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Clifford Algebra and Hypercomplex Number as well as Their Applications in Physics

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Abstract

The Clifford algebra is a unification and generalization of real number, complex number, quaternion, and vector algebra, which accurately and faithfully characterizes the intrinsic properties of space-time, providing a unified, standard, elegant, and open language and tools for numerous complicated mathematical and physical theories. So it is worth popularizing in the teaching of undergraduate physics and mathematics. Clifford algebras can be directly generalized to 2^n-ary associative algebras. In this generalization, the matrix representation of the orthonormal basis of space-time plays an important role. The matrix representation carries more information than the abstract definition, such as determinant and the definition of inverse elements. Without this matrix representation, the discussion of hypercomplex numbers will be difficult. The zero norm set of hypercomplex numbers is a closed set of special geometric meanings, like the light-cone in the realistic space-time, which has no substantial effect on the algebraic calculus. The physical equations expressed in Clifford algebra have a simple formalism, symmetrical structure, standard derivation, complete content. Therefore, we can hope that this magical algebra can complete a new large synthesis of modern science.

Keywords

Quaternion, Hypercomplex Number, Supercomplex, Clifford Algebra, Geometric Algebra, Maxwell Equations, Dirac Equation

1. Introduction

The development of the number systems had gone through a long and difficult historical process, not as simple as it seems today. The improvement of the concept of "number" is full of legends. After Pythagoras and Hippasus found that the diagonal length and edge length of a square is irreducible, Eudoxus (400BC-347BC) established the irreducible theory. Until the 19th century, after R. Dedekind (1831-1916), G. Cantor (1846-1918) and K. Weierstrass (1815-1897) completed the rigorous irrational number theory, the theory of real numbers was strictly established [1].

The introduction of "imaginary number" has also experienced many difficulties, and at the beginning, many mathematicians hold resistance attitude against the imaginary number. It was not until the second half of the 18th century that K. F. Gauss (1777-1855) found a geometric representation of complex numbers, that is, they correspond to the points of a plane, then the imaginary number got a specific geometric interpretation and wide application in practical problems, that this new number system was widely recognized [1]. From the perspective of algebra, rational, real, and complex numbers are all closed for addition, subtraction, multiplication and division operations and without zero factors, and the calculations satisfy the associative, distributive, and commutative laws.

By the 19th century, people had a clear understanding of number systems and algebraic algorithm, and had studied the legitimacy of symbolic operations. This paved the way for the later development of abstract algebras, especially of Boolean algebra. How to extend the superiority of complex numbers in the plane to 3-dimensional space was a difficult problem in front of people, and many famous mathematicians were looking for the "3-ary numbers" [1]. The Irish mathematician W. R. Hamilton (1805-1865) was also joined in the search for 3-ary numbers due to the practical needs in physics. He fought intermittently for the 3-ary numbers for 15 years, always struggling about how to define its multiplication. At the dusk of October 16, 1843, Hamilton had a flash of inspiration that the commutative law of multiplication must be abandoned and an ordered array with 4 real numbers was needed, and then quaternion was born.

Having real numbers, complex numbers and quaternions, a natural idea is to similarly expand the number system by abandoning the algebraic rules as little as possible. But in 1878, an important theorem, proved by the German mathematician F. G. Frobenius (1849-1917), gave a negative conclusion. The theorem shows that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are the only finite-dimensional division algebras over $\mathbb{R}$ without zero factors. Subsequently, a generalized Frobenius' theorem showed that if the associative law of multiplication is abandoned, the division algebra without zero factors leaves only the octonion or Cayley number [1].

Clifford algebra is a kind of algebraic systems rooted in geometry, which was introduced by William Kingdon Clifford (1845-1879). The element of Clifford algebra represented by the Grassmann basis is the unification and generalization of real, complex, quaternion and vector algebra (H. G. Grassmann, 1809-1877). Clifford algebra accurately and faithfully describes the intrinsic properties of space-time, and can realize the convenient conversion between flat space-time and manifold. Clifford algebra only depends on a few simple concepts such as number fields, vector space etc., providing a unified, standard, elegant and open language and tool for numerous complicated mathematical and physical theories [2] [3] [4] [5] [6]. In recent years, Clifford algebra has made brilliant achievements in differential geometry, theoretical physics, classical analysis and other aspects, and has been widely used engineering, such as robotics and computer vision [7] [8] [9] [10]. Clifford algebra also has applications in mathematical mechanization such as machine proof of geometric theorem. Currently, geometric automatic proof has become an important application of Clifford algebra. The algorithm of Clifford Algebra is arithmetic-like operations, well understood by everyone. This feature is very useful for pedagogical purposes, and promoting Clifford algebra in high schools and universities will greatly improve the efficiency of students in learning the basic knowledge of mathematics and physics [5] [6].

2. Clifford Algebras and Hypercomplex Numbers

For n dimensional Minkowski space-time $\mathbb{M}^{p,q}(p+q=n)$ over number field $\mathbb{F}$, the metric corresponding to the orthonormal basis $\{\mathbf{e}_a\}$ and coframe $\{\mathbf{e}^a\}$ is given by $(\eta_{ab})=(\eta^{ab})=\mathrm{diag}(I_p,-I_q)$. The **Clifford algebra** defined on $\mathbb{M}^{p,q}$ is an algebra satisfying the following multiplication rules,

$$\mathbf{e}_a\mathbf{e}_b+\mathbf{e}_a\mathbf{e}_b=2\eta_{ab},\quad \mathbf{e}^a\mathbf{e}^b+\mathbf{e}^a\mathbf{e}^b=2\eta^{ab}. \tag{2.1}$$

Because of the close relation between Clifford algebra and geometry, it is also called **geometric algebra**, and the product $\mathbf{e}_a\mathbf{e}_b$ is called **Clifford product** or **geometric product**. The geometric product is bilinear, *i.e.*, for any vectors $\mathbf{x}=x^a\mathbf{e}_a=x_a\mathbf{e}^a$ and $\mathbf{y}=y^a\mathbf{e}_a=y_a\mathbf{e}^a$, we have

$$\mathbf{xy}=(x^a\mathbf{e}_a)(y^b\mathbf{e}_b)=x^ay^b(\mathbf{e}_a\mathbf{e}_b)=x_ay_b(\mathbf{e}^a\mathbf{e}^b),\quad \forall x^a,y^a\in\mathbb{F}.$$

In this paper the Einstein's summation is adopted.

By (2.1) we have

$$\mathbf{xy}=x^ay^b\left(\frac{1}{2}(\mathbf{e}_a\mathbf{e}_b+\mathbf{e}_b\mathbf{e}_a)+\frac{1}{2}(\mathbf{e}_a\mathbf{e}_b-\mathbf{e}_b\mathbf{e}_a)\right)=\mathbf{x}\cdot\mathbf{y}+\mathbf{x}\wedge\mathbf{y}, \tag{2.2}$$

where $\mathbf{x}\wedge\mathbf{y}$ is the **Grassmann product** or **exterior product** of vectors. The geometric meaning of the exterior product is the area of a parallelogram constructed by the vectors $\mathbf{x}$ and $\mathbf{y}$. The symmetrization of the geometric product of the basis $\mathbf{e}^a$ yields the unit or zero element, and the antisymmetrization one yields the exterior product of basis. Since the exterior product on an n dimensional space-time has 2^n **basis elements**, which form the basis of **Grassmann algebra** (also called **exterior algebra**),

$$\mathcal{K}=C_0I+C_a\mathbf{e}^a+C_{ab}\mathbf{e}^{ab}+C_{abc}\mathbf{e}^{abc}+\cdots+C_{12\cdots n}\mathbf{e}^{12\cdots n}, \tag{2.3}$$

where $\forall C_{ab\cdots}\in\mathbb{F}$, $\mathbf{e}^{ab\cdots c}=\mathbf{e}^a\wedge\mathbf{e}^b\wedge\cdots\wedge\mathbf{e}^c$, $a<b<\cdots<c$. (2.3) is called **Clifford-Grassmann numbers**, which is equivalent to the Clifford algebra $C\ell_{p,q}$ in the sense of linear algebra.

The representation of the sum (2.3) is not unique due to the antisymmetry of Grassmann product. For example, the term $C_{ab}\boldsymbol{e}^{ab}$ usually adopts the convention $a<b$, but sometimes adopts the antisymmetrical form $\frac{1}{2!}C_{ab}\boldsymbol{e}^{ab}$ with

$C_{ab}=-C_{ba}$. In 1 + 3 dimensional space-time we also use the 3-d vector form

$$\vec{E}=\left(C_{01},C_{02},C_{03}\right),\quad \vec{B}=\left(C_{23},C_{31},C_{12}\right),$$

The results of all these representations are equivalent. Since the Clifford algebra is isomorphic to matrix algebras, the basis vectors can be represented by a set of special square matrices, so that the geometric algebra transforms into the familiar matrix algebra. The following theorem provides a canonical matrix representation for the orthonormal basis $\{\mathbf{e}_a\}$ of the space-time.

Denote the Pauli matrices by

$$\sigma^a \equiv \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}$$

$$\tilde{\sigma}^0=\sigma^0=I,\quad \tilde{\sigma}^k=-\sigma^k,\quad k=1,2,3,$$

and higher order Dirac-γ matrices by

$$\gamma^a=\begin{pmatrix}0 & \tilde{\vartheta}^a\\ \vartheta^a & 0\end{pmatrix}\equiv\Gamma^a\left(m\right), \tag{2.4}$$

in which $m\ge 1$,

$$\vartheta_a=\operatorname{diag}\left(\overbrace{\sigma_a,\sigma_a,\cdots,\sigma_a}^{m}\right),\quad \tilde{\vartheta}_a=\operatorname{diag}\left(\overbrace{\tilde{\sigma}_a,\tilde{\sigma}_a,\cdots,\tilde{\sigma}_a}^{m}\right).$$

γ^a (2.4) forms the complex matrix representation of the basis vectors $\mathbf{e}^a$ or the generators of Clifford algebra $C\ell_{1,3}$.

Denote m-order identity matrix by I_m. For any square matrix $A=\left(A_{ab}\right)$, denote the block matrix by

$$A\otimes I_m=\left(A_{ab}I_m\right),\quad \left[A,B,C,\cdots\right]=\operatorname{diag}\left(A,B,C,\cdots\right).$$

Clearly we have $I_2\otimes I_2=I_4$, $I_2\otimes I_2\otimes I_2=I_8$ and so on, where $\otimes$ means the Kronecker product of matrices. For $a\in\{0,1,2,3\}$, $\Gamma^a\left(m\right)$ are all $4m\times 4m$ square matrices, they constitute a set of basis of $\mathbb{M}^{1,3}$. For Clifford algebra $C\ell_{p+q}$, we have the following theorem of complex matrix representation [11].

Theorem 1 1) *In the case of neglecting an imaginary factor i, for the generators of* $C\ell_{4m}$, *there exists the following unique matrix representation in the equivalent sense*

$$\begin{aligned}\Bigg\{&\Gamma^a\left(n\right),\ \left[\Gamma^a\left(\frac{n}{2^2}\right),-\Gamma^a\left(\frac{n}{2^2}\right)\right]\otimes I_2,\\ &\left[\left[\Gamma^a\left(\frac{n}{2^4}\right),-\Gamma^a\left(\frac{n}{2^4}\right)\right],-\left[\Gamma^a\left(\frac{n}{2^4}\right),-\Gamma^a\left(\frac{n}{2^4}\right)\right]\right]\otimes I_{2^2},\\ &\left[\left[\Gamma^a\left(\frac{n}{2^6}\right),-\Gamma^a\left(\frac{n}{2^6}\right),-\Gamma^a\left(\frac{n}{2^6}\right),\Gamma^a\left(\frac{n}{2^6}\right)\right],\right.\\ &\left.-\left[\Gamma^a\left(\frac{n}{2^6}\right),-\Gamma^a\left(\frac{n}{2^6}\right),-\Gamma^a\left(\frac{n}{2^6}\right),\Gamma^a\left(\frac{n}{2^6}\right)\right]\right]\otimes I_{2^3},\cdots\Bigg\}.\end{aligned} \tag{2.5}$$

In which $n=2^{m-1}N$, N *is any given positive integer. All matrices are*

$2^{m+1}N \times 2^{m+1}N$ *type.*

2) *For* $C\ell_{4m+1}$, *besides* (2.5) *we have another generator*

$$\gamma^{4m+1} = \left[\left[\left[E,-E\right],-\left[E,-E\right]\right],-\left[\left[E,-E\right],-\left[E,-E\right]\right],\cdots\right] \tag{2.6}$$

where $E = \left[I_{2k}, -I_{2l}\right], (kl \neq 0)$. *If and only if* $k = l$, *this representation can be uniquely expanded as generators of* $C\ell_{4m+4}$.

3) *The generators of* $C\ell_{4m+2}$ *or* $C\ell_{4m+3}$ *can be represented by* $4m+2$ *or* $4m+3$ *matrices from the matrix representation of the* $C\ell_{4m+4}$ *generators.*

4) *For* $C\ell_j, (j = 2,3)$, *besides to select the matrices from the basis of* $C\ell_4$ *for the representation, we also have the following matrix representation*

$$\gamma^a \in \left\{\operatorname{diag}\left(\vartheta_k, -\xi_k\right), k = 1,2,3\right\} \tag{2.7}$$

where

$$\vartheta_k = \operatorname{diag}\left(\overbrace{\sigma_k, \sigma_k, \cdots, \sigma_k}^{m}\right), \quad \xi_k = \operatorname{diag}\left(\overbrace{\sigma_k, \sigma_k, \cdots, \sigma_k}^{n-m}\right) \tag{2.8}$$

and $m\left(0 \le m \le n\right)$ *is independent of k.*

Then we obtain all complex matrix representations for generators of $C\ell_{p,q}$ explicitly.

By the isomorphism of Clifford algebra and matrix algebra, we can directly use the γ^a matrices defined in (2.5) as basis vectors, and this brings great convenience in calculation. According to matrix algebra, the Clifford-Grassmann numbers (2.3) also satisfy the associative law of multiplication and the distributive law for addition, but like quaternons, there is no commutative law of multiplication. In (2.3), each grade-*k* term is a rank-*k* tensor, for examples, $C_0 I \in \Lambda^0$ is a true scalar, $C_a \gamma^a \in \Lambda^1$ is a true vector, $C_{ab}\gamma^{ab} \in \Lambda^2$ is rank-2 antisymmetric tensor, also called bivector, $C_{a_1 a_2 \cdots a_k}\gamma^{a_1 a_2 \cdots a_k} \in \Lambda^k$ is a *k*-vector and so on. The canonical matrix basis set is given by

$$I,\ \gamma^a,\ \gamma^{ab} = \gamma^a \wedge \gamma^b,\ \gamma^{abc} = \gamma^a \wedge \gamma^b \wedge \gamma^c,\ \cdots,\ \gamma^{12\cdots n}. \tag{2.9}$$

In particular, for the 1 + 3 dimensional Minkowski space-time, we have the lowest order complex matrix representation of the basis of Clifford algebra $C\ell_{1,3}$ as

$$I_4, \quad \gamma^a, \quad \gamma^{ab}, \quad \gamma^{abc} = \mathrm{i}\varepsilon^{abcd}\gamma_d \gamma^{0123}, \quad \gamma^{0123} = -\mathrm{i}\gamma^5, \tag{2.10}$$

in which $\gamma^5 = \operatorname{diag}\left(I_2, -I_2\right)$ and $\varepsilon^{0123} = 1$. In the case without confusion, we also use 1 to represent the identity matrix *I*. So we have a Clifford-Grassmann number as

$$\mathcal{K} = sI_4 + A_a\gamma^a + H_{ab}\gamma^{ab} + Q_a\gamma^a\gamma^{0123} + p\gamma^{0123}, \quad \left(s, p, A_a, \cdots \in \mathbb{F}\right). \tag{2.11}$$

Let $\vec{E} = \left(H_{01}, H_{02}, H_{03}\right)$ and $\vec{B} = \left(H_{23}, H_{31}, H_{12}\right)$, we have the determinant of $\mathcal{K}$,

$$\begin{aligned}\det\left(\mathcal{K}\right) = &\left(s^2 + p^2 - \mathbf{A}^2 - \mathbf{Q}^2\right)^2 + 4\left(A_a Q^a\right)^2 - \left(s^2 + p^2\right)^2 \\ &+ \left(p^2 - s^2 + \vec{E}^2 - \vec{B}^2\right)^2 + 4\left(\vec{E}\cdot\vec{B} - sp\right)^2 - 4\mathbf{A}^2\mathbf{Q}^2 + \Delta,\end{aligned} \tag{2.12}$$

where $\mathbf{A}^2 = A_a A^a = A_0^2 - \vec{A}^2, \mathbf{Q}^2 = Q_a Q^a = Q_0^2 - \vec{Q}^2$ and

$$\begin{aligned}\Delta = & -2\left(\vec{E}^2+\vec{B}^2\right)\left(\mathbf{A}^2+\mathbf{Q}^2\right)+4\left(\vec{E}^2+\vec{B}^2\right)\left(A_0^2+Q_0^2\right) \\ & -4\left[\left(\vec{A}\cdot\vec{E}\right)^2+\left(\vec{A}\cdot\vec{B}\right)^2+\left(\vec{Q}\cdot\vec{E}\right)^2+\left(\vec{Q}\cdot\vec{B}\right)^2\right] \\ & +8\left[\left(A_0\vec{A}+Q_0\vec{Q}\right)\cdot\left(\vec{B}\times\vec{E}\right)\right] \\ & +8s\left[\left(\vec{Q}\times\vec{A}\right)\cdot\vec{E}+\left(Q_0\vec{A}-A_0\vec{Q}\right)\cdot\vec{B}\right] \\ & +8p\left[\left(\vec{Q}\times\vec{A}\right)\cdot\vec{B}-\left(Q_0\vec{A}-A_0\vec{Q}\right)\cdot\vec{E}\right],\end{aligned} \tag{2.13}$$

in which $\vec{A}=\left(A_1,A_2,A_3\right)$, $\vec{Q}=\left(Q_1,Q_2,Q_3\right)$. From (2.12) we learn that, $\det\left(\mathcal{K}\right)\in\mathbb{F}$ is independent of the imaginary unit "±i" in the basis matrix, that is, the matrix representation of the basis elements (2.10) is actually independent of the number field $\mathbb{F}$.

In the domain $\left\{\det\left(\mathcal{K}\right)\neq 0\right\}$, we define the reciprocal of $\mathcal{K}$ as the matrix $\mathcal{K}^{-1}$. Since $\mathcal{K}^{-1}\left(s\right)$ can be expressed as Taylor series of s^{-1} at $s=\infty$, $\mathcal{K}^{-1}$ is also a Clifford-Grassmann number with basis matrices (2.10). This means that the basis (2.10) is closed for all algebraic calculation of matrix. Thus, we can generally define analytic functions and equations of Clifford-Grassmann numbers, such as

$$\mathcal{H}=\mathcal{N}e^{\mathcal{W}}+\mathcal{Y}e^{\mathcal{A}r}\sin\left(\omega\mathcal{T}\right)\mathcal{M}\mathcal{A}^{-n}\mathcal{J}^m$$

where $\left(\mathcal{H},\mathcal{N},\mathcal{Y},\cdots\right)$ are all Clifford numbers with coefficients in field $\mathbb{F}$. Therefore, the Clifford algebra is actually a supercomplex system with basis (2.10) or (2.9).

In the case $\mathbb{F}=\mathbb{R}$, which is the most used case, defining the norm $\|\mathcal{K}\|=\sqrt[4]{\left|\det\left(\mathcal{K}\right)\right|}$, $\|\mathcal{K}\|$ is an invariant scalar under the transformation of rotation, reflection, and translation of the coordinate system [12]. For example, under rotational transformation $\mathcal{R}\in\mathrm{Spin}_{p,q}$, we have transformation $\mathcal{K}'=\mathcal{R}\mathcal{K}\mathcal{R}^{-1}$ and $\|\mathcal{R}\|=1$, and then a set of orthogonal basis $\gamma^{ab\cdots c}$ transform into another set of orthogonal basis $\tilde{\gamma}^{ab\cdots c}=\mathcal{R}\gamma^{ab\cdots c}\mathcal{R}^{-1}$. By the multiplication rule of the matrix determinants, we have $\|\mathcal{K}'\|=\|\mathcal{K}\|$ and the modular law $\|\mathcal{K}\mathcal{L}\|=\|\mathcal{K}\|\cdot\|\mathcal{L}\|$. The zero norm set $\left\{\|\mathcal{K}\|=0\right\}$ is a closed set. For the number systems $\mathbb{R},\mathbb{C}$ and $\mathbb{H}$, this definition of norm is the same as the usual length. If relaxing the requirement of zero factor condition $\|\mathcal{K}\|=0\Leftrightarrow\mathcal{N}=0$, we will obtain infinite noncommutative associative division algebras. Since the zero norm set $\left\{\det\left(\mathcal{K}\right)=0\right\}$ is some analytic hypersurfaces similar to light cones, relaxing this requirement has little influence on the algebraic operations and leads to much less problems than abandoning the associativity. Instead, these zero hypersurfaces, like light cones, may have special geometric and physical significance and deserve careful investigation.

If we adopt the metric convention $\eta_{ab}=\mathrm{diag}\left(-1,1,1,1\right)$, we check the similarities and differences between the determinant of the Clifford-Grassmann numbers in $C\ell_{3,1}$ and $C\ell_{1,3}$. Making transformation $\tilde{\gamma}^a=\mathrm{i}\gamma^a$, we transform $C\ell_{1,3}$ into $C\ell_{3,1}$. In this case we have $\tilde{\gamma}^{ab}=-\gamma^{ab}$ and $\tilde{\gamma}^{0123}=\gamma^{0123}$. Then for

Clifford-Grassmann number in $C\ell_{3,1}$

$$\begin{aligned}\tilde{\mathcal{K}} &= sI_4 + A_a\tilde{\gamma}^a + H_{ab}\tilde{\gamma}^{ab} + Q_a\tilde{\gamma}^a\tilde{\gamma}^{0123} + p\tilde{\gamma}^{0123} \\ &= sI_4 + (\mathrm{i}A_a)\gamma^a + (-H_{ab})\gamma^{ab} + (\mathrm{i}Q_a)\gamma^a\gamma^{0123} + p\gamma^{0123},\end{aligned} \tag{2.14}$$

comparing (2.14) with (2.12), we obtain the determinant of $\tilde{\mathcal{K}}$ as

$$\begin{aligned}\det\left(\tilde{\mathcal{K}}\right) &= \left(s^2 + p^2 - \mathbf{A}^2 - \mathbf{Q}^2\right)^2 + 4\left(A_aQ^a\right)^2 - \left(s^2 + p^2\right)^2 \\ &\quad + \left(p^2 - s^2 + \vec{E}^2 - \vec{B}^2\right)^2 + 4\left(\vec{E}\cdot\vec{B} - sp\right)^2 - 4\mathbf{A}^2\mathbf{Q}^2 + \Delta',\end{aligned} \tag{2.15}$$

where $\mathbf{A}^2 = A_aA^a = \vec{A}^2 - A_0^2, \mathbf{Q}^2 = Q_aQ^a = \vec{Q}^2 - Q_0^2$ and

$$\begin{aligned}\Delta' = &-2\left(\vec{E}^2 + \vec{B}^2\right)\left(\mathbf{A}^2 + \mathbf{Q}^2\right) - 4\left(\vec{E}^2 + \vec{B}^2\right)\left(A_0^2 + Q_0^2\right) \\ &+ 4\left[\left(\vec{A}\cdot\vec{E}\right)^2 + \left(\vec{A}\cdot\vec{B}\right)^2 + \left(\vec{Q}\cdot\vec{E}\right)^2 + \left(\vec{Q}\cdot\vec{B}\right)^2\right] \\ &- 8\left[\left(A_0\vec{A} + Q_0\vec{Q}\right)\cdot\left(\vec{B}\times\vec{E}\right)\right] \\ &- 8s\left[\left(\vec{Q}\times\vec{A}\right)\cdot\vec{E} + \left(Q_0\vec{A} - A_0\vec{Q}\right)\cdot\vec{B}\right] \\ &- 8p\left[\left(\vec{Q}\times\vec{A}\right)\cdot\vec{B} - \left(Q_0\vec{A} - A_0\vec{Q}\right)\cdot\vec{E}\right].\end{aligned} \tag{2.16}$$

Comparing $\det\left(\tilde{\mathcal{K}}\right)$ with $\det\left(\mathcal{K}\right)$ we find that, vectors $\mathbf{A}$ and $\mathbf{Q}$ in the determinant appear in quadratic forms. Except for the cross terms (Δ, Δ'), the difference is that the signs of quadratic terms of $\mathbf{A}$ and $\mathbf{Q}$ are reversed, which can be also regarded as a result of different conventions of metric. However, the cross terms (Δ, Δ') lead to different objective effects that cannot be eliminated by a linear transformation of parameters in $\mathbb{R}$. In physics, the sign convention of metric $(+,-,-,-)$ or $(-,+,+,+)$ is just an artificial choice and should not result in objective effects. But $\det(\mathcal{K})$ is not equivalent to $\det\left(\tilde{\mathcal{K}}\right)$ in logic. This situation will produce a serious problem in philosophy. Either the metric sign $(+,-,-,-)$ is essentially not equivalent to $(-,+,+,+)$ and the realistic space-time corresponds to only one case, so our arbitrary choice of the metric sign is wrong. Or the terms of different signs in two determinants do not exist in physics.

If the odd grade terms and even grade terms do not appear in the same Clifford-Grassmann number, then this **objectivity problem** will be avoidable. For example, let

$$\mathcal{L} = A_a\gamma^a + Q_a\gamma^a\gamma^{0123}, \quad \mathcal{N} = sI_4 + H_{ab}\gamma^{ab} + p\gamma^{0123},$$

by (2.12) or (2.15) we have

$$\det(\mathcal{L}) = \left(\mathbf{A}^2 - \mathbf{Q}^2\right)^2 + 4\left(A_aQ^a\right)^2, \tag{2.17}$$

$$\det(\mathcal{N}) = \left(p^2 - s^2 + \vec{E}^2 - \vec{B}^2\right)^2 + 4\left(\vec{E}\cdot\vec{B} - sp\right)^2. \tag{2.18}$$

In this case, the norms have concise and symmetric forms, and are positive semi-definite. The zero norm condition of $\mathcal{L}$ is that the two vectors are orthogonal to each other and of equal length.

Over the real field $\mathbb{R}$, $\left\{\mathcal{N} \in C\ell_{3,0}; \|\mathcal{N}\| \neq 0\right\}$ constitutes a class of associative "octonion" or "biquaternion". For the case of $C\ell_{0,3}$, $\mathrm{i}\sigma^a$ cannot form a suitable basis, because imaginary unit i appears in the determinant, so we need a higher-order representation matrix (2.10). In this case we have the Clifford-Grassmann number as

$$\mathcal{N} = sI_4 + E_a\gamma^a + B_a\gamma^a\gamma^{123} + p\gamma^{123}, \quad \left(s, p, E_a, B_a \in \mathbb{F}\right). \tag{2.19}$$

Its determinant is given by

$$\det\left(\mathcal{N}\right) = \left(p^2 + s^2 + \vec{E}^2 + \vec{B}^2\right)^2 - 4\left(\vec{E}\cdot\vec{B} + sp\right)^2. \tag{2.20}$$

(2.20) is different from (2.18), so they are different hypercomplex numbers in logic. Over real field $\mathbb{R}$, we have $\det\left(\mathcal{N}\right) \ge 0$. For case

$$\left\{p = s, \vec{E} = \vec{B}\right\} \text{ or } \left\{p = -s, \vec{E} = -\vec{B}\right\}, \tag{2.21}$$

we have $\det\left(\mathcal{N}\right) = 0$. This is another kind of "octonion".

Similarly, we can analyze other hypercomplex numbers in real $C\ell_{p,q}$. For $C\ell_{0,2}$ we have

$$\mathbf{Q} = sI_4 + E_a\gamma^a + p\gamma^{12}, \quad \det\left(\mathbf{Q}\right) = \left(s^2 + p^2 + \vec{E}^2\right)^2,$$

where $s, p, E_a \in \mathbb{R}, a = 1, 2$. This is a quaternion. For $C\ell_{0,1}$ we have

$$\mathbf{Q} = sI_2 + p\mathrm{i}\sigma^a, \quad \det\left(\mathbf{Q}\right) = s^2 + p^2,$$

where $a \in \{1, 2, 3\}$. This is a complex number. For $C\ell_{1,0}$, we have

$$\mathbf{Q} = sI_2 + p\sigma^a, \quad \det\left(\mathbf{Q}\right) = s^2 - p^2. \tag{2.22}$$

This is a "hyperbolic number".

The relations between several simple Clifford algebras and hypercomplex numbers are discussed, and the matrix representation (11) and the determinant (12) play a key role. Obviously, the similar discussion is also suitable for the general case of $C\ell_{p,q}$, and infinite new hypercomplex numbers of interesting can be constructed. The matrix representations of the Clifford algebras carry more important information than the abstract definitions, such as the definitions of inverse element and norm.

3. Cyclic and Commutative Number System

If the zero-factor condition of $\|a\| = 0 \Leftrightarrow a = 0$ is relaxed, many new division algebras with associativity can be defined by matrix algebra. For example, we can construct an interesting series of commutative "n-ary cyclic numbers" as follows. Denote n square matrices by

$$\mathbf{e}_0 = I_n, \quad \mathbf{e}_m = \begin{pmatrix} 0 & I_{n-m} \\ I_m & 0 \end{pmatrix}, \quad 1 \le m \le n-1. \tag{3.1}$$

$\mathbf{e}_k$ constitutes a set of bases of the n-dimensional real vector space $\mathbb{R}^n$. According to the matrix algebra we have multiplication rules of basis vectors as follows

$$\mathbf{e}_j\mathbf{e}_k=\mathbf{e}_k\mathbf{e}_j=\mathbf{e}_m,\quad m=j+k \bmod n, \tag{3.2}$$

$$\mathbf{e}_m=\mathbf{e}_1^m,\quad \mathbf{e}_1^n=I_n,\quad 0\le m<n. \tag{3.3}$$

Thus, $\{\mathbf{e}_m \mid m=0,1,\cdots,n-1\}$ is a matrix representation of n-element cyclic group.

For n-ary number

$$A_n=\sum_{k=0}^{n-1}a_k\mathbf{e}_k,\quad (a_0,a_1,\cdots,a_{n-1})\in\mathbb{R}^n, \tag{3.4}$$

the reciprocal is defined as the inverse matrix A_n^{-1} of the matrix A_n. In this way, it is easy to verify that n-ary numbers form a commutative number system in the domain $\{\det(A_n)\neq 0\}$ according to matrix algebra. If $n=2$, the set $\{A_2=a\mathbf{e}_0+b\mathbf{e}_1\}$ is the hyperbolic numbers (2.22). In the case $n=3$, we have

$$\det(A_3)=\frac{1}{2}(a_0+a_1+a_2)\left[(a_0-a_1)^2+(a_1-a_2)^2+(a_2-a_1)^2\right], \tag{3.5}$$

$$\{\det(A_3)=0\}=\{a_0+a_1+a_2=0\}\cup\{a_0=a_1=a_2\}. \tag{3.6}$$

In the case $n=4$, we have

$$\det(A_4)=\left[(a_0-a_2)^2+(a_1-a_3)^2\right]\left[(a_0+a_2)^2-(a_1+a_3)^2\right], \tag{3.7}$$

$$\{\det(A_4)=0\}=\{a_0+a_1+a_2+a_3=0\}\cup\{a_0=a_2,a_1=a_3\}\cup\{a_0+a_2=a_1+a_3\}. \tag{3.8}$$

In general case, we have the following theorem.

Theorem 2. *Let* $w=\exp\left(\frac{2\pi}{n}\mathrm{i}\right)$, *we have* $w^n=1$ *and* $\bar{w}=w^{-1}$. *For n-ary number* (3.4), *denoting*

$$R_k=a_0+a_1w^k+a_2w^{2k}+\cdots+a_{n-1}w^{(n-1)k},\ k=0,1,2,\cdots,\left[\frac{n}{2}\right], \tag{3.9}$$

then the determinant has the following factorization,

$$\det(A_{2m+1})=(R_1\bar{R}_1)(R_2\bar{R}_2)\cdots(R_m\bar{R}_m)(a_0+a_1+\cdots+a_{2m}), \tag{3.10}$$

$$\det(A_{2m+2})=(R_1\bar{R}_1)(R_2\bar{R}_2)\cdots(R_m\bar{R}_m)\left[(a_0+a_2+\cdots)^2-(a_1+a_3+\cdots)^2\right]. \tag{3.11}$$

Proof. Let $W_k=\mathrm{diag}\left(1,w^k,w^{2k},\cdots,w^{(n-1)k}\right)$, we have $|\det(W_k)|=1$, so $|\det(A_nW_k)|=|\det(A_n)|$. Adding up the five columns of the matrix A_nW_k, we find that the determinant $\det(A_n)$ has a factor R_k. Since $\det(A_n)$ is a real number, so $\bar{R}_k$ is also a factor of $\det(A_n)$ if R_k is a complex. Since all R_k are different numbers for arbitrary $\{a_j\}$, thus we proved the theorem.

By the theorem we find

$$\{\det(A_n)=0\}\supset\{a_0+a_1+\cdots+a_{n-1}=0\}\cup\{a_0=a_1=\cdots=a_{n-1}\}. \tag{3.12}$$

The zero norm set $\{\det(A_n)=0\}$ are some lower dimensional surfaces in $\mathbb{R}^n$. In contrast to the usual number system $\mathbb{C}$ and $\mathbb{Q}$, the determinants are not positive definite. In the case $n=2N$, by (3.3) we find the even grade numbers $\{\tilde{A}=a_0I_n+a_2\mathbf{e}_2+\cdots+a_{n-2}\mathbf{e}_{n-2}\}$ form a subalgebra.

4. Some Applications in Physics

4.1. Geometric Meaning of Hypercomplex Numbers

In the above section we discussed the Clifford algebra and the relation to the hypercomplex numbers from an algebraic perspective. In fact, we can also discuss Clifford algebras from a geometric perspective [6] [11]. By introducing the inner, exterior and geometric products of the vectors, Clifford algebra accurately and faithfully describes the intrinsic properties of a space-time. The concepts and methods describing the flat space-time and curved one are completely the same, thus Clifford algebra becomes a unified language and standard tool for dealing with geometric and physical problems. We show how the Clifford algebra implements these virtues by examples.

In geometry and physics, we often use curvilinear coordinate systems or study problems in curved space-time. In this case, the line element vector in the neighborhood of a given point $\mathbf{x}$ is described by

$$\mathrm{d}\mathbf{x} = \gamma_\mu \mathrm{d}x^\mu = \gamma^\mu \mathrm{d}x_\mu = \gamma_a \delta X^a = \gamma^a \delta X_a, \tag{4.1}$$

in which γ_a is used as the local orthonomal basis in tangent space, and γ^a is the coframe. The **distance** $\mathrm{d}s = |\mathrm{d}\mathbf{x}|$ and **oriented volume** $\mathrm{d}V_k$ are respectively defined as

$$\mathrm{d}\mathbf{x}^2 = \frac{1}{2}\left(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu\right) \mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ab} \delta X^a \delta X^b, \tag{4.2}$$

$$\mathrm{d}V_k = \mathrm{d}\mathbf{x}_1 \wedge \mathrm{d}\mathbf{x}_2 \wedge \cdots \wedge \mathrm{d}\mathbf{x}_k = \gamma_{\mu\nu\cdots\omega} \mathrm{d}x_1^\mu \mathrm{d}x_2^\nu \cdots \mathrm{d}x_k^\omega, \ (1 \le k \le 4), \tag{4.3}$$

where $\eta_{ab} = \gamma_a \cdot \gamma_b = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric and $g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu$ is the Riemann metric.

$$\gamma_{\mu\nu\cdots\omega} = \gamma_\mu \wedge \gamma_\nu \wedge \cdots \wedge \gamma_\omega \in \Lambda^k\left(\mathbb{M}^{1,3}\right)$$

is the Grassmann basis, which represents the unit and direction of k dimensional volume $\mathrm{d}V_k$. In the Cartesian coordinate system, the norm $\|\gamma_{ab\cdots c}\| = 1$. In the curved space-time, the Clifford-Grassmann number has the following form

$$\mathcal{C} = C_0 I + C_\mu \gamma^\mu + C_{\mu\nu}\gamma^{\mu\nu} + \cdots + C_{12\cdots n}\gamma^{12\cdots n}, \ \left(\forall C_{\mathbf{k}}(\mathbf{x}) \in \mathbb{R}\right) \tag{4.4}$$

The geometric meanings of elements $\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{y}$ and $\mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y}$ are given in **Figure 1**.

In what follows, we mainly discuss 1 + 3 dimensional space-time. For clearness, it is necessary to clarify the conventions and notations frequently used in the following. We take $c = 1$ as unit of speed. For indices, we use the Greek characters for curvilinear coordinates, Latin characters for local Minkowski coordinates and $\{i, j, k, l\}$ for spatial indices. The Pauli and Dirac matrices in curvilinear coordinate system are given by

$$\sigma^\mu = f^\mu_a \sigma^a, \quad \tilde{\sigma}_\mu = f_\mu^{\ a} \tilde{\sigma}_a, \quad \gamma_\mu = f_\mu^{\ a}\gamma_a, \quad \gamma^\mu = f^\mu_a \gamma^a, \tag{4.5}$$

$$\gamma^a = \begin{pmatrix} 0 & \tilde{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vartheta^5 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}. \tag{4.6}$$

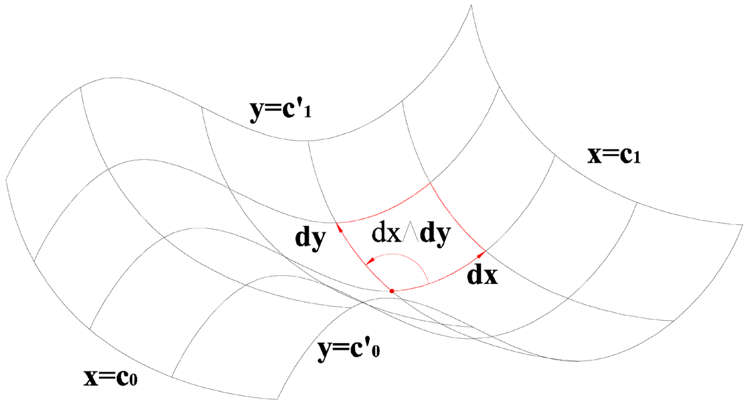

Figure 1. The geometric meanings of vectors $\mathrm{d}\mathbf{x},\mathrm{d}\mathbf{y}$ and $\mathrm{d}\mathbf{x}\wedge\mathrm{d}\mathbf{y}$.

where $a,\mu\in\{0,1,2,3\}$, f_a^μ, f_μ^a are the **frame coefficients** satisfying

$$f_\mu^{\ a} f_\nu^{\ b}\eta_{ab}=g_{\mu\nu},\quad f_a^\mu f_b^\nu\eta^{ab}=g^{\mu\nu}. \tag{4.7}$$

For frame coefficients, the first index is always for curvilinear coordinate, and the second index for Minkowski index of the tangent space-time.

In equivalent sense, γ^a forms the unique representation for generators of $C\ell_{1,3}$. Since the Clifford algebra is isomorphic to matrix algebra, we need not distinguish matrix γ^a with tetrad γ^a. Thus we have **Clifford relations**

$$\begin{cases}\gamma_a\gamma_b+\gamma_b\gamma_a=2\eta_{ab}, & \gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu},\\ \gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}, & \gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}.\end{cases} \tag{4.8}$$

In physics and geometry, the multiplication of variables is usually Clifford products, but only by projecting the variables onto Grassmann basis, their physical and geometric meanings become clear. Therefore, in Clifford algebra the relations between Clifford product such as $\gamma_a\gamma_b$ and Grassmann product $\gamma_a\wedge\gamma_b$ become important. For Grassmann basis, it is easy to check the following relations between Clifford products and Grassmann products [6]

$$\gamma^\mu\cdot\gamma^{\theta_1\theta_2\cdots\theta_k}=g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k}-g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k}+\cdots+(-1)^{k+1}g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}},$$

$$\gamma^{\theta_1\theta_2\cdots\theta_k}\cdot\gamma^\mu=(-1)^{k+1}g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k}+(-1)^k g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k}+\cdots+g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}}.$$

$$\gamma^\mu\gamma^{\theta_1\theta_2\cdots\theta_k}=\gamma^\mu\cdot\gamma^{\theta_1\theta_2\cdots\theta_k}+\gamma^{\mu\theta_1\cdots\theta_k}, \tag{4.9}$$

$$\gamma^{\theta_1\theta_2\cdots\theta_k}\gamma^\mu=\gamma^{\theta_1\theta_2\cdots\theta_k}\cdot\gamma^\mu+\gamma^{\theta_1\cdots\theta_k\mu}. \tag{4.10}$$

$$\gamma_{a_1a_2\cdots a_{n-1}}=\varepsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}\gamma^{a_n},$$

$$\gamma_{a_1a_2\cdots a_{n-2}}=\frac{1}{2!}\varepsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}\gamma^{a_{n-1}a_n},$$

$$\gamma_{a_1a_2\cdots a_{n-k}}=\frac{1}{k!}\varepsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}\gamma^{a_{n-k+1}\cdots a_n}.$$

Similarly, we can define **multi-inner product** $\odot^k$ of Clifford algebra as follows,

$$\gamma^{\mu\nu}\odot\gamma^{\alpha\beta}=g^{\mu\beta}\gamma^{\nu\alpha}-g^{\mu\alpha}\gamma^{\nu\beta}+g^{\nu\alpha}\gamma^{\mu\beta}-g^{\nu\beta}\gamma^{\mu\alpha},$$

$$\gamma^{\mu\nu} \odot^2 \gamma^{\alpha\beta} = g^{\mu\beta} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta}, \quad \gamma^{\mu\nu} \odot^k \gamma^{\alpha\beta} = 0, (k > 2). \quad \cdots$$

$$\gamma^{\mu\nu}\gamma^{\alpha\beta} = \gamma^{\mu\nu} \odot^2 \gamma^{\alpha\beta} + \gamma^{\mu\nu} \odot \gamma^{\alpha\beta} + \gamma^{\mu\nu\alpha\beta}. \tag{4.11}$$

More geometric meanings of hypercomplex numbers can be found in [6] [11].

4.2. Dynamics of Vector and Maxwell Equations

The dynamical equations in physics are usually in form of Clifford-Grassmann numbers, we can directly establish some dynamical equation by Clifford algebra. Now we take vector field $\mathbf{A} = \gamma^\mu A_\mu(\mathbf{x})$ as example to show how Clifford algebra works in physics. Denote covariant differential operator by $\mathbf{D} = \gamma^\alpha \nabla_\alpha$, then we have Clifford calculus [6] [11]

$$\begin{aligned} \mathbf{DA} &= \gamma^\alpha \gamma^\beta \nabla_\alpha A_\beta = \left(g^{\alpha\beta} + \gamma^{\alpha\beta}\right) \nabla_\alpha A_\beta \\ &= \nabla_\mu A^\mu + \frac{1}{2}\gamma^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \equiv H + \frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu}. \end{aligned} \tag{4.12}$$

Clearly, $\mathbf{DA} \in \Lambda^0 \cup \Lambda^2 \subset C\ell_{1,3}$. By relations between products

$$\gamma^\alpha \gamma^{\mu\nu} = g^{\alpha\mu}\gamma^\nu - g^{\alpha\nu}\gamma^\mu + \gamma^{\alpha\mu\nu}, \quad \gamma^{\mu\nu}\gamma^\alpha = g^{\alpha\nu}\gamma^\mu - g^{\alpha\mu}\gamma^\nu + \gamma^{\alpha\mu\nu}, \tag{4.13}$$

we have

$$\begin{aligned} \mathbf{D}^2\mathbf{A} &= \gamma^\alpha \partial_\alpha H + \frac{1}{2}\gamma^\alpha \gamma^{\mu\nu} \nabla_\alpha F_{\mu\nu} \\ &= \gamma^\alpha \partial_\alpha H + g^{\alpha\mu}\gamma^\nu \nabla_\alpha F_{\mu\nu} + \frac{1}{2}\gamma^{\alpha\mu\nu}\nabla_\alpha F_{\mu\nu} \\ &= \gamma_\alpha \left(\partial^\alpha H - \nabla_\mu F^{\alpha\mu}\right) + \frac{\mathrm{i}\gamma_\varpi \gamma^5}{2\sqrt{g}}\left(\varepsilon^{\alpha\mu\nu\varpi}\nabla_\alpha F_{\mu\nu}\right). \end{aligned} \tag{4.14}$$

Therefore $\mathbf{D}^2\mathbf{A} \in \Lambda^1 \cup \Lambda^3$ and $\mathbf{D}^3\mathbf{A} \in \Lambda^0 \cup \Lambda^2 \cup \Lambda^4$. Since the term $\gamma^{\alpha\mu\nu}$ automatically vanishes in $\mathbf{D}^2\mathbf{A}$, namely $\varepsilon^{\alpha\mu\nu\varpi}\nabla_\alpha F_{\mu\nu} = \varepsilon^{\alpha\mu\nu\varpi}\nabla_\alpha \nabla_\mu A_\nu \equiv 0$, and the dynamical equations in physics should be closed in $\mathbf{D}^2$, so we have

$$\mathbf{D}^2\mathbf{A} = -b^2\mathbf{A} + e\mathbf{q} = \gamma_\alpha\left(-b^2 A^\alpha + e q^\alpha\right). \tag{4.15}$$

The above equations are the dynamics for a 4 dimensional vector in the form of Clifford-Grassmann numbers.

Comparing (4.15) with (4.14), we obtain the first-order dynamics of the vector $\mathbf{A}$

$$\begin{cases} \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}, \quad \nabla_\alpha A^\alpha = H \\ \partial^\mu H + \nabla_\nu F^{\nu\mu} + b^2 A^\mu = e q^\mu. \end{cases} \tag{4.16}$$

All of the variables have covariant forms and clear physical significance. If the Lorenz gauge condition $H = 0$ holds, for the long distance interaction $b = 0$, (4.16) gives the complete Maxwell equations with current conservation laws $q^\mu_{;\mu} = 0$. The above derivation shows the hypercomlex structure of physical variables and the convenience of Clifford algebras.

4.3. Eigen Equation in the Form of Clifford Algebra

We now examine the relation of the Clifford algebra to the spinor field equations. In the curvilinear coordinate system, Dirac equation with electromagnetic potential and nonlinear potential is given by [6] [13] [14]

$$\gamma^{\mu}\left(\hat{p}_{\mu}+\frac{1}{2}\hbar\gamma^{5}\Omega_{\mu}\right)\phi+\mathrm{i}\gamma^{5}F_{\vartheta}\phi=\left(mc-F_{\gamma}\right)\phi, \tag{4.17}$$

in which the parameters and operators are defined as

$$\hat{p}_{\mu}=\mathrm{i}\hbar\left(\partial_{\mu}+\Upsilon_{\mu}\right)-eA_{\mu},\quad \Upsilon_{\mu}=\frac{1}{2}f_{a}^{\ \nu}\left(\partial_{\mu}f_{\nu}^{\ a}-\partial_{\nu}f_{\mu}^{\ a}\right),$$

$$\Omega^{\alpha}=-\frac{1}{2}\left(\varepsilon^{dabc}f_{d}^{\ \alpha}f_{a}^{\ \mu}f_{b}^{\ \nu}\right)\partial_{\mu}f_{\nu}^{\ e}\eta_{ce}=\frac{1}{4\sqrt{g}}\varepsilon^{\alpha\mu\nu\omega}\eta_{ab}f_{\omega}^{\ a}\left(\partial_{\nu}f_{\mu}^{\ b}-\partial_{\mu}f_{\nu}^{\ b}\right).$$

Υ_{μ} is Keller connection, Ω_{μ} is Gu-Nester potential which is a pseudo vector. If the metric can be diagonalized, we have $\Omega_{\mu}\equiv 0$. The nonlinear potential $F=F\left(\breve{\gamma},\breve{\vartheta}\right)$ is a function of the following quadratic scalar and pseudo scalar

$$\breve{\gamma}=\phi^{+}\gamma^{0}\phi,\quad \breve{\vartheta}=\phi^{+}\vartheta^{5}\phi,\quad F_{\gamma}=\frac{\partial F}{\partial\breve{\gamma}},\quad F_{\vartheta}=\frac{\partial F}{\partial\breve{\vartheta}}. \tag{4.18}$$

the coefficients in (4.17) belong to $\Lambda^{0}\cup\Lambda^{1}\cup\Lambda^{3}\cup\Lambda^{4}\subset C\ell_{1,3}$.

To get the Hamiltonian formalism of (4.17), the time t must be the global cosmic time, that is to say, the Hamiltonian formalism can be clearly expressed only in Gu's natural coordinate system (NCS) [15]

$$\mathrm{d}s^{2}=g_{tt}\mathrm{d}t^{2}-\bar{g}_{kl}\mathrm{d}x^{k}\mathrm{d}x^{l},\quad \mathrm{d}\tau=\sqrt{g_{tt}}\mathrm{d}t=f_{t}^{\ 0}\mathrm{d}t,\quad \mathrm{d}V=\sqrt{\bar{g}}\mathrm{d}^{3}x. \tag{4.19}$$

In which $\mathrm{d}s$ is the 4-d length of line element, $\mathrm{d}\tau$ is the Newton's absolute cosmic time element and $\mathrm{d}V$ is the absolute volume element of space at time t. The NCS generally exists and the global simultaneity is unique. Only in NCS can we clearly establish the Hamiltonian formalism and calculate the Noether charges of a spinor.

The NCS is different from the Gaussian coordinate system that is valid only in the neighborhood of the initial Cauchy hypersurface. The isodistant translating hypersurface will deform soon, so that the metric $\mathrm{d}s^{2}=\mathrm{d}t^{2}+g_{kl}\mathrm{d}x^{k}\mathrm{d}x^{l}$ becomes invalid. The NCS is also different from the Einstein's lift moving along a geodesic, namely the co-moving coordinate system, as this requires the lift to be an infinitesimal volume in curved space-time. While NCS holds unconditionally and globally, and its time is objective cosmic time. g_{tt} represents gravity and cannot be merged into the time coordinate t.

In NCS, to lift and lower the index of a vector means

$$\Upsilon^{0}=g^{00}\Upsilon_{0},\quad \Upsilon^{k}=-g^{kl}\Upsilon_{l},\quad g_{kl}g^{ln}=\delta_{k}^{n}. \tag{4.20}$$

Then the eigen equation of (4.17) can be rewritten in the following Hamiltonian formalism

$$\mathrm{i}\alpha^{0}\left(\partial_{t}+\Upsilon_{t}\right)\phi=\hat{H}\phi, \tag{4.21}$$

$$\hat{H}=-\alpha^{k}\hat{p}_{k}+eA_{0}\alpha^{0}+\left(mc-F_{\gamma}\right)\gamma^{0}-F_{\vartheta}\vartheta^{5}-\Omega_{\mu}\hat{S}^{\mu}, \tag{4.22}$$

where $\hat{H}$ is the Hamiltonian in curved space-time, and $\left(\alpha^{\mu},\hat{S}^{\mu}\right)$ are respectively current and spin operators defined by

$$\alpha^{\mu}\equiv\gamma^{0}\gamma^{\mu}=\mathrm{diag}\left(\sigma^{\mu},\tilde{\sigma}^{\mu}\right),\quad \hat{S}^{\mu}\equiv\frac{1}{2}\hbar\alpha^{\mu}\gamma^{5}=\frac{1}{2}\hbar\,\mathrm{diag}\left(\sigma^{\mu},-\tilde{\sigma}^{\mu}\right). \tag{4.23}$$

$\vec{S}\in\Lambda^{3}$ is the usual spin of the spinor.

To get the eigen state of energy of a particle, we must separate the drifting motion of the particle by a local Lorentz transformation to its comoving coordinate system. Thus we get eigen equation

$$\hat{H}\phi=\mathrm{i}\alpha^{0}\left(\partial_{t}+\Upsilon_{t}\right)\phi=E\phi,\quad \left(E\in\mathbb{R}\right). \tag{4.24}$$

In (4.24) $\hat{H}\in C\ell_{4,0}$ with the generators $\left(\gamma^{0},\alpha^{k}\right)$, so we introduce the generators of $C\ell_{4,0}$ as follows,

$$\vartheta^{a}=\left(\gamma^{0},\alpha^{k}\right),\quad \vartheta_{a}=\delta_{ab}\vartheta^{b},\quad \vartheta^{a}\vartheta^{b}+\vartheta^{b}\vartheta^{a}=2\delta^{ab}, \tag{4.25}$$

$$\vartheta^{\mu}=t_{a}^{\mu}\vartheta^{a},\quad \vartheta_{\mu}=t_{\mu}^{\ a}\vartheta_{a},\quad \vartheta_{\mu}\vartheta_{\nu}+\vartheta_{\nu}\vartheta_{\mu}=2g_{\mu\nu}=2\mathrm{diag}\left(1,g_{kl}\right), \tag{4.26}$$

$$t_{\mu}^{\ a}t_{\nu}^{\ b}\delta_{ab}=g_{\mu\nu},\quad t_{\mu}^{\ a}t_{b}^{\mu}=\delta_{b}^{a},\quad t_{\mu}^{\ a}t_{a}^{\nu}=\delta_{\mu}^{\nu}. \tag{4.27}$$

The complete bases of $C\ell_{4,0}$ and their relations are given by

$$I,\quad \vartheta^{a},\quad \vartheta^{ab}\equiv\vartheta^{a}\wedge\vartheta^{b},\quad \vartheta^{abc}=-\varepsilon^{abcd}\vartheta_{d}\vartheta^{5},\quad \vartheta^{0123}=\vartheta^{5}. \tag{4.28}$$

By (4.25)-(4.28), the Hamiltonian can be rewritten in Clifford algebra $C\ell_{4,0}$ as

$$\hat{H}=eA_{0}\sqrt{g^{00}}+\vartheta^{\mu}\hat{P}_{\mu}+\frac{\mathrm{i}\hbar}{2}\Omega_{\mu}\vartheta^{0\mu}\vartheta^{5}+\frac{\mathrm{i}\hbar}{2}\sqrt{g^{00}}\Omega_{0}\vartheta^{0}\vartheta^{5}-F_{\vartheta}\vartheta^{5}, \tag{4.29}$$

in which the Clifford numbers

$$A_{0}\in\Lambda^{0},\ \hat{P}_{\mu}\in\Lambda^{1},\ \vec{\Omega}\in\Lambda^{2},\ \Omega_{0}\in\Lambda^{3},\ F_{\vartheta}\in\Lambda^{4}. \tag{4.30}$$

We find the grade of some quantities in hyperbolic system γ^{μ} has been changed in elliptic system ϑ^{μ}. For example, $\left(\vec{\Omega},\vec{S}\right)\in\Lambda^{3}$ in system γ^{μ} have been converted into $\left(\vec{\Omega},\vec{S}\right)\in\Lambda^{2}$ in system ϑ^{μ}. The 4-vector momentum $\hat{P}_{\mu}$ is redefined as

$$\hat{P}_{\mu}=\left(mc-F_{\gamma},-\mathrm{i}\hbar\left(\partial_{k}+\Upsilon_{k}\right)-eA_{k}\right). \tag{4.31}$$

In $\hat{P}_{\mu}$ the sign of eA_{k} should be changed, this is because the Minkowski metric $\eta^{kk}=-1$ in $C\ell_{1,3}$ has been converted into $\delta^{kk}=1$ in $C\ell_{4,0}$. So we have $\hat{P}^{k}\phi=\hat{P}_{k}\phi\to mv_{k}\phi$.

4.4. Integrable Conditions for Dirac Equations

The common method to solve the eigensolution of (4.24) is the method of separation variables, which is based on the following well-known theorem [16].

Theorem 3. *For two linear Hermitian operators* A,B *on a vector space with finite degeneracy, if they commute with each other* $\left[A,B\right]=0$, *then they have common eigenvectors.*

For any Hamiltonian operator $\hat{H}$, if we can construct the following Hermitian operators set [17],

$$\hat{H}_{1}=H_{1}\left(\partial_{1}\right),\quad \hat{H}_{2}=H_{2}\left(\partial_{1},\partial_{2}\right),\quad \cdots,\quad \hat{H}=H_{n}\left(\partial_{1},\partial_{2},\cdots,\partial_{n}\right), \tag{4.32}$$

which form an Abelian Lie algebra

$$\left[\hat{H}_j,\hat{H}_k\right]=0,\quad \left(j,k=1,\cdots,n\right), \tag{4.33}$$

then (4.33) forms the integrable conditions of the eigen equation $\hat{H}\phi=E\phi$, and the eigen solutions can be completely solved by means of separating variables. The operators of observables in such sets have a joint system of eigenstates, and their eigenvalues determine quantum numbers of each state. It is well known that in the relativistic Coulomb problem the Hamiltonian and the angular momentum do not generate a complete set of commuting observables, and an additional operator is needed to complete the set. So the existence of complete commutative operator chain forms the integrable condition for Dirac equation or Schrödinger equation.

Since (4.29) is a hypercomplex operator in $C\ell_{4,0}$, the operator equation (4.33) can be converted into an algebraic equation in $C\ell_{4,0}$. The derived auxiliary commutative operator is a hypercomplex operator, so each term is a covariant tensor and has special physical significance. Now we take spherical coordinate system as example to show the concepts and the advantage of Clifford algebra. For static potential $A_0=A_0\left(r,\theta\right)$ and magnetic field $\vec{B}=B_z\vartheta^z\vartheta^5$, we have $\vec{A}=A\left(r,\theta\right)\vartheta^\varphi$ and

$$\left(g_{\mu\nu}\right)=\mathrm{diag}\left(1,1,r^2,r^2\sin^2\theta\right),\quad \left(t_\mu^a\right)=\mathrm{diag}\left(1,1,r,r\sin\theta\right). \tag{4.34}$$

$$\Upsilon_\mu=\left(0,\frac{1}{r},\frac{1}{2}\cot\theta,0\right),\quad \Omega_\mu=0. \tag{4.35}$$

$$\hat{P}_\mu=\left(mc-F_\gamma,-\mathrm{i}\hbar\left(\partial_r+\frac{1}{r}\right),-\mathrm{i}\hbar\left(\partial_\theta+\frac{1}{2}\cot\theta\right),-\mathrm{i}\hbar\partial_\varphi-eA\right). \tag{4.36}$$

In the case of diagonal metric we have $\Omega^\mu=0$, and the Hamiltonian becomes

$$\hat{H}=eA_0+\vartheta^\mu\hat{P}_\mu-F_\vartheta\vartheta^5. \tag{4.37}$$

Then we simply have

$$\hat{L}_z=-\mathrm{i}\hbar\partial_\varphi,\quad \left[\hat{H},\hat{L}_z\right]=0. \tag{4.38}$$

Different from the usual operator $\hat{J}_z=-\mathrm{i}\hbar\partial_\varphi+\frac{1}{2}\hbar S_3$, the spin $\hat{S}$ vanishes in the operator, this is because the spinor ϕ has been implicitly made the following spin-$\frac{1}{2}$ transformation [6],

$$\psi=\Pi\psi',\quad \tilde{\psi}=\Pi^*\tilde{\psi}', \tag{4.39}$$

in which

$$\Pi=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{i}\exp\left[-\frac{\mathrm{i}}{2}\left(\theta+\varphi+\frac{\pi}{2}\right)\right] & \exp\left[\frac{\mathrm{i}}{2}\left(\theta-\varphi+\frac{\pi}{2}\right)\right]\\ -\exp\left[-\frac{\mathrm{i}}{2}\left(\theta-\varphi+\frac{\pi}{2}\right)\right] & -\mathrm{i}\exp\left[\frac{\mathrm{i}}{2}\left(\theta+\varphi+\frac{\pi}{2}\right)\right]\end{pmatrix}$$

and $\Pi^*=\Pi$.

Now we look for the additional Hermitian commutator

$$\hat{T}\left(\partial_\theta,\partial_\varphi\right)=T^0+T^\theta\hat{P}_\theta+T^\varphi\hat{P}_\varphi, \tag{4.40}$$

in which $\left(T^0,T^\theta,T^\varphi\right)$ are all Hermitian matrices. By $\left[\hat{L}_z,\hat{T}\right]=0$, we get $\partial_\varphi\hat{T}=0$. This means $\left(T^0,T^\theta,T^\varphi\right)$ should be independent of φ. We examine $\left[\hat{H},\hat{T}\right]=0$. For any eigen solution ϕ, we have $\hat{P}_\varphi\phi=\left(m_z\hbar-eA\right)\phi$. This means $\hat{P}_\varphi$ can be regarded as an ordinary function for any eigen solution. Then we have integrable condition as

$$\begin{aligned}0=\left[\hat{H},\hat{T}\right]&=\left[\hat{H},\hat{T}\right]_A-\mathrm{i}\hbar\vartheta^\mu\partial_\mu\hat{T}+\mathrm{i}\hbar T^\theta\partial_\theta\hat{H}\\&=\left[\vartheta^r,T^\theta\right]\hat{P}_r\hat{P}_\theta+\left[\vartheta^\theta,T^\theta\right]\hat{P}_\theta^2+\left[\vartheta^r,\hat{T}\right]\hat{P}_r+\cdots,\end{aligned} \tag{4.41}$$

where $\left[\hat{H},\hat{T}\right]_A$ stands for algebraic commutation taking ∂_μ as ordinary numbers. The notation $\left[\hat{H},\hat{T}\right]_A$ is convenient for calculation, especially for programming. Then we get integrable conditions

$$\left[\vartheta^r,T^\theta\right]=\left[\vartheta^\theta,T^\theta\right]=\left[\vartheta^r,\hat{T}\right]=\cdots=0. \tag{4.42}$$

By Hermiticity of $\hat{T}$ and (4.28), T^θ can be represented by the following Clifford algebra

$$T^\theta=X+Y_\mu\vartheta^\mu+\mathrm{i}Z_{\mu\nu}\vartheta^{\mu\nu}+\mathrm{i}U_\mu\vartheta^\mu\vartheta^{0r\theta\varphi}+V\vartheta^{0r\theta\varphi}, \tag{4.43}$$

in which all coefficients are real functions of (r,θ) and $Z_{\mu\nu}=-Z_{\nu\mu}$. Then by (4.9)-(4.11) and (4.34), we have

$$\begin{aligned}\frac{1}{2}\left[\vartheta^r,T^\theta\right]&=Y_\mu\vartheta^{r\mu}+\mathrm{i}Z_{\mu\nu}\vartheta^r\odot\vartheta^{\mu\nu}+\mathrm{i}U_\mu\left(\vartheta^r\odot\vartheta^\mu\right)\vartheta^{0r\theta\varphi}+V\vartheta^r\odot\vartheta^{0r\theta\varphi}\\&=Y_\mu\vartheta^{r\mu}+2\mathrm{i}Z_{r\nu}g^{rr}\vartheta^\nu+\mathrm{i}U_rg^{rr}\vartheta^{0r\theta\varphi}-Vg^{rr}\vartheta^{0\theta\varphi}.\end{aligned} \tag{4.44}$$

By $\left[\vartheta^r,T^\theta\right]=0$ we get

$$Y_0=Y_\theta=Y_\varphi=Z_{0r}=Z_{r\theta}=Z_{r\varphi}=U_r=V=0.$$

$$T^\theta=X+Y_r\vartheta^r+\mathrm{i}Z_{0\theta}\vartheta^{0\theta}+\mathrm{i}Z_{0\varphi}\vartheta^{0\varphi}+\mathrm{i}Z_{\theta\varphi}\vartheta^{\theta\varphi}+\mathrm{i}\left(U_0\vartheta^0+U_\theta\vartheta^\theta+U_\varphi\vartheta^\varphi\right)\vartheta^{0r\theta\varphi}.$$

By $\left[\vartheta^\theta,T^\theta\right]=0$ we get

$$\frac{1}{2}\left[\vartheta^\theta,T^\theta\right]=Y_r\vartheta^{\theta r}+2\mathrm{i}\left(-Z_{0\theta}g^{\theta\theta}\vartheta^0-Z_{0\varphi}\vartheta^{0\theta\varphi}+Z_{\theta\varphi}g^{\theta\theta}\vartheta^\varphi\right)+\mathrm{i}U_\theta g^{\theta\theta}\vartheta^{0r\theta\varphi}=0,$$

$$T^\theta=X+\mathrm{i}\left(U_0\vartheta^0+U_\varphi\vartheta^\varphi\right)\vartheta^{0r\theta\varphi}=X+\mathrm{i}U_0\vartheta^{r\theta\varphi}-\mathrm{i}U_\varphi g^{\varphi\varphi}\vartheta^{0r\theta}.$$

Similarly, by Clifford calculus of (4.41) we finally derive the integrable conditions as

$$\hat{T}=\mathrm{i}r\left(\vartheta^{0r\theta}\hat{P}_\theta+\vartheta^{0r\varphi}\hat{P}_\varphi\right). \tag{4.45}$$

$$A_0=V(r),\quad A=A(\theta),\quad \partial_\theta F_\gamma=0,\quad F_\vartheta=0. \tag{4.46}$$

According to

$$A=A(\theta),\quad r\vartheta^{0r\theta}=\vartheta^{012},\quad r\vartheta^{0r\varphi}=\frac{1}{\sin\theta}\vartheta^{013}, \tag{4.47}$$

we find $\hat{T}$ is actually independent of *r*. $\hat{T}\in\Lambda^3$ is an additional invariant, whose physical meaning is still unclear. In Minkowski space-time the condition for vector potential becomes [17]

$$\vec{A} = \vartheta^{\varphi} A_{\varphi} = \frac{1}{r} U(\theta)(-\sin\varphi, \cos\varphi, 0), \quad U \equiv \frac{A(\theta)}{\sin\theta}. \tag{4.48}$$

5. Discussion and Conclusion

The development of the number systems has undergone a long and difficult history, not as simple as it seems now. The number system that abandons the associative law has little practical value, because the result of a line of products must be determined by the order of the multiplication. In comparison, the presence of a zero factor is a repairable defect. For a hypercomplex number composed of real Clifford algebras, whose zero norm set is a closed set of special geometric significance, like a light cone in realistic space-time. If the realistic 4-dimensional space-time is not hyperbolic, but is an elliptic world without zero factor like the quaternions, then the world will be lifeless. It is only because of this hyperbolic nature that space-time has an alive soul and abundant structures. From this perspective, the existence of a zero norm set cannot be regarded as a defect of hypercomplex numbers, because it has no substantial influence on algebraic operations. Moreover, the relationship between zero norm set and the metric sign convention of basic space-time is an issue worthy of further study.

Clifford algebra describes complex and quaternions in a unified way, and can be directly extended to 2^n-ary associative algebras. In this generalization, the orthonormal basis matrix (2.5) and the set of Grassmann basis elements (2.9) constructed from it play a key role. Matrix representation carries more information than the abstract definitions, such as the determinants and definition of inverse numbers. Without the matrix representation (2.5), the discussion for hypercomplex numbers will be difficult. The relations between line elements defined by (4.1)-(4.3) directly generalize the Clifford algebras to curved space-time and differential geometry [6] [11], which bring great convenience for learning and research. The Clifford algebras convert the complicated mathematical operations into mechanical calculations—simple, intuitive and error-free, which can be well mastered by middle school students. Because of the isomorphism between the Clifford algebra and the matrix algebra, the conclusions obtained from the matrix algebra are also correct for Clifford algebra. In this way, the subtle and abstract concepts and problems are crystallized, and numbers and shapes are uniformly described.

Hamilton's original motivation for discovering quaternions was to describe the Maxwell equations. Since realistic space-time is not an elliptic space described by quaternions, the vector and scalar parts in the Maxwell equations have to be described separately, and this weakness causes the quaternion's failure in the competition with vector algebra. It can be seen from this paper that, Clifford algebra has no limitation of quaternions, it is directly defined on the basic space-time and describes geometry and physics without superfluous or absent content. From the derivation of (4.12)-(4.16), we find that the fundamental dynamical equations in physics are actually determined by the space-time structure. If expressing physical equations by Clifford algebras, they look simplicity in formalism, symmetry in structure, standard in derivation and completeness in content. Therefore, we can hope that this magical algebra will complete a new big synthesis in science [6].

From the above discussion, we find that there are some significant topics for hypercomplex numbers that deserve further study. For the Clifford-Grassmann number (2.3), the zero norm set is closed lower dimensional analytic surfaces that, like light cones, may have some interesting properties not yet revealed. If the differential geometry is represented in the form of Clifford algebras, as shown in [6] [11], the proof of theorems may be simpler, the expression may be clearer, and some new results may be found. For the n-ary cyclic number (3.4), it has not previously been taken as a number system, so its geometric and physical significance is completely unclear. Its zero norm set is simple and symmetric with respect to coordinates a_k, but the geometric significance is also unclear and requires further investigation.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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