsilon c_{13}\lambda_1^{2m}}{2}+\frac{c_{15}+c_{16}}{2}+\frac{c_{23}(1-\mu_0)\lambda_1^{2m}}{2}\right.\\&\left.+c_{19}+2c_{20}+c_{22}\right)\left\|D^k\zeta_j\right\|^2+\left(\frac{\beta-2}{2}\varepsilon+\frac{1-\mu_0}{2}\right)\left\|D^{2m+k}\eta_j\right\|^2\\&+\left(\frac{\varepsilon c_{14}-4}{2}\beta\lambda_1^{2m}+2\varepsilon^2+\frac{c_{17}+c_{18}}{2}+\frac{c_{24}(1-\mu_0)\lambda_1^{2m}}{2}+c_{21}\right)\left\|D^k\sigma_j\right\|^2\\&+\left(\frac{\varepsilon^2}{2}+\frac{c_{15}+c_{17}}{2}+c_{19}+c_{21}\right)\left\|D^k\xi_j\right\|^2+\left(\frac{\varepsilon^2}{2}+\frac{c_{16}+c_{18}}{2}\right)\left\|D^k\eta_j\right\|^2\end{aligned} \tag{25}$$

Order

$$\begin{aligned}\ell=\min\Bigg\{&\frac{2-\beta}{2}\varepsilon+\frac{\mu_0-1}{2},-\frac{\beta\varepsilon c_{13}\lambda_1^{2m}}{2}-\frac{c_{15}+c_{16}}{2}\\&+\frac{c_{23}(\mu_0-1)\lambda_1^{2m}}{2}-c_{19}-2c_{20}-c_{22},\\&\frac{4-\varepsilon c_{14}}{2}\beta\lambda_1^{2m}-2\varepsilon^2-\frac{c_{17}+c_{18}}{2}+\frac{c_{24}(\mu_0-1)\lambda_1^{2m}}{2}-c_{21}\Bigg\},\end{aligned}$$

$$\Upsilon=\max\left\{\frac{\varepsilon^2}{2}+\frac{c_{15}+c_{17}}{2}+c_{19}+c_{21},\frac{\varepsilon^2}{2}+\frac{c_{16}+c_{18}}{2}\right\},$$

To all time τ, there are $0<t_k=\frac{k}{2m+k}<1$, making

$$\begin{aligned}\sum_{j=1}^N\left(F'(\boldsymbol{\varphi})y_j,y_j\right)_{E_k}&=\sum_{j=1}^N\left(\left(-\boldsymbol{P}(\boldsymbol{\varphi})+\boldsymbol{\Gamma}_1(\boldsymbol{\varphi})+\boldsymbol{\Gamma}_2(\boldsymbol{\varphi})\right)y_j,y_j\right)\\&\le-N\ell+\Upsilon\sum_j^N\left(\left\|D^k\xi_j\right\|^2+\left\|D^k\eta_j\right\|^2\right)\\&\le-N\ell+2\Upsilon\sum_j^N\lambda_j^{t_k-1},\end{aligned}$$

If $\frac{1}{N}\sum_j^N\lambda_j^{t_k-1}<\frac{\ell}{2\Upsilon}$,

$$q_N(s)=\sup_{\varphi_0\in A_k}\sup_{\chi_j\in E_k}\left\{\frac{1}{t}\int_0^t \boldsymbol{Tr}F'(\boldsymbol{\varphi}(\tau))\circ\boldsymbol{P}_N^*(\tau)\,\mathrm{d}\tau\right\}\le-N\left(\ell-\frac{2\Upsilon}{N}\sum_j^N\lambda_j^{t_k-1}\right),$$

then $q_N=\lim_{s\to\infty}\sup q_N(s)<0$.

Theorem 5 is proved.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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