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On the Paradox of the Duality of Autoregressive and Moving Average Processes

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Abstract

A widely held view in time series analysis is the concept of duality that a finite order stationary autoregressive process of order p (AR(p)) is equivalent to an infinite order moving average (MA) process and a finite order invertible moving average of order q (MA(q)) is equivalent to an infinite order autoregressive (AR) process. The purpose of this paper is to demonstrate that the concept is not universally true. Thus, a finite order stationary autoregressive process of order p (AR(p)) can be written as an finite order moving average process and a finite order moving average process of order q (MA(q)) can be written as a finite order stationary autoregressive process. The regions of breakdown of concept of duality were determined for $p = q = 1, 2$ using method of moments. The method involves equating non-zero autocovariances of the stationary AR(p) to the equivalent non-zero autocovariances of the invertible MA(p) to determine the region of non-duality. In such region of breakdown in duality, 1) both the Autocorrelation function and the Partial Autocorrelation function of the AR process and MA process cuts off after equal lags 2) a finite AR model can be adequately represented by a finite MA model of equal order and conversely with the same error variance and 3) negative values of the parameters of the AR process are equal in magnitude but opposite in direction to the parameters of the equivalent MA process and conversely. Empirical examples (simulation and real life examples) were used to illustrate these. Therefore, it has been recommended that caution should be exercised in using the concept of duality in time series analysis until future research proves otherwise.

Keywords

Duality, Non-Duality, Method of Moments, Quadratic Inequality, Stationarity Region, Invertibility Region

1. Introduction

The concept of duality is an old topic and a fruitful idea that can be found in many areas of Mathematics and other related disciplines [1] [2] [3]. Duality is a principle that gives two different points of view to the same object [1] [4] [5]. These points of view constitute an instance of opposition or contrast between concepts that are interchangeable, such that all results in one formulation also hold in the dual formulation [6]-[11]. As a consequence, the concept of duality involves symmetry within a system such that a theorem remains valid if certain objects, relations or operations are interchanged [12].

In a mathematical system, the concept of duality forms bases for describing and identifying the property of algebraic structures, theorems and/or expressions, being dual to each other. In his views, [13] stated that the concept of duality involves two objects plus a relation between them that is symmetric, such that one can get from one object to the other and vice versa. Some examples where the principle of duality applies in Mathematics include the linear duality in plane geometry and linear algebra, duality for Abelian groups and Non-Abelian groups, duality for non-linear geometry and Fourier theory, etc. [1]. Similarly, in Physics, the concept of duality has been applied to describe and identify the properties of variables in space. These include mirror symmetry, position momentum, quantum mechanics and electromagnetism [1] [6] [14].

Dual Aspect Concept, also known as Duality Principle, is a fundamental convention of accounting that forms basis for double entry accounting system. Under the accounting system, transactions are classified into debit or credit, loss or profit, etc. which enables every transaction to have dual effects [15] [16]. These dual effects ensure that all aspects of a transaction are accounted for in the financial statements.

In Statistics duality has been used to describe the relationship between models for two classes of stationary time series (the autoregressive and moving average processes). A given stationary autoregressive process of order p (AR(p)) was denoted by [17];

$$\boldsymbol{\varphi}_p(B)X_t = e_t \tag{1.1}$$

we can write (1.1) as

$$X_t = \boldsymbol{\psi}(B)e_t \tag{1.2}$$

such that

$$\boldsymbol{\varphi}_p(B)\boldsymbol{\psi}(B) = 1 \tag{1.3}$$

In (1.1) and (1.2), $e_t \sim N(0,\sigma^2)$ and

$$\boldsymbol{\varphi}_p(B) = 1 + \phi_1 B + \phi_2 B^2 + \cdots + \phi_p B^p \tag{1.4}$$

$$\boldsymbol{\psi}(B) = 1 + \psi_1 B^1 + \psi_2 B^2 + \cdots \tag{1.5}$$

That is, a finite order stationary AR process (1.1) is equivalent to an infinite order moving average process (1.2).

Conversely, for a given invertible moving average process of order q, (MA(q)) denoted by

$$X_t = \boldsymbol{\theta}_q(B)e_t \tag{1.6}$$

we can write (1.6) as

$$e_t = \boldsymbol{\theta}_q^{-1}(B)X_t = \prod(B)X_t \tag{1.7}$$

such that

$$\boldsymbol{\theta}_q(B)\prod(B) = 1 \tag{1.8}$$

In (1.6) and (1.7), $e_t \sim N(0,\sigma^2)$ and

$$\boldsymbol{\theta}_q(B) = 1 + \theta_1 B^1 + \theta_2 B^2 + \cdots + \theta_q B^q \tag{1.9}$$

$$\prod(B) = 1 + \pi_1 B^1 + \pi_2 B^2 + \cdots \tag{1.10}$$

is an infinite but converging series. That is, a finite order invertible MA process (1.6) is equivalent to an infinite order AR process (1.7).

The dual relationship also exists in their autocorrelation functions (ACFs) and partial autocorrelation functions (PACFs) [17]. For an AR(p) process, while the ACF tails off, the PACF cuts off after lag p, but for MA(q) process, the ACF cuts off after lag q while the PACF tails off.

The works of [18]-[27] outlined other properties of the duality principle among these models. Using the duality principle, [28] showed that if AR and MA processes of the same order are simulated from the same sequence of errors using the same parameter values, then to a close approximation, the least squares estimates calculated from the MA series will tend to underestimate the true parameter values while those from AR series will tend to over-estimate them by the same amount. Thus, suggesting that the duality principle between AR(p) is equivalent to an infinite order moving average (MA) process and MA(q) processes is equivalent to an infinite order autoregressive (AR) process may be universally true.

However, certain real life examples have shown that this principle may not be strictly true. Thereby raising some doubts about the universality of duality in time series analysis. The question that we now ask is, "Is the duality between models for stationary AR(p) and invertible MA(q) processes universally true? If not, under what condition(s) will the principle break down? This question is what this study intends to address. Therefore, the ultimate objective of this study is to determine the parameter region within which the principle of duality between models for stationary AR(p) and invertible MA(q) processes breaks down. This region has been designated as the 'Non-Duality' region in this study. For the translation from stationary AR(p) process to invertible MA(p) process and, from invertible MA(p) process to stationary AR(p) process, the method adopted to determine the non-duality region was discussed in Section 2. In Section 3, the study illustrated the breakdown in duality principle between stationary AR(p) and invertible MA(p) processes using empirical examples (real life data and simulated series examples) while Section 4 is the concluding remark.

2. Methodology

The methodology adopted to determine the Non-Duality region is the method of moments. The method involves equating non-zero autocovariances of the stationary Autoregressive Process of order p (AR(p)), to corresponding non-zero autocovariances of the invertible Moving Average Process of order p (MA(p)). This is because as noted by [17], the dual relationship between models for stationary AR(p) and invertible MA(q) processes also exists between the autocorrelation functions (ACFs) and partial autocorrelation functions (PACFs). For a given sequence of errors, our bases for determining breakdown in duality is to evaluate

$$\left| R^{\mathrm{AR}(p)}(k) - R^{\mathrm{MA}(p)}(k) \right| = h_k,\ k = 0,1,\cdots,p \tag{2.1}$$

where $R^{\mathrm{AR}(p)}(k)$ is the lag k autocovariance of the stationary AR(p) process, $R^{\mathrm{MA}(p)}(k)$ is the lag k autocovariance of the invertible MA(p) process and h_k is the degree of precision. The principle of duality is considered to have broken down if $h_k \approx 0 \quad \forall k$ to some degree of approximation. The parameters of stationary AR(p) process and invertible MA(p) process for which $h_k \approx 0$ give the region where duality breaks down (Non-duality region). Hence, for various degrees of approximations allowable for h_k, the real values of the parameters for the Non-Duality region are determined.

2.1. Non-Duality between AR(1) and MA(1) Processes

For the model and autocovariances of stationary autoregressive process of order one (AR(1)) [20]:

$$\tilde{X}_t = \phi_1 \tilde{X}_{t-1} + e_t\,;\ e_t \sim N\left(0,\sigma^2\right),\ \ -1 < \phi_1 < 1 \tag{2.2}$$

$$R^{\mathrm{AR}(1)}(k) = \begin{cases} \dfrac{\sigma^2}{1-\phi_1^2},\ k = 0 \\ \dfrac{\phi_1\sigma^2}{1-\phi_1^2},\ \ k = 1 \\ \phi_1 R(k-1),\ k > 1 \end{cases} \tag{2.3}$$

and for invertible moving average process of order one (MA(1)) [20]:

$$\tilde{X}_t = e_t - \theta_1 e_{t-1}; e_t \sim N\left(0,\sigma^2\right),\ \tilde{X}_t = X_t - \mu,\ -1 < \theta_1 < 1 \tag{2.4}$$

$$R^{\mathrm{MA}(1)}(k) = \begin{cases} \sigma^2\left(1+\theta_1^2\right),\ k = 0 \\ -\theta_1\sigma^2,\ k = 1 \\ 0,\ k > 1 \end{cases} \tag{2.5}$$

where $\tilde{X}_t = X_t - \mu$ and $\mu = E\left(X_t\right)$. By equating non-zero autocovariances of AR(p) and MA(p) processes: $p = 1$, we obtain

$$\frac{\sigma^2}{1-\phi_1^2} = \sigma^2\left(1+\theta_1^2\right) \tag{2.6}$$

$$\frac{\phi_1\sigma^2}{1-\phi_1^2} = -\theta_1\sigma^2 \tag{2.7}$$

By combining (2.6) and (2.7), we obtain

$$\phi_1 = \frac{-\theta_1}{1+\theta_1^2} \Rightarrow \phi_1\theta_1^2 + \theta_1 + \phi_1 = 0 \tag{2.8}$$

Note that for invertible MA(1) process, the autocorrelation function of MA(1) process at lag 1 $-0.5 < \rho_1^{\text{MA}(1)} < 0.5$. This implies that $-0.5 < \frac{-\theta_1}{1+\theta_1^2} < 0.5$. Hence, for Non-duality, it is expected that $-0.5 < \phi_1 < 0.5$, since $\phi_1 = \frac{-\theta_1}{1+\theta_1^2}$.

In solving Equation (2.4), two cases arise.

Case I: Non-Duality Region in movement from stationary AR(1) process to invertible MA(1) process.

The solutions for θ_1 in the resulting quadratic Equation (2.8) are

$$\theta_1 = \frac{-1 \pm \sqrt{1-4\phi_1^2}}{2\phi_1} \tag{2.9}$$

For real values $D = 1-4\phi_1^2 \ge 0$, $\Rightarrow 4\phi_1^2 \ge 1$, $\phi_1^2 \ge \frac{1}{4}$ and $-\frac{1}{2} < \phi_1 < \frac{1}{2}$.

For $-1 < \theta_1 < 1$, the acceptable value of θ_1 is

$$\theta_1 = \frac{-1 + \sqrt{1-4\phi_1^2}}{2\phi_1} \tag{2.10}$$

Therefore when moving from stationary AR(1) process to invertible MA(1) process, the region of breakdown of duality is determined using (2.1), and illustrated in Table 1.

Case II: Non-Duality Region in movement from invertible MA(1) process to stationary AR(1) process.

Table 1. Regions of ϕ_1 and θ_1 for non-duality of AR(1) and MA(1) processes.

Degree of approximation allowable for h_k	Region of Non-duality		Remark
	AR(1) Process	MA(1) Process	
	ϕ_1	θ_1	
1	$[-0.38, 0.38]$	$[-0.46, 0.46]$	Negative values ϕ_1 are mapped into positive values of θ_1 and vice versa
2	$[-0.24, 0.24]$	$[-0.26, 0.26]$	Negative values ϕ_1 are mapped into positive values of θ_1 and vice versa
3	$[-0.14, 0.14]$	$[-0.14, 0.14]$	Negative values ϕ_1 are mapped into positive values of θ_1 and vice versa
4	$[-0.08, 0.08]$	$[-0.08, 0.08]$	Negative values ϕ_1 are mapped into positive values of θ_1 and vice versa

When it is assumed that the process is invertible MA(1), that is θ_1 is known, and it is required to determine the region of Non-Duality, ϕ_1 is determined in terms of θ_1 subject to $-1<\phi_1<1$ using Equation (2.8). Hence, when moving from invertible MA(1) process to stationary AR(1) process, the region of breakdown of duality is determined using (2.1), and illustrated in Table 2.

Remark 2.1: As shown in Table 1 and Table 2, if we allow beyond two or more decimal places, the intervals of breakdown of duality are the same for AR(1) process and MA(1) process. However, beyond three or more decimal places, the intervals are quite small and whose significance may be questionable. Therefore, we recommend that for breakdown of duality, one to three decimal places of the absolute difference between the lag k corresponding autocovariances of stationary AR(p) process and that for the invertible MA(p) process may be allowed.

2.2. Non-Duality between AR(2) and MA(2) Processes

For the model and autocovariances of stationary autoregressive process of order two (AR(2)) [20]:

$$\tilde{X}_t=\phi_1\tilde{X}_{t-1}+\phi_2\tilde{X}_{t-2}+e_t;e_t\sim N\left(0,\sigma^2\right),\ \phi_2-\phi_1<1,\phi_2+\phi_1<1,-1<\phi_2<1 \quad (2.11)$$

$$R^{\text{AR}(2)}(k)=\begin{cases}\dfrac{(1-\phi_2)\sigma^2}{(1+\phi_2)(1-\phi_2+\phi_1)(1-\phi_2-\phi_1)}, k=0\\ \dfrac{\phi_1\sigma^2}{(1+\phi_2)(1-\phi_2+\phi_1)(1-\phi_2-\phi_1)}; k=1\\ \dfrac{\left(\phi_1^2+\phi_2(1-\phi_2)\right)\sigma^2}{(1+\phi_2)(1-\phi_2+\phi_1)(1-\phi_2-\phi_1)}; k=2\\ \phi_1R(k-1)+\phi_2R(k-2), \qquad k>2\end{cases} \quad (2.12)$$

and for invertible moving average process of order one (MA(1)) [20]:

Table 2. Regions of θ_1 and ϕ_1 for non-duality of MA(1) and AR(1) Processes.

Degree of approximation allowable for h_k	Region of Non-duality		Remark
	MA(1) Process	AR(1) Process	
	θ_1	ϕ_1	
1	$[-0.47,0.47]$	$[-0.38,0.38]$	Negative values θ_1 are mapped into positive values of ϕ_1 and vice versa
2	$[-0.26,0.26]$	$[-0.24,0.24]$	Negative values θ_1 are mapped into positive values of ϕ_1 and vice versa
3	$[-0.14,0.14]$	$[-0.14,0.14]$	Negative values θ_1 are mapped into positive values of ϕ_1 and vice versa
4	$[-0.08,0.08]$	$[-0.08,0.08]$	Negative values θ_1 are mapped into positive values of ϕ_1 and vice versa

$$\tilde{X}_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} \,;\; e_t \sim N\left(0,\sigma^2\right),\, \theta_2 - \theta_1 < 1,\, \theta_2 + \theta_1 < 1,\, -1 < \theta_2 < 1 \quad (2.13)$$

$$R^{\mathrm{MA}(2)}\left(k\right) = \begin{cases} \sigma^2\left(1+\theta_1^2+\theta_2^2\right),\, k=0 \\ -\theta_1\left(1-\theta_2\right)\sigma^2,\, k=1 \\ -\theta_2\sigma^2,\, k=2 \\ 0,\, k>2 \end{cases} \quad (2.14)$$

By equating corresponding non-zero autocovariances from stationary AR(2) process and invertible MA(2) process respectively, we obtain

$$\frac{1-\phi_2}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} = 1+\theta_1^2+\theta_2^2 \quad (2.15)$$

$$\frac{\phi_1}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} = -\theta_1\left(1-\theta_2\right) \quad (2.16)$$

$$\frac{\phi_1^2+\phi_2\left(1-\phi_2\right)}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} = -\theta_2 \quad (2.17)$$

From (2.15), we have

$$\frac{1}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} = \frac{1+\theta_1^2+\theta_2^2}{1-\phi_2} \quad (2.18)$$

From (2.16), we have

$$\frac{1}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} = \frac{-\theta_1\left(1-\theta_2\right)}{\phi_1} \quad (2.19)$$

From (2.17)

$$\frac{1}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} = \frac{-\theta_2}{\phi_1^2+\phi_2\left(1-\phi_2\right)} \quad (2.20)$$

Equating the right hand sides of (2.18), (2.19) and (2.20), we obtain

$$\frac{-\theta_1\left(1-\theta_2\right)}{\phi_1} = \frac{-\theta_2}{\phi_1^2+\phi_2\left(1-\phi_2\right)} = \frac{1}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} \quad (2.21)$$

Hence, Equation (2.21) is the relational equation connecting ϕ_1, ϕ_2, θ_1 and θ_2. Using transitivity rule, these three equations can be used to obtain the desired solutions. That is

$$\frac{-\theta_1\left(1-\theta_2\right)}{\phi_1} = \frac{-\theta_2}{\phi_1^2+\phi_2\left(1-\phi_2\right)} \quad (2.22)$$

$$\frac{-\theta_1\left(1-\theta_2\right)}{\phi_1} = \frac{1}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} \quad (2.23)$$

$$\frac{-\theta_2}{\phi_1^2+\phi_2\left(1-\phi_2\right)} = \frac{1}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} \quad (2.24)$$

In considering the above three equations, two cases arise resulting in the

Non-Duality region.

Case I: Non-Duality Region in movement from stationary AR(2) process to invertible MA(2) process.

Given the stationary AR(2) process, we determine θ_1 and θ_2 (in terms ϕ_1 and ϕ_2) that must satisfy the invertibility conditions explicitly from the relational Equation (2.24) as

$$\theta_2 = \frac{-\left(\phi_1^2 + \phi_2\left(1-\phi_2\right)\right)}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} = \frac{-\phi_1^2 - \phi_2\left(1-\phi_2\right)}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)} \quad (2.25)$$

Substituting (2.25) into (2.23), we obtain θ_1 as

$$\theta_1 = \frac{-\phi_1}{\left(\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)\right) + \left(\phi_1^2 + \phi_2\left(1-\phi_2\right)\right)} \quad (2.26)$$

From the invertibility conditions of MA(2) process, $|\theta_2| < 1$ implies that

$$|\theta_2| = \left|\frac{-\left(\phi_1^2 + \phi_2\left(1-\phi_2\right)\right)}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)}\right| < 1 \quad (2.27)$$

The expression (2.27) is added to the stationarity conditions to obtain the Non-duality region/conditions for movement from AR(2) to MA(2) processes as

$$\left.\begin{array}{l} \left.\begin{array}{l} \phi_2 - \phi_1 < 1 \\ \phi_2 + \phi_1 < 1 \\ -1 < \phi_2 < 1 \end{array}\right\} \text{stationarity conditions} \\ \left|\dfrac{-\left(\phi_1^2 + \phi_2\left(1-\phi_2\right)\right)}{\left(1+\phi_2\right)\left(1-\phi_2+\phi_1\right)\left(1-\phi_2-\phi_1\right)}\right| < 1 \end{array}\right\} \text{Non-duality region} \quad (2.28)$$

This region (2.28) is illustrated graphically in **Figure 1**. Then, using (2.1), the region of breakdown of duality when moving from stationary AR(2) process to invertible MA(2) process is determined. This region determined is shown in **Table 3**.

Table 3. Regions of (ϕ_1, ϕ_2) and (θ_1, θ_2) for non-duality of AR(2) and MA(2) processes.

Degree of approximation allowable for h_k	Region of Non-duality				Remark
	AR(2) Process		MA(2) Process		
	ϕ_1	ϕ_2	θ_1	θ_2	
1	$[-0.30, 0.30]$	$[-0.30, 0.30]$	$[-0.37, 0.37]$	$[-0.43, 0.32]$	positive values (ϕ_1, ϕ_2) are mapped into negative values of (θ_1, θ_2) and vice versa
2	$[-0.30, 0.30]$	$[-0.20, 0.20]$	$[-0.30, 0.30]$	$[-0.22, 0.20]$	positive values (ϕ_1, ϕ_2) are mapped into negative values of (θ_1, θ_2) and vice versa
3	$[-0.10, 0.10]$	-0.10	$[-0.10, 0.10]$	0.09	positive values (ϕ_1, ϕ_2) are mapped into negative values of (θ_1, θ_2) and vice versa

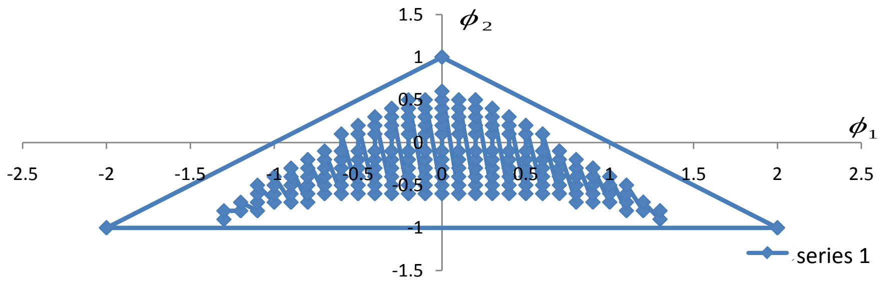

Figure 1. The shaded portion under the triangular region (stationarity region) specifies the non-duality region for movement from AR(2) Process to MA(2) process.

Case II: Non-Duality region in movement from invertible MA(2) process to stationary AR(2) process.

Given the invertible MA(2) process, we determine the values of (ϕ_1 and ϕ_2) that lead to break down in duality from Equation (2.22). Solving for ϕ_1 and ϕ_2, the expression reduces to the quadratic Equation

$$\phi_1^2 - \left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]\phi_1 + \phi_2(1-\phi_2) = 0 \tag{2.29}$$

Equation (2.29) is a quadratic equation in both ϕ_1 and ϕ_2. By either solving for ϕ_1 first in (2.29) or ϕ_2 first in (2.29), Case II breaks into two-sub cases. Case II (a) involves solving for ϕ_1 first in (2.29) and we obtain

$$\phi_1 = \frac{\frac{\theta_2}{\theta_1(1-\theta_2)} \pm \sqrt{\left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]^2 - 4\phi_2(1-\phi_2)}}{2} \tag{2.30}$$

for fixed θ_1 and θ_2 with $-1 < \phi_2 < 1$. For real roots of ϕ_1, it is requested that the discriminant

$$D = 4\phi_2^2 - 4\phi_2 + \left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]^2 \geq 0 \tag{2.31}$$

For fixed values of θ_1 and θ_2 satisfying the invertibility conditions, Equation (2.31) is a quadratic inequality in ϕ_2. First, we solve for ϕ_2 in

$$4\phi_2^2 - 4\phi_2 + \left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]^2 = 0 \tag{2.32}$$

to obtain

$$\phi_2 = \frac{1 \pm \sqrt{1 - \left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]^2}}{2} \tag{2.33}$$

Since (2.26) is a quadratic inequality, the roots of (2.33) split the interval for ϕ_2, $\phi_2 \in (-1,1)$, into three intervals [29] [30]. If we define

$$W = \frac{\theta_2}{\theta_1(1-\theta_2)}. \tag{2.34}$$

then, the intervals are $\left(-1,\frac{1-\sqrt{1-W^2}}{2}\right)$, $\left[\frac{1-\sqrt{1-W^2}}{2},\frac{1+\sqrt{1-W^2}}{2}\right]$ and $\left(\frac{1+\sqrt{1-W^2}}{2},1\right)$.

From (2.33), it is clear that for two distinct real roots of ϕ_2;

$$\left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]^2<1 \tag{2.35}$$

$$\Rightarrow\left|\frac{\theta_2}{\theta_1(1-\theta_2)}\right|<1 \tag{2.36}$$

Similarly, Case II (b) involves solving for ϕ_2 first in (2.29) to obtain

$$\phi_2=\frac{1\pm\sqrt{1+4\phi_1\left[\phi_1-\left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]\right]}}{2} \tag{2.37}$$

for fixed θ_1 and θ_2 with $-2<\phi_1<2$. For real roots of ϕ_2,

$$1+4\phi_1\left[\phi_1-\left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]\right]\geq 0 \tag{2.38}$$

Equation (2.38) is a quadratic inequality on ϕ_1. Solving for ϕ_1, we use

$$1+4\phi_1\left[\phi_1-\left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]\right]=0 \tag{2.39}$$

to obtain

$$\phi_1=\frac{\frac{\theta_2}{\theta_1(1-\theta_2)}\pm\sqrt{\left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]^2-1}}{2} \tag{2.40}$$

for fixed θ_1 and θ_2. Since (2.38) is a quadratic inequality, (2.40) split the interval for ϕ_1, $\phi_1\in(-2,2)$, into three intervals [30]. The intervals are: $\left(-2,\frac{W-\sqrt{W^2-1}}{2}\right)$, $\left[\frac{W-\sqrt{W^2-1}}{2},\frac{W+\sqrt{W^2-1}}{2}\right]$ and $\left(\frac{W+\sqrt{W^2-1}}{2},2\right)$

From (2.40), it is clear that for distinct real roots of ϕ_1;

$$\left[\frac{\theta_2}{\theta_1(1-\theta_2)}\right]^2>1 \tag{2.41}$$

$$\Rightarrow\left|\frac{\theta_2}{\theta_1(1-\theta_2)}\right|>1 \tag{2.42}$$

Equation (2.36) and (2.42) combined with the invertibility conditions gives what we call the Non-Duality conditions/region when going from MA(2) to AR(2) processes. The Non-Duality conditions/region when moving from MA(2) to AR(2) processes is given as

$$\left.\begin{array}{l}\left.\begin{array}{l}\theta_2-\theta_1<1\\ \theta_2+\theta_1<1\\ -1<\theta_2<1\end{array}\right\}\text{invertibility conditions}\\ \left|\dfrac{\theta_2}{\theta_1\left(1-\theta_2\right)}\right|\neq 1\end{array}\right\}\text{Non-duality region} \tag{2.43}$$

See **Figure 2** for illustration of Equation (2.43). Then, using (2.1), the region of breakdown of duality when moving from invertible MA(2) process to stationary AR(2) process for case IIa and case IIb are determined and illustrated in **Table 4** and **Table 5** respectively.

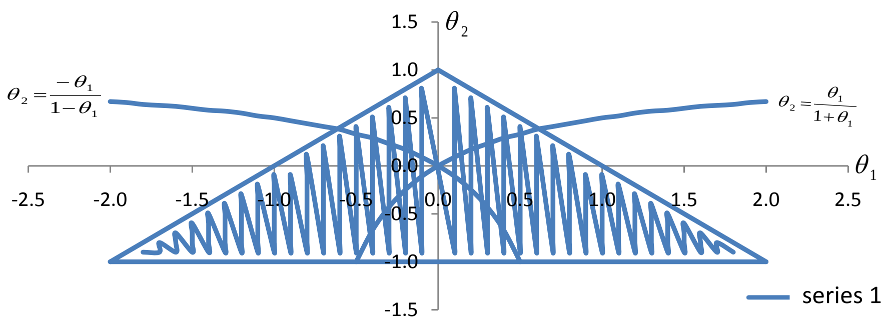

Figure 2. The shaded portion under the triangular region (invertibility region) specifies the non-duality region for movement from MA(2) Process to AR(2) process.

Table 4. Regions of (θ_1,θ_2) and (ϕ_1,ϕ_2) for non-duality of MA(2) and AR(2) processes.

Degree of approximation allowable for h_k	Region of Non-duality				Remark
	MA(2) Process		AR(2) Process		
	θ_1	θ_2	ϕ_1	ϕ_2	
1	0.10	−0.11	$[-0.12,-0.07]$	$[0.07,0.12]$	positive values (θ_1,θ_2) are mapped into negative values of (ϕ_1,ϕ_2) and vice versa
2	0.10	−0.11	−0.10	0.10	positive values (θ_1,θ_2) are mapped into negative values of (ϕ_1,ϕ_2) and vice versa

Table 5. Regions of (θ_1,θ_2) and (ϕ_1,ϕ_2) for non-duality of MA(2) and AR(2) processes.

Degree of approximation allowable for h_k	Region of Non-duality				Remark
	MA(2) Process		AR(2) Process		
	θ_1	θ_2	ϕ_1	ϕ_2	
1	−0.11	−0.22	$[0.09,0.11]$	$[0.17,0.21]$	positive values (θ_1,θ_2) are mapped into negative values of (ϕ_1,ϕ_2) and vice versa
2	−0.11	−0.22	0.10	0.20	positive values (θ_1,θ_2) are mapped into negative values of (ϕ_1,ϕ_2) and vice versa

3. Empirical Examples

This section presents some empirical examples to illustrate the results obtained in Section 2. Both simulated and real life time series data were used for the illustration. The first set of simulations consist of one thousand (1000) stationary AR(2) series of 120 observations each, simulated using $e_t \sim N(0,\sigma^2)$ and $(\varphi_1,\varphi_2)=(-0.30,-0.30)$ which satisfy the non-duality conditions defined in Equation (2.28). The estimates of the autocorrelation function, acf (ρ_k) and the partial autocorrelation function, pacf (ϕ_{kk}) of the simulated series cut off after lag two. These indicate that the patterns in the series may be adequately described by either the stationary AR(2) process or the invertible MA(2) process. Since the series were simulated from stationary AR(2) process, an MA(2) model was fitted to the series. The estimate of the acf and the pacf of the residuals from the fitted MA(2) model all lie within 2 standard deviations from mean (zero) indicating that the fitted model is adequate. The estimates of the means and variances of the residuals are also not significantly different from 0 and 1 respectively. These indicate that the model adequately describes the pattern in the simulated series. The estimates of θ_1 and θ_2, obtained using MINITAB 17 series software are shown in **Table 6** (for ten simulated series only for want of space). As **Table 6** shows, the estimates also: (i) satisfy the invertibility conditions and (ii) compare favourably well, in absolute terms, with those of stationary AR(2) process but with opposite signs. These indicate that a stationary AR(2) process can be adequately represented by an invertible MA(2) process contrary to the widely held duality concept.

The second set of simulations consists of one thousand (1000) invertible MA(2) series of 120 observations each, simulated using $e_t \sim N(0,\sigma^2)$ and $(\theta_1,\theta_2)=(0.37,0.27)$ which satisfy the non-duality conditions defined in Equation (2.43). The estimates of the autocorrelation function, acf (ρ_k) and the partial autocorrelation function, pacf (ϕ_{kk}) of the simulated series cut off after lag two. These indicate that the patterns in the series may be adequately described by either the invertible MA(2) process or the stationary AR(2) process. Since the series were simulated from invertible MA(2) process, the AR(2) model was fitted to the series. The estimates of the mean, variance, acf and the pacf of the residuals from the fitted AR(2) model indicate that the model adequately describes the pattern in the simulated series. The estimates of ϕ_1 and ϕ_2 obtained using MINITAB 17 series software shown in **Table 7** (for ten simulations for want of space) also: (i) satisfy the stationarity conditions and (ii) compare favourably well, in absolute terms, with those of invertible MA(2) process with opposite signs. These indicate that an invertible MA(2) process can be adequately represented by a stationary AR(2) process contrary to the widely held duality concept.

Furthermore, the breakdown of duality was illustrated using the real life data on the monthly average exchange rate of Naira per unit of CFA currency collected from Central Bank of Nigeria for the period January 2002 to December 2013 [31] [32]. The estimates of ρ_k and ϕ_{kk} of the original series (X_t) shown

Table 6. Estimates of parameters of MA(2) model fitted to simulated AR(2) process with $\phi_1 = -0.30$, $\phi_2 = -0.30$ and $e_t \sim N(0,1)$.

S/No.	i	AR(2) Model				i	MA(2) Model			
		$\hat{\phi}_i$	$std(\hat{\phi}_i)$	t-value	$\hat{\sigma}^2$		$\hat{\theta}_i$	$std(\hat{\theta}_i)$	t-value	$\hat{\sigma}^2$
1	1	−0.3241	0.0943	−3.44	1.1220	1	0.3105	0.0954	3.25	1.1020
	2	−0.3612	0.0953	−3.79		2	0.3286	0.0958	3.43	
2	1	−0.2162	0.0961	−2.25	1.0890	1	0.2218	0.0974	2.28	1.0900
	2	−0.3130	0.0964	−3.25		2	0.2563	0.0975	2.63	
3	1	−0.2127	0.0974	−2.18	0.8725	1	0.2718	0.0975	2.79	0.8520
	2	−0.3010	0.0973	−3.09		2	0.2762	0.0975	2.83	
4	1	−0.3098	0.0944	−3.28	0.9103	1	0.3323	0.0979	3.39	0.9220
	2	−0.3606	0.0945	−3.82		2	0.2468	0.0981	2.52	
5	1	−0.2619	0.0934	−2.81	1.1780	1	0.2009	0.0960	2.09	1.2210
	2	−0.3873	0.0934	−4.15		2	0.3390	0.0963	3.52	
6	1	−0.2957	0.0909	−3.25	0.8452	1	0.2356	0.0946	2.49	0.8851
	2	−0.4419	0.0916	−4.82		2	0.3531	0.0953	3.71	
7	1	−0.2177	0.0961	−2.26	1.0073	1	0.2896	0.0961	3.02	0.9750
	2	−0.3064	0.0962	−3.18		2	0.3080	0.0961	3.21	
8	1	−0.2970	0.0949	−3.13	0.9185	1	0.3579	0.0972	3.68	0.9009
	2	−0.3395	0.0951	−3.57		2	0.2726	0.0972	2.80	
9	1	−0.2634	0.0953	−2.76	1.092	1	0.2035	0.0983	2.07	1.1390
	2	−0.3477	0.0956	−3.64		2	0.2255	0.0983	2.29	
10	1	−0.3409	0.0947	−3.60	0.9556	1	0.3399	0.0992	3.43	0.9664
	2	−0.3476	0.0949	−3.66		2	0.2084	0.1017	2.05	

Table 7. Estimates of parameters of AR(2) model fitted to simulated MA(2) model with $\theta_1 = 0.37$, $\theta_2 = 0.27$ and $e_t \sim N(0,1)$.

S/No.	i	MA(2) Model				i	AR(2) Model			
		$\hat{\theta}_i$	$std(\hat{\theta}_i)$	t-value	$\hat{\sigma}^2$		$\hat{\phi}_i$	$std(\hat{\phi}_i)$	t-value	$\hat{\sigma}^2$
1	1	0.3427	0.0981	3.49	1.0136	1	−0.2615	0.0967	−2.70	1.0450
	2	0.2862	0.0984	2.91		2	−0.3528	0.0971	−3.63	
2	1	0.3513	0.0943	3.72	1.0510	1	−0.2013	0.0964	−2.09	1.1560
	2	0.3412	0.0943	3.62		2	−0.3011	0.0966	−3.12	
3	1	0.3902	0.0929	4.20	1.1770	1	−0.3004	0.0953	−3.15	1.2770
	2	0.3940	0.0933	4.23		2	−0.3500	0.0953	−3.67	
4	1	0.3629	0.0967	3.75	1.0590	1	−0.2879	0.0973	−2.96	1.1230
	2	0.2958	0.0970	3.05		2	−0.2709	0.0973	−2.78	

Continued

5	1	0.3322	0.0977	3.40	0.9375	1	−0.2472	0.0970	−2.55	0.9769
	2	0.2830	0.0977	2.90		2	−0.2901	0.0994	−2.92	
6	1	0.4133	0.0975	4.24	0.9956	1	−0.3220	0.0963	−3.34	1.0670
	2	0.2674	0.0991	2.70		2	−0.3053	0.0962	−3.17	
7	1	0.2955	0.0983	3.01	0.9634	1	−0.2575	0.0967	−2.66	0.9719
	2	0.2476	0.0990	2.50		2	−0.2996	0.0983	−3.05	
8	1	0.3655	0.0987	3.70	1.0770	1	−0.3685	0.0933	−3.95	1.0540
	2	0.2279	0.0987	2.31		2	−0.3906	0.0941	−4.15	
9	1	0.3554	0.0976	3.64	1.2330	1	−0.3564	0.0954	−4.74	1.2280
	2	0.2808	0.0978	2.87		2	−0.3401	0.0961	−3.54	
10	1	0.2389	0.0958	2.49	1.0290	1	−0.2598	0.0940	−2.76	1.0260
	2	0.3301	0.0963	3.43		2	−0.3869	0.0941	−4.11	

in **Table 8** suggest that the series requires differencing to remove non-stationarity in mean. The estimates of ρ_k and ϕ_{kk} of the first order differenced series $\left((1-B)X_t\right)$ shown in **Table 8**, indicates that both ρ_k and ϕ_{kk} cut off after lag one. This suggests that the pattern in the differenced series may be adequately described by either AR(1) process or MA(1) process. The estimates ($\hat{\rho}_k$ and $\hat{\phi}_{kk}$) of the residuals from the fitted models (AR(1) and MA(1)) also shown in **Table 8**, indicate that both models adequately describe the pattern in the series [33]. The estimate $\left(\hat{\phi}_1 = -0.2369\right)$ of ϕ_1 in the AR(1) process, shown in **Table 9** satisfies the stationarity condition, while the estimate $\left(\hat{\theta}_1 = 0.2581\right)$ of θ_1 in MA(1) process also shown in **Table 9** satisfy the invertibility condition. In addition, both estimates ($\hat{\phi}_1$ and $\hat{\theta}_1$) compare favourably well in absolute terms, but with opposite signs. Here again contrary to the widely held duality concept, these indicate that a stationary AR(1) process can be adequately represented by an invertible MA(1) process.

The second real life data used to illustrate the breakdown of duality is the monthly All Shares Index of the Nigeria Stock Exchange (NSE), for the period January 1, 1985 to December 31, 2007, collected from the CBN Statistical Bulletin [31] [32] [34]. Following the procedure of [35] [36], the original series was shown to require logarithmic transformation to stabilize its variance. The estimates of ρ_k and ϕ_{kk} of the transformed series (Y_t) shown in **Table 10**, suggest that the series requires differencing to remove non-stationarity in mean. The estimates ($\hat{\rho}_k$ and $\hat{\phi}_{kk}$) of acf and pacf of the first order differenced series (Z_t) also shown in **Table 10**, indicate that both $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ cut off after lag five. These suggest that the pattern in the series may be adequately described by either AR(5) model or MA(5) model. Both models were fitted to the series and the estimates ($\hat{\rho}_k$ and $\hat{\phi}_{kk}$) of acf and pacf of the residuals from the fitted models shown in **Table 10**, indicate that both models adequately describe the pattern in the series. However, some of the estimates of the parameters of the fitted

Table 8. The estimates of ρ_k and ϕ_{kk} of X_t, $(1-B)X_t$ and residuals from the fitted models (AR(1) and MA(1) Models).

k	CFA							
	X_t		$(1-B)X_t$		Residuals (e_t)			
					AR(1)		MA(1)	
	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	$\hat{\rho}_k$	$\hat{\phi}_{kk}$
1	0.94	0.94	−0.25	−0.25	−0.03	−0.03	−0.01	−0.01
2	0.90	0.13	−0.05	−0.11	−0.09	−0.09	−0.03	−0.03
3	0.86	0.03	0.08	0.04	0.07	0.06	0.06	0.06
4	0.82	−0.05	−0.04	−0.01	−0.03	−0.04	−0.03	−0.03
5	0.78	−0.01	−0.02	−0.03	−0.02	−0.01	−0.02	−0.02
6	0.74	0.01	0.04	0.02	0.02	0.01	0.02	0.01
7	0.71	0.04	−0.07	−0.06	−0.07	−0.07	−0.07	−0.07
8	0.68	0.07	0.01	−0.02	−0.01	−0.01	−0.01	−0.01
9	0.66	0.00	0.00	−0.01	0.00	−0.02	−0.01	−0.02
10	0.63	−0.02	−0.03	−0.03	−0.04	−0.04	−0.05	−0.05
11	0.61	−0.02	−0.03	−0.05	−0.07	−0.08	−0.08	−0.09
12	0.59	0.00	−0.12	−0.16	−0.15	−0.17	−0.16	−0.18
13	0.57	0.06	−0.02	−0.10	−0.07	−0.10	−0.09	−0.10
14	0.55	0.04	−0.08	−0.16	−0.10	−0.15	−0.12	−0.15
15	0.54	0.06	−0.02	−0.10	−0.04	−0.08	−0.05	−0.07
16	0.54	0.02	0.00	−0.07	0.00	−0.06	−0.01	−0.05
17	0.53	−0.01	0.01	−0.03	−0.01	−0.04	0.00	−0.03
18	0.52	−0.02	−0.08	−0.12	−0.05	−0.10	−0.04	−0.08
19	0.51	0.06	0.14	0.06	0.16	0.11	0.16	0.11
20	0.49	−0.06	0.08	0.12	0.10	0.09	0.11	0.10
21	0.47	−0.09	−0.08	−0.02	−0.04	−0.02	−0.03	−0.03
22	0.45	−0.01	0.13	0.10	0.11	0.10	0.12	0.08
23	0.42	−0.08	−0.06	−0.04	−0.04	−0.08	−0.04	−0.09
24	0.39	−0.01	−0.02	−0.06	−0.04	−0.07	−0.04	−0.08
25	0.36	−0.03	−0.01	−0.13	−0.03	−0.14	−0.02	−0.13
26	0.35	0.09	−0.03	−0.13	0.00	−0.08	0.01	−0.07
27	0.34	0.10	0.13	0.06	0.15	0.11	0.15	0.11
28	0.31	−0.22	0.07	0.09	0.09	0.06	0.09	0.06
29	0.28	−0.01	−0.11	−0.04	−0.09	−0.06	−0.08	−0.06
30	0.26	0.01	0.06	0.01	0.04	0.03	0.04	0.03
31	0.24	0.00	0.01	0.05	0.01	0.05	0.00	0.05
32	0.22	−0.04	−0.06	0.01	−0.07	0.00	−0.07	0.00
33	0.20	0.02	−0.01	0.00	−0.04	0.00	−0.04	0.00
34	0.18	0.02	−0.04	0.00	−0.04	0.00	−0.05	0.00
35	0.18	0.03	0.01	0.01	0.01	0.01	0.00	0.00
36	0.18	0.06						

Table 9. Estimates of parameters of AR(1) process and MA(1) process fitted to the monthly average exchange rate of Naira per unit of CFA currency.

AR(1) Model				MA(1) Model			
Coefficient	Estimate	Std Error	t-value	Coefficient	Estimate	Std Error	t-value
ϕ_1	−0.2369	0.0816	−2.90	θ_1	0.2581	0.0811	3.18
$\hat{\sigma}^2$		0.0002		$\hat{\sigma}^2$		0.0002	

Table 10. The estimates of ρ_k and ϕ_{kk} of X_t, $(1-B)X_t$ and residuals from the fitted models (AR(5) process and MA(5) process).

	NSE ALL SHARES INDEX							
k	$Y_t = \log_e X_t$		$Z_t = (1-B)Y_t$		Residuals (e_t)			
					AR(5)		MA(5)	
	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	$\hat{\rho}_k$	$\hat{\phi}_{kk}$
1	0.99	0.99	0.19	0.19	0.00	0.00	0.00	0.00
2	0.98	−0.01	0.15	0.12	0.01	0.01	−0.01	−0.01
3	0.97	0.00	0.06	0.01	−0.02	−0.02	0.02	0.02
4	0.96	−0.01	−0.02	−0.06	0.00	0.00	0.00	0.00
5	0.95	−0.01	0.18	0.19	0.01	0.01	0.01	0.01
6	0.93	−0.03	0.04	−0.02	−0.01	−0.01	0.03	0.03
7	0.92	−0.01	0.02	−0.04	−0.05	−0.05	−0.01	−0.01
8	0.91	−0.01	0.08	0.08	0.07	0.07	0.08	0.08
9	0.90	−0.01	0.05	0.05	0.04	0.04	0.02	0.02
10	0.89	0.00	0.06	−0.01	0.02	0.02	0.05	0.05
11	0.88	0.00	−0.01	−0.04	−0.03	−0.03	−0.03	−0.03
12	0.87	0.01	−0.03	−0.02	−0.02	−0.01	−0.01	−0.01
13	0.86	0.01	−0.08	−0.10	−0.12	−0.12	−0.10	−0.11
14	0.84	0.00	0.04	0.07	0.03	0.02	0.03	0.02
15	0.83	−0.01	0.04	0.04	0.01	0.02	0.02	0.01
16	0.82	−0.01	0.10	0.08	0.11	0.11	0.11	0.11
17	0.81	−0.02	0.02	−0.03	−0.01	−0.02	−0.01	−0.02
18	0.80	0.02	0.02	0.03	0.02	0.01	0.01	0.00
19	0.79	0.00	0.07	0.05	0.07	0.07	0.06	0.06
20	0.78	0.00	0.08	0.05	0.10	0.10	0.09	0.10
21	0.77	0.00	−0.02	−0.09	−0.05	−0.03	−0.05	−0.04
22	0.76	0.00	−0.03	−0.02	0.00	0.01	0.00	−0.01
23	0.75	−0.01	−0.07	−0.05	−0.04	−0.03	−0.04	−0.04
24	0.74	−0.01	−0.09	−0.11	−0.04	−0.07	−0.04	−0.07
25	0.73	−0.01	−0.11	−0.13	−0.10	−0.12	−0.10	−0.12
26	0.72	−0.01	−0.10	−0.04	−0.07	−0.08	−0.05	−0.08

Continued

27	0.71	−0.02	−0.13	−0.08	−0.10	−0.10	−0.10	−0.10
28	0.70	0.00	−0.06	0.00	0.01	−0.02	0.00	−0.02
29	0.69	0.00	−0.11	−0.05	−0.10	−0.09	−0.10	−0.08
30	0.68	0.04	−0.03	0.04	0.00	−0.01	0.01	0.01
31	0.67	−0.01	0.02	0.08	0.03	0.04	0.02	0.05
32	0.66	−0.01	−0.02	0.04	0.02	0.04	0.01	0.04
33	0.65	−0.01	−0.04	−0.02	−0.03	0.01	−0.03	0.02
34	0.63	−0.04	−0.02	0.01	0.00	0.00	−0.01	0.01
35	0.62	0.01	−0.03	−0.01	−0.05	−0.05	−0.06	−0.05
36	0.61	−0.02	0.11	0.10	0.14	0.11	0.13	0.12
37	0.60	−0.02	0.03	0.00	0.03	0.05	0.03	0.05
38	0.59	−0.01	−0.02	−0.07	−0.02	−0.05	−0.03	−0.05
39	0.58	−0.01	−0.04	−0.06	−0.04	−0.07	−0.05	−0.08
40	0.57	0.00	0.00	0.04	0.04	0.02	0.04	0.02
41	0.56	−0.02	−0.01	−0.01	−0.02	0.01	−0.02	0.01
42	0.54	−0.03	−0.03	−0.04	−0.03	−0.04	−0.04	−0.04
43	0.53	−0.02	−0.06	0.01	−0.04	0.02	−0.04	0.00
44	0.52	0.00	−0.04	0.03	−0.01	0.03	0.00	0.02
45	0.51	0.00	−0.05	−0.02	−0.05	−0.02	−0.05	−0.01
46	0.50	0.01	−0.02	−0.02	−0.02	−0.02	−0.01	−0.02
47	0.48	−0.02	0.01	0.05	0.03	0.05	0.03	0.04
48	0.47	0.00	−0.04	−0.05	−0.01	−0.01	−0.01	−0.02
49	0.46	0.01	−0.08	−0.08	−0.08	−0.06	−0.09	−0.06
50	0.45	0.00	0.00	−0.01	0.01	−0.03	0.01	−0.04
51	0.44	−0.01	0.03	0.01	0.04	0.01	0.03	0.00
52	0.43	0.01	0.06	−0.02	0.09	0.01	0.08	0.02
53	0.42	−0.01	−0.03	−0.06	−0.03	−0.05	−0.03	−0.05
54	0.41	0.00	−0.05	−0.05	−0.05	−0.06	−0.05	−0.06
55	0.40	−0.01	−0.05	−0.05	−0.03	−0.07	−0.03	−0.06
56	0.38	0.00	−0.01	−0.02	0.03	−0.04	0.02	−0.03
57	0.37	−0.01	−0.06	−0.03	−0.07	−0.06	−0.08	−0.06
58	0.36	0.00	−0.03	0.03	−0.03	−0.01	−0.02	−0.01
59	0.35	−0.01	0.00	0.05	0.02	0.03	0.02	0.04
60	0.34	0.00	0.02	0.05	0.03	0.03	0.03	0.03
61	0.33	0.01	0.03	0.05	0.01	0.05	0.01	0.05
62	0.32	0.00	0.03	0.06	0.04	0.07	0.04	0.07

models shown in **Table 11** are not significant. Therefore, the subset models were fitted to the series following the selection of subset time series models by [37]

Table 11. Estimates of parameters of AR(5) model and MA(5) model fitted to the monthly all shares index of the Nigeria Stock Exchange (NSE).

AR(5)				MA(5)			
Coefficient	**Estimate**	**Std Error**	**t-value**	**Coefficient**	**Estimate**	**Std Error**	**t-value**
Constant ϕ_0	0.0136	0.0029	**4.75**	**Constant** θ_0	0.0228	0.0042	5.42
ϕ_1	0.1745	0.0599	**2.91**	θ_1	−0.1741	0.0602	−2.89
ϕ_2	0.1220	0.0607	**2.01**	θ_2	−0.1756	0.0612	−2.87
ϕ_3	−0.0015	0.0612	**−0.02**	θ_3	−0.0156	0.0621	−0.25
ϕ_4	−0.0895	0.0607	**−1.47**	θ_4	0.0656	0.0612	1.07
ϕ_5	0.1965	0.0603	**3.26**	θ_5	−0.1711	0.0606	−2.82
$\hat{\sigma}^2$		0.0022		$\hat{\sigma}^2$		0.0022	

Table 12. Estimates of parameters of subset AR(5) model and subset MA(5) model for the models fitted to the monthly all shares index of the Nigeria Stock Exchange (NSE).

AR(5) SUBSET				**MA(5) SUBSET**			
Coefficient	**Estimate**	**Std Error**	**t-value**	**Coefficient**	**Estimate**	**Std Error**	**t-value**
Constant ϕ_0	0.0126	0.0036	**3.47**	**Constant** θ_0	0.0202	0.0038	5.31
ϕ_1	0.1697	0.0599	**2.83**	θ_1	−0.1648	0.0404	−4.07
ϕ_2	0.1092	0.0602	**1.82**	θ_2	−0.1793	0.0454	−3.95
ϕ_5	0.1801	0.0594	**3.03**	θ_5	−0.1774	0.0388	−4.57
$\hat{\sigma}^2$		0.0022		$\hat{\sigma}^2$		0.0022	

and the non-linear least squares method of estimation of parameters by [38]. The estimates of the parameters of the subset AR(5) model and the subset MA(5) model shown in **Table 12** compare favourably well in absolute terms with opposite signs. Contrary to the widely held duality concept, these indicate that an AR(5) series can be adequately represented by an MA(5) model and conversely.

4. Conclusions

In summary, this study has demonstrated analytically and empirically that the concept of duality that a finite order stationary autoregressive process of order p (AR(p)) stationary is equivalent to an infinite order moving average (MA) process and invertible moving average of order q (MA(q)) is equivalent to an infinite order autoregressive (AR) process is not universally true. Specifically, this study constructed regions of breakdown of duality starting from AR(p) to MA(p) processes and from MA(p) to AR(p) processes for $p = 1, 2$. These regions, designated as "Non-duality" regions in this study, have been illustrated mathematically and graphically. In these regions (a) both the Autocorrelation function and the Partial Autocorrelation function of the AR process and MA process cuts off after equal lags (b) a finite AR model can be adequately represented by a finite MA model of equal order and conversely with the same error variance and (c) negative values of the parameters of the AR process are mapped into positive values of the parameters of the equivalent MA process and conversely.

In view of these, it has been recommended that the concept of duality should be treated with caution in analysis of time series data.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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