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Shock Fronts in Non-Polar Electro-Conducting Media

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Abstract

We analyze the propagation of electromagnetic fronts in unbounded electric conductors. Our model is based on the Maxwell model of electromagnetism, which includes the displacement current and Ohm's law in its simplest forms. The shock-like electromagnetic front is a propagating surface, across which the electric and magnetic fields, as well as their higher temporal and spatial derivatives, experience finite jumps. The shock-like fronts are essentially different as compared with the weak fronts; in particular, the bulk Maxwell equations are essentially insufficient for the analysis of the shock-like fronts, and they should be amended we the physical jump conditions. We choose these additional conditions by using conditions similar to those suggested by Heaviside. We derive the basic shock intensity relationships implied by this model.

Keywords

Electric Current in Conductors, Boundary Value Problems, Exact Solution, Ray-Equations

1. Introduction

Establishing the self-consistent mathematical model of electromagnetic waves (and of light, in particular) is the culmination of the Maxwell model of electromagnetism [1]. Maxwell himself has not seen recognition of his model by the greatest electricians of his time, including Kelvin and Helmholtz. However, forthcoming developments in science eventually proved to be instrumental in thousands of applications.

In the absence of electric and magnetic polarization, the bulk system of Maxwell equations includes (in its contemporary form) 4 partial differential equations for the distributed electric current and the electric and magnetic fields. This system is basically linear (the only source of nonlinearity can appear in the generalized Ohms law for the high amplitude currents.) Many approaches to solving Maxwell equations and the explicit exact solutions can be found in thousands of textbooks and monographs.

The most close to our approach is the approaches of the monographs of Luneburg [2] and Born and Wolfe [3]. These two monographs contain the sections dealing with the electromagnetic wave-fronts treated as surfaces of discontinuities of electromagnetic fields. For other media, similar methods were used by Hadamard [4], Levi-Civita [5], Thomas [6], and Keller [7]. In their publications, the interested readers can find further references and historic discussions.

The key instrument in analysis of wavefronts is the so-called compatibility conditions. In 20th century, the biggest contributions in their usage belong to the outstanding geometers and mathematical physicists first usage of this approach in electromagnetism belongs to Luneburg [2], as well as Born and Wolf [3].

In the recent paper Grinfeld and Grinfeld [8], we explored an evolution of the so-call wek fronts in electric conductors. What still remains unexplored, is the dynamics of the shock-like wave fronts in dissipative electro-conducting media in the framework of the full Maxwell model. In this paper, we fill this gap of the classical analysis.

Compatibility conditions have been discussed in detail in the monographs of Michael Grinfeld [9], and Pavel Grinfeld [10]. From these monographs, we borrow required results and notation. We use standard tensorial notation and operations in the form, presented in the same monographs [9] [10]. The Latin (spatial) indices $i, j, k, \cdots$ run the values 1, 2, 3); the Greek (surface) indices $\alpha, \beta, \gamma, \cdots$ run the values 1, 2. The space is referred to the immobile coordinates z^i; δ_i^j is the spatial Kronecker delta; z_{ij} and z^{ij} are the co- and contravariant components of the metrics, which is used for raising and lowering ("juggling") spatial indices, and for definition of covariant differentiation ∇_i; z_{ijk} is the Levi-Civita skew-symmetric tensor.

2. The Bulk Master System of Equations

The bulk Maxwell equations read

$$\nabla_i E^i = 4\pi Q \tag{1.1}$$

$$\nabla_i H^i = 0 \tag{1.2}$$

$$\frac{1}{c}\frac{\partial H^i}{\partial t} = -z^{ijk}\nabla_j E_k \tag{1.3}$$

$$\frac{\partial E^i}{\partial t} + 4\pi I^i = c z^{ijk}\nabla_j H_k \tag{1.4}$$

where c is a speed of light, E^i is an electric field, H^i is a magnetic field; Q and I^i are the bulk free charge and electric current.

If there were non-zero charges Q at $t = 0$, they charges will decay exponentially at $t > 0$ (Landau, Lifshits [11]) Therefore, we assume up-front that $Q(z,t) = 0$.

We assume that the electric field E^i and electric current I^i are connected by the nonlinear anisotropic Ohm's law

$$I^i = \chi^i(E_k) \tag{1.5}$$

where σ is the electroconducitity constant.

Eliminating electric current I^i between Equations (1.4) and (1.5) we get

$$\frac{\partial E^i}{\partial t} + 4\pi\chi^i(E_k) = cz^{ijk}\nabla_j H_k \tag{1.6}$$

3. The Heaviside Shock Conditions and Their Implications

The bulk system of Maxwell Equations (1.1) - (1.4) is sufficient when analyzing smooth solutions. However, when the electromagnetic field and its derivative experience discontinuities, Equations (1.1) - (1.4) become meaningless in some points of space-time and the description, based on these partial differential equations, appears to be essentially incomplete. This incompleteness should be compensated by some additional relationships.

The additional relationships are different for different types of singularities. For the so-called weak discontinuities, the fields E^i and H^i are continuous everywhere but not their first temporal and spatial derivatives. On the contrary, the first derivatives experience finite jumps on some moving singular surfaces, called wavefronts. No additional physical ideas should be used for analysis of weak discontinuities. The physical Equations (1.1) - (1.6), in principle, fully describe the weak discontinuities. But still some additional relationships should be used at the fronts. They are called the compatibility relationships. We explored this sort of weak fronts in our earlier publication [8].

When dealing with electromagnetic shock waves, not only first derivatives but also the fields themselves experience finite jumps. Therefore, geometric and geometric boundary conditions significantly change as compared with the weak fronts, as it follows from comparison of the compatibility relations of [8], and the relations below. Moreover, for shock fronts, it is not sufficient to use only the geometric and kinematic compatibility conditions. For this sort of singular surfaces, we have to amend the bulk Maxwell equations with their analogies at the front.

There are different approaches to establishing additional conditions across the shock waves. We will dwell on the additional conditions known to Heaviside (Born and Wolfe [3]). We will call them the Heaviside conditions; they read:

$$C\left[H^i\right]_-^+ = cz^{ijk}\left[E_k\right]_-^+ N_j, \tag{2.1}$$

$$C\left[E^i\right]_-^+ = -cz^{ijk}\left[H_k\right]_-^+ N_j, \tag{2.2}$$

If $C \neq 0$, the Heaviside conditions (2.1), (2.2) obviously imply the following relationships

$$\left[H^i \right]_-^+ N_i = 0, \tag{2.3}$$

$$\left[E^i \right]_-^+ N_i = 0. \tag{2.4}$$

In order to prove Equations (2.3) and (2.4) we contract Equations (2.1) and (2.2) with N_i and use the identity $z^{ijk} v_i v_j \equiv 0$ which is valid for arbitrary vector v_i.

In order to prove Equations (2.3, 2.4) we contract Equations (2.1, 2.2) with N_i and use the identity $z^{ijk} v_i v_j \equiv 0$ which is valid for arbitrary vector v_i.

Contracting Equations (2.1), (2.2) with the vectors $[E_i]_-^+$ and $[H_i]_-^+$, respectively, we get one and the same identity

$$[E_i]_-^+ \left[H^i \right]_-^+ = 0 \tag{2.5}$$

At last, contracting Equations (2.1), (2.2) with the vectors $[H_i]_-^+$ and $[E_i]_-^+$, respectively, we get the identity

$$[E_i]_-^+ \left[E^i \right]_-^+ = [H_i]_-^+ \left[H^i \right]_-^+ \tag{2.6}$$

The Equations (2.3) - (2.6) are very important for the future and self-explanatory.

4. The Compatibility Conditions for Shock Fronts

In addition to the physical shock-front conditions (2.1), (2.2), the following condition is implied by the geometry and kinematics only:

$$\begin{aligned}
\left[\nabla_j E^i \right]_-^+ &= A^i N_j + z_{j.}^{\alpha} \nabla_\alpha a^i \\
\left[\partial_t E^i \right]_-^+ &= -C A^i + \frac{\delta a^i}{\delta t} \\
\left[\nabla_j H^i \right]_-^+ &= B^i N_j + z_{j.}^{\alpha} \nabla_\alpha b^i \\
\left[\partial_t H^i \right]_-^+ &= -C B^i + \frac{\delta b^i}{\delta t}
\end{aligned} \tag{3.1}$$

where the zero and first order jump vectors are defined as

$$\begin{aligned}
a^i(\xi,t) \equiv \left[E^i \right]_-^+, \; A^i(\xi,t) \equiv \left[\nabla_m E^i \right]_-^+ N^m, \\
b^i(\xi,t) \equiv \left[H^i \right]_-^+, \; B^i(\xi,t) \equiv \left[\nabla_m H^i \right]_-^+ N^m
\end{aligned} \tag{3.2}$$

We notice that the jumps vectors are defined on the moving shock fronts only.

Let us present the zero order jump vectors a^i, b^i in the following form:

$$a^i(\xi,t) = \left[E^i \right]_-^+ \equiv l_E \tau_E^i, \; b^i(\xi,t) = \left[H^i \right]_-^+ \equiv l_H \tau_H^i, \tag{3.3}$$

where τ_{Ei}, τ_{Hi} are the polarization unit vectors, and $l_E > 0, l_H > 0$ are the amplitudes of those vectors.

In view of the relationship (2.6), we get

$$|l_E| = |l_H| = l \tag{3.4}$$

if $C \neq 0$.

Combining Equations (3.3), (3.4), we get

$$a^i(\xi,t) = l\tau_E^i,\ b^i(\xi,t) = l\tau_H^i, \tag{3.5}$$

5. Shock-Front Velocity and the Zeroth Order Jump Vector

Let us rewrite Equations (2.1), (2.2) as follows

$$Cb^i = cz^{ijk}a_k N_j, \tag{4.1}$$

$$Ca^i = -cz^{ijk}b_k N_j \tag{4.2}$$

$$a^i N_i = b^i N_i = 0 \tag{4.3}$$

or

$$\begin{bmatrix} cz^{ijk}N_j & -Cz^{ik} \\ Cz^{ik} & cz^{ijk}N_j \end{bmatrix}\begin{bmatrix} a_k \\ b_k \end{bmatrix} = \begin{bmatrix} 0^i \\ 0^i \end{bmatrix} \tag{4.4}$$

Combining Equations (4.2), (4.3), we get

$$C^2 = c^2 \tag{4.5}$$

If we change the normal orientation N_j to the opposite, the front velocity will change to the opposite also. Let us choose the normal N_j orientation in such a way that the velocity C is positive:

$$C = c > 0 \tag{4.6}$$

Per Equations (4.1) - (4.3), all 3 vectors, $b_n = [H_n]_-^+$, $a_n = [E_n]_-^+$, N_j, are mutually orthogonal, and we get the relationships

$$a_k = -z_{k..}^{.mn}b_n N_m,\ b_k = z_k^{.mn}a_n N_m \tag{4.7}$$

and

$$\tau_E^i = -z_{...}^{imn}\tau_{Hn}N_m,\ \tau_H^i = z_{.}^{imn}\tau_{En}N_m \tag{4.8}$$

6. Evolution of Shock-Wave Intensity along Rays

Calculating the jumps of the terms in the bulk master system (1.1) - (1.4) and using the compatibility conditions (3.1) - (3.3), we arrive at the following relationships:

$$A^i N_i + z_{i.}^{\alpha}\nabla_\alpha a^i = 0 \tag{5.1}$$

$$B^i N_i + z_{i.}^{\alpha}\nabla_\alpha b^i = 0 \tag{5.2}$$

$$\frac{1}{c}\left(-CB^i + \frac{\delta b^i}{\delta t}\right) = -z^{ijk}\left(A_k N_j + z_{j.}^{\alpha}\nabla_\alpha a_k\right) \tag{5.3}$$

$$\frac{1}{c}\left(-CA^i + \frac{\delta a^i}{\delta t}\right) + \frac{4\pi}{c}\left[\chi^i(E)\right]_-^+ = z^{ijk}\left(B_k N_j + z_{j.}^{\alpha}\nabla_\alpha b_k\right) \tag{5.4}$$

Using Equation (4.6), we can rewrite Equations (5.3), (5.4) as follows:

$$-B^i + \frac{1}{c}\frac{\delta b^i}{\delta t} = -z^{ijk}\left(A_k N_j + z_{j.}^{\alpha}\nabla_\alpha a_k\right) \tag{5.5}$$

$$-A^i + \frac{1}{c}\frac{\delta a^i}{\delta t} + \frac{4\pi}{c}\left[\chi^i(E)\right]_-^+ = z^{ijk}\left(B_k N_j + z_{j.}^{\alpha}\nabla_\alpha b_k\right) \tag{5.6}$$

and then as

$$B^{i} = z^{ijk} A_k N_j + \frac{1}{c}\frac{\delta b^{i}}{\delta t} + z^{ijk} z^{\cdot\alpha}_{j.} \nabla_\alpha a^{\cdot}_{k} \tag{5.7}$$

$$A^{i} = -z^{ijk} B_k N_j + \frac{1}{c}\frac{\delta a^{i}}{\delta t} + \frac{4\pi}{c}\left[\chi^{i}(E)\right]_{-}^{+} - z^{ijk} z^{\cdot\alpha}_{j.} \nabla_\alpha b^{\cdot}_{k} \tag{5.8}$$

Equation (5.7) can be rewritten as follows

$$B_k = z^{\cdot jm}_{k..} A_m N_j + \frac{1}{c}\frac{\delta b_k}{\delta t} + z^{\cdot jm}_{k..} z^{\cdot\alpha}_{j.} \nabla_\alpha a^{\cdot}_{m} \tag{5.9}$$

Eliminating B_k between Equations (5.8), (5.9), we get

$$\begin{aligned} &A^{i} + z^{ijk} z^{\cdot jm}_{k..} A_m N_j \\ &= \frac{4\pi}{c}\left[\chi^{i}(E)\right]_{-}^{+} + \frac{1}{c}\frac{\delta a^{i}}{\delta t} - z^{ijk}\frac{1}{c}\frac{\delta b_k}{\delta t} N_j - z^{ijk} z^{\cdot jm}_{k..} z^{\cdot\alpha}_{j.} \nabla_\alpha a^{\cdot}_{m} N_j - z^{ijk} z^{\cdot\alpha}_{j.} \nabla_\alpha b^{\cdot}_{k} \end{aligned} \tag{5.10}$$

as implied by the following chain:

$$\begin{aligned} &A^{i} = -z^{ijk} B_k N_j + \frac{1}{c}\frac{\delta a^{i}}{\delta t} + \frac{4\pi}{c}\left[\chi^{i}(E)\right]_{-}^{+} - z^{ijk} z^{\cdot\alpha}_{j.} \nabla_\alpha b^{\cdot}_{k} \\ &\to A^{i} = -z^{ijk}\left(z^{\cdot nm}_{k..} A_m N_n + \frac{1}{c}\frac{\delta b_k}{\delta t} + z^{\cdot nm}_{k..} z^{\cdot\alpha}_{n.} \nabla_\alpha a^{\cdot}_{m} \right) N_j + \frac{1}{c}\frac{\delta a^{i}}{\delta t} \\ &\qquad + \frac{4\pi}{c}\left[\chi^{i}(E)\right]_{-}^{+} - z^{ijk} z^{\cdot\alpha}_{j.} \nabla_\alpha b^{\cdot}_{k} \\ &\to A^{i} + z^{ijk} z^{\cdot nm}_{k..} A_m N_n N_j \\ &\quad = \frac{4\pi}{c}\left[\chi^{i}(E)\right]_{-}^{+} + \frac{1}{c}\frac{\delta a^{i}}{\delta t} - z^{ijk}\frac{1}{c}\frac{\delta b_k}{\delta t} N_j - z^{ijk} z^{\cdot nm}_{k..} z^{\cdot\alpha}_{n.} \nabla_\alpha a^{\cdot}_{m} N_j - z^{ijk} z^{\cdot\alpha}_{j.} \nabla_\alpha b^{\cdot}_{k} \end{aligned}$$

We, then, get, using Equations (4.7), (5.10),

$$\frac{1}{c}\frac{\delta a^{i}}{\delta t} - z^{ijk}\frac{1}{c}\frac{\delta b_k}{\delta t} N_j = \frac{2}{c}\frac{\delta a^{i}}{\delta t} \tag{5.11}$$

as implied by the chain:

$$\begin{aligned} &\frac{1}{c}\frac{\delta a^{i}}{\delta t} - z^{ijk}\frac{1}{c}\frac{\delta b_k}{\delta t} N_j \\ &= \frac{1}{c}\frac{\delta a^{i}}{\delta t} - z^{ijk}\frac{1}{c}\frac{\delta\left(z_k^{\cdot mn} a_n N_m\right)}{\delta t} N_j \\ &= \frac{1}{c}\frac{\delta a^{i}}{\delta t} - z^{ijk} z_k^{\cdot mn}\frac{1}{c}\frac{\delta\left(a_n N_m\right)}{\delta t} N_j \\ &= \frac{1}{c}\frac{\delta a^{i}}{\delta t} - \left(z^{im} z^{jn} - z^{in} z^{jm}\right)\frac{1}{c}\frac{\delta\left(a_n N_m\right)}{\delta t} N_j \\ &= \frac{1}{c}\frac{\delta a^{i}}{\delta t} - \left(z^{im} z^{jn} - z^{in} z^{jm}\right)\frac{1}{c}\left(N_m \frac{\delta a_n}{\delta t} + a_n \frac{\delta N_m}{\delta t}\right) N_j \\ &= \frac{1}{c}\left\{ \frac{\delta a^{i}}{\delta t} - \left(z^{im} z^{jn} - z^{in} z^{jm}\right) N_m N_j \frac{\delta a_n}{\delta t} - \left(z^{im} z^{jn} - z^{in} z^{jm}\right) N_j \frac{\delta N_m}{\delta t} a_n \right\} \\ &= \frac{1}{c}\left\{ \frac{\delta a^{i}}{\delta t} - \left(N^{i} N^{n} - z^{in}\right)\frac{\delta a_n}{\delta t} - N^{i}\frac{\delta N_m}{\delta t} a_n \right\} \end{aligned}$$

$$= \frac{1}{c}\left\{2\frac{\delta a^i}{\delta t} - N^i N^n \frac{\delta a_n}{\delta t} - N^i \frac{\delta N_m}{\delta t} a_n\right\}$$
$$= \frac{1}{c}\left\{2\frac{\delta a^i}{\delta t} + N^i \frac{\delta N^n}{\delta t} a_n - N^i \frac{\delta N_m}{\delta t} a_n\right\}$$
$$= \frac{2}{c}\frac{\delta a^i}{\delta t}$$

We, then, get the relationship

$$A^i + z^{ijk} z_{k..}^{.nm} A_m N_n N_j = A_m N^i N^m \tag{5.12}$$

as implied by the chain

$$\begin{aligned} A^i + z^{ijk} z_{k..}^{.nm} A_m N_n N_j &= A^i + \left(z^{in} z^{jm} - z^{im} z^{jn}\right) A_m N_n N_j \\ &= A^i + \left(N^i N^m - z^{im}\right) A_m \\ &= A_m N^i N^m \end{aligned}$$

Also, we get the relationship

$$-z^{ijk} z_{k..}^{.nm} z_{n.}^{.\alpha} \nabla_\alpha a_m^{\cdot} N_j - z^{ijk} z_{j.}^{.\alpha} \nabla_\alpha b_k^{\cdot} = -a^i b_\alpha^\alpha - z_{..}^{n\alpha} \nabla_\alpha a_n N^i \tag{5.13}$$

as implied by the chain

$$\begin{aligned} &-z^{ijk} z_{k..}^{.nm} z_{n.}^{.\alpha} \nabla_\alpha a_m^{\cdot} N_j - z^{ijk} z_{j.}^{.\alpha} \nabla_\alpha b_k^{\cdot} \\ &= -\left(z^{in} z^{jm} - z^{im} z^{jn}\right) z_{n.}^{.\alpha} \nabla_\alpha a_m^{\cdot} N_j - z^{ijk} z_{j.}^{.\alpha} \nabla_\alpha b_k^{\cdot} \\ &= -\left(z^{in} z^{jm} z_{n.}^{.\alpha} N_j - z^{im} z^{jn} z_{n.}^{.\alpha} N_j\right) \nabla_\alpha a_m^{\cdot} - z^{ijk} z_{j.}^{.\alpha} \nabla_\alpha b_k^{\cdot} \\ &= -z^{in} z_{n.}^{.\alpha} N^m \nabla_\alpha a_m^{\cdot} - z^{ijk} z_{j.}^{.\alpha} \nabla_\alpha b_k^{\cdot} \\ &= -z^{in} z_{n.}^{.\alpha} N^m \nabla_\alpha a_m^{\cdot} - z^{ijk} z_{j.}^{.\alpha} \nabla_\alpha \left(z_k^{.mn} a_n N_m\right) \\ &= -z^{in} z_{n.}^{.\alpha} N^m \nabla_\alpha a_m^{\cdot} - z^{ijk} z_k^{.mn} z_{j.}^{.\alpha} \nabla_\alpha \left(a_n N_m\right) \\ &= -z^{in} z_{n.}^{.\alpha} N^m \nabla_\alpha a_m^{\cdot} - z^{ijk} z_k^{.mn} z_{j.}^{.\alpha} \left(\nabla_\alpha a_n N_m + a_n \nabla_\alpha N_m\right) \\ &= -z^{in} z_{n.}^{.\alpha} N^m \nabla_\alpha a_m^{\cdot} - \left(z^{im} z^{jn} - z^{in} z^{jm}\right) z_{j.}^{.\alpha} \left(\nabla_\alpha a_n N_m - a_n b_{\alpha\beta} z_{m.}^{.\beta}\right) \\ &= -z^{in} z_{n.}^{.\alpha} N^m \nabla_\alpha a_m^{\cdot} - \left(z^{im} z^{jn} - z^{in} z^{jm}\right) z_{j.}^{.\alpha} \nabla_\alpha a_n N_m + \left(z^{im} z^{jn} - z^{in} z^{jm}\right) z_{j.}^{.\alpha} a_n b_{\alpha\beta} z_{m.}^{.\beta} \\ &= -a^i b_\alpha^\alpha - z_{..}^{n\alpha} \nabla_\alpha a_n N^i \end{aligned}$$

Combining Equations (5.11) - (5.13), we can transform Equation (5.10) as follows

$$A_m N^i N^m = \frac{4\pi}{c}\left[\chi^i (E)\right]_-^+ + \frac{2}{c}\frac{\delta a^i}{\delta t} - a^i b_\alpha^\alpha - z_{..}^{n\alpha} \nabla_\alpha a_n N^i \tag{5.14}$$

Contracting Equation (5.14) with $a_i^{\cdot}$, we get

$$\frac{\delta\left(a^i a_i\right)}{\delta t} - a_i a^i c b_\alpha^\alpha + 4\pi a_i \left[\chi^i (E)\right]_-^+ = 0 \tag{5.15}$$

Contracting Equation (5.14) with $b_i^{\cdot}$, we get

$$\frac{4\pi}{c}\left[\chi^i (E)\right]_-^+ b_i + \frac{2}{c}\frac{\delta a^i}{\delta t} b_i = 0 \tag{5.16}$$

In linear case, we can rewrite Equations (5.15), (5.16)

$$\frac{\delta\left(a^i a_i\right)}{\delta t} - a_i a^i c b_\alpha^\alpha + 4\pi\chi^{ij} a_i a_j = 0 \tag{5.17}$$

$$\frac{\delta a^i}{\delta t} b_i + 2\pi\chi^{ij} a_j b_i = 0 \tag{5.18}$$

In the case of isotropic linear conductor we get, by definition, $\chi^i\left(E\right) = \chi E^i$; we, then, get

$$\frac{\delta}{\delta t} \ln \frac{|a|^2}{J} = -4\pi\chi \tag{5.19}$$

as implied by the chain

$$\frac{\delta\left(a^i a_i\right)}{\delta t} - a_i a^i c b_\alpha^\alpha + 4\pi a_i \left[\chi^i\left(E\right)\right]_-^+ = 0$$

$$\frac{\delta}{\delta t} \ln\left(a^i a_i\right) - \frac{\delta}{\delta t} \ln J = -4\pi\chi$$

$$\frac{\delta}{\delta t} \ln \frac{|a|^2}{J} = -4\pi\chi$$

In Equation (5.19), the quantity J is the ray-divergence (see [2] [3] [5] [7] [8] [9] [10]).

Also, in the case $\chi^i\left(E\right) = \chi E^i$, Equation (5.18) reads

$$\frac{4\pi}{c} \chi a^i b_i + \frac{2}{c} \frac{\delta a^i}{\delta t} b_i = 0 \tag{5.20}$$

and in view of Equation (4.7) we get

$$\frac{\delta a^i}{\delta t} b_i = 0 \tag{5.21}$$

Also, we get

$$\frac{\delta a^i}{\delta t} = \frac{1}{2}\left(c b_\alpha^\alpha - 4\pi\chi\right) a^i \tag{5.22}$$

$$\frac{\delta b^i}{\delta t} = \frac{1}{2}\left(c b_\alpha^\alpha - 4\pi\chi\right) b^i \tag{5.23}$$

as implied by the analysis below.

Let us look for the derivative $\delta a^i / \delta t$ in the form:

$$\frac{\delta a^i}{\delta t} = B_1 a^i + B_2 b^i + B_3 N^i \tag{5.24}$$

where B_1, B_2, B_3 are certain coefficients.

Let us use the relationships

$$\frac{\delta a^i}{\delta t} a_i = B_1 |a|^2, \frac{\delta a^i}{\delta t} b_i = B_2 |b|^2, \frac{\delta a^i}{\delta t} N_i = B_3 \tag{5.25}$$

We, then, get

$$B_1 = \frac{1}{2}\left(c b_\alpha^\alpha - 4\pi\chi\right), B_2 = 0, B_3 = 0 \tag{5.26}$$

Using Equations (5.26), we arrive at the relationship (5.22).

Now, using the chain

$$\begin{aligned}\frac{\delta b^i}{\delta t} &= \frac{\delta}{\delta t}\left(z^{imn}_{\dots} a_n N_m\right) = z^{imn}_{\dots} N_m \frac{\delta a_n}{\delta t} \\ &= \frac{1}{2}\left(cb^\alpha_\alpha - 4\pi\chi\right) z^{imn}_{\dots} N_m a_n = \frac{1}{2}\left(cb^\alpha_\alpha - 4\pi\chi\right) b^i\end{aligned} \tag{5.27}$$

we arrive at the relationship (5.23).

7. Anisotropic Conductivity

Consider the case $\chi^i(E) = \chi^{ij} E_j$; we then get

$$\frac{\delta\left(a^i a_i\right)}{\delta t} - a_i a^i cb^\alpha_\alpha + 4\pi\chi^{ij} a_i a_j = 0 \tag{6.1}$$

and then

$$a_i \frac{\delta a^i}{\delta t} = \frac{1}{2} cb^\alpha_\alpha a_i a^i - 2\pi\chi^{ij} a_i a_j \tag{6.2}$$

Thus, instead of Equations (5.22), (5.23) we arrive at the relationships

$$\frac{\delta a^i}{\delta t} = \left(\frac{1}{2} cb^\alpha_\alpha - 2\pi\chi^{km} \frac{a_k a_m}{|a|^2}\right) a^i \tag{6.3}$$

and

$$\frac{\delta b^i}{\delta t} = \left(\frac{1}{2} cb^\alpha_\alpha - 2\pi\chi^{km} \frac{a_k a_m}{|a|^2}\right) b^i \tag{6.4}$$

We proceed as

$$\begin{aligned}\frac{\delta b^i}{\delta t} &= \frac{\delta}{\delta t}\left(z^{imn}_{\dots} a_n N_m\right) = z^{imn}_{\dots} N_m \frac{\delta a_n}{\delta t} \\ &= z^{ipq}_{\dots} N_p \left(cb^\alpha_\alpha/2 - 2\pi\chi^{km} \frac{a_k a_m}{|a|^2}\right) a_q \\ &= \left(cb^\alpha_\alpha/2 - 2\pi\chi^{km} \frac{a_k a_m}{|a|^2}\right) b^i\end{aligned} \tag{6.5}$$

Using Equation (3.5), we can rewrite Equations (6.3), (6.4) as follows

$$l\frac{\delta \tau^i_E}{\delta t} + \frac{\delta l}{\delta t}\tau^i_E = \left(cb^\alpha_\alpha/2 - 2\pi\chi^{km}\tau_{Ek}\tau_{Em}\right) l\tau^i_E \tag{6.6}$$

and

$$l\frac{\delta \tau^i_H}{\delta t} + \frac{\delta l}{\delta t}\tau^i_H = \left(cb^\alpha_\alpha/2 - 2\pi\chi^{km}\tau_{Ek}\tau_{Em}\right) l\tau^i_H \tag{6.7}$$

Then, we get

$$\frac{\delta \tau^i_E}{\delta t} + \frac{\delta \ln l}{\delta t}\tau^i_E = \left(cb^\alpha_\alpha/2 - 2\pi\chi^{km}\tau_{Ek}\tau_{Em}\right)\tau^i_E \tag{6.8}$$

and

$$\frac{\delta \tau_H^i}{\delta t} + \frac{\delta \ln l}{\delta t} \tau_H^i = \left(c b_\alpha^\alpha / 2 - 2\pi \chi^{km} \tau_{Ek} \tau_{Em} \right) \tau_H^i \tag{6.9}$$

Equation (6.8) implies

$$\frac{\delta \ln l}{\delta t} = \frac{1}{2} c b_\alpha^\alpha - 2\pi \chi^{km} \tau_{Ek} \tau_{Em} \tag{6.10}$$

and

$$\frac{\delta \tau_E^i}{\delta t} = 0 \tag{6.11}$$

Similarly, we get

$$\frac{\delta \tau_H^i}{\delta t} = 0 \tag{6.12}$$

Thus, the directors τ_E^i and τ_H^i of jumps of the electric and magnetic fields remain unchanged along each of the rays.

8. Conclusion

We investigated electromagnetic shock-fronts, propagating within non-polarizable and non-magnetizable conductors. Our analysis relies on the Heaviside jump condition and on the so-called geometric and kinematic compatibility conditions. The Ohm's resistance does not influence the speed of the shock-fronts—they still propagate with the speed of light in vacuum. Evolution of the jump vectors of electric and magnetic fields is described in terms of the rays; the rays are the two-parameter sets of lines, which are orthogonal to the consecutive positions of the shock-fronts. It was demonstrated that the unit directors τ_E^i and τ_H^i of the jumps of the electric and magnetic fields remain unchanged along each of the rays. At the same time, the amplitudes of the jumps of the electric and magnetic fields change due to two reasons: the rays divergence and Ohmic dissipation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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