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Toric Realisation of Quantum Systems via Clifford Structures

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Abstract

We present a geometric approach to describe quantum information systems. Precisely, we elaborate a genuine correspondence between toric geometry and qubit systems using the construction of Clifford graph algebras. This mapping may be explored to investigate qubit information applications using toric geometry considered as a potent tool to understand high energy physics including black holes and string theory. Explicitly, we examine in some details the cases of two and three qubits, and we find that they are associated with certain Clifford graph structures. Mixed states are also discussed.

Keywords

Qubit Information Systems, Clifford Graph Algebras, Toric Geometry

1. Introduction

Quantum information combined with toric geometry is considered a powerful tool to study complex varieties used in modern physics like string theory and related models [1] [2]. As well, toric geometry has been also used to build mirror manifolds providing an outstanding way to understand the extension of T-duality in the presence of D-branes moving near toric Calabi-Yau singularities using combinatorial calculations. Also, it has been remarked that such a combination can be exploited to understand a special class of black hole solutions obtained from type II superstrings on local Calabi-Yau manifolds [3] [4]. Recently, the black hole physics has found a place in quantum information theory using qubit building blocks. Precisely, many connections have been established in the context of STU black holes as proposed in Refs [5] [6].

Studies have been realized to develop new geometric approach to deal with qubit information systems using hypercube graph theory and toric geometry [7] [8]. These investigations have brought new understanding of the fundamental physics associated with qubits and theirs supersymmetric extensions. These latter are connected to various theories including D-branes, toric geometry and supermanifolds. Precisely, a nice interplay between the black holes and qubits has been discussed using higher-dimensional supergravity models [9] [10]. Moreover, Clifford algebras play a key role in the link between quantum information and physics of black holes, rigorously dyonic black classes obtained from lower dimensional toroidal compactification are mapped to Clifford algebras using vee product [11]. Moreover, the superpotential depending on four quaternionic fields has been considered as an element of a particular Clifford class [12].

The aim of this paper is to contribute to this program by introducing a mapping between Clifford graph algebras, toric geometry and its relation to quantum information systems. The main objective is to deal with qubit systems using geometry and graph theory which is considered a potent tool to grasp modern physics such as string theory and related models. As an illustration, we examine lower-dimensional qubit systems. In particular, we consider in some details the cases of one, two and three qubits. Concretely, we find that they are linked with $CP^1 \times CP^1$ and $CP^1 \times CP^1 \times CP^1$ toric varieties respectively. Using a geometric procedure, we show that the qubit physics can be converted into a scenario working with toric data of such manifolds with the help of Clifford graph algebras. Mixed states are also approached.

The present paper is organized as follows. In Section 2, we give an overview of the construction of Clifford graph algebras. Section 3 provides materials on toric geometry which is used to discuss qubit information systems. The connection between these later and Clifford graph algebra is explored in Section 4 where we focus on a one-to-one correspondence that binds them all. Section 4 is devoted to some concluding remarks.

2. Construction of Clifford Graph Algebras

The strong character of Clifford algebras gives the possibility of using them as a mapping between quantum computation, and high energy physics [13]. They can play a key role in supersymmetry as supersymmetric description of qubits systems [14]. To make a link between graph theory and quantum information systems, we present an overview of a useful construction of Clifford graph algebras.

Let A is a classical Clifford algebra, then it is an unital algebra over $\mathbb{C}$ with n generators $e_1, e_2, \cdots, e_n$ which verify the following relations

$$\begin{cases} e_i^2 = -1 & \text{for any } i \\ e_i e_j = -e_j e_i & \text{for } i \neq j \end{cases} \tag{2.1}$$

Every monomial in this algebra either commute or anticommutes with generators e_i. The center of this algebra is none other than the subalgebra of elements that commute with all elements. Moreover, a graph G with m vertices and no multiple edges and no loops can be associated with an algebras A_G over $\mathbb{C}$, with m generators corresponding to the vertices as

$$\begin{cases} e_i^2 = 1 & \text{for any } i \\ e_i e_j = -e_j e_i & \text{if there is an edge between the } i^{\text{th}} \text{ and } j^{\text{th}} \text{ vertices} \\ e_i e_j = e_j e_i & \text{if there is no edge between the } i^{\text{th}} \text{ and } j^{\text{th}} \text{ vertices} \end{cases} \tag{2.2}$$

In the present work, we are interested in such a construction of graphs called Clifford graph algebras which may be used to describe quantum information systems. Precisely, we focus on the dimension of the center of Clifford algebra. More precisely, it has been shown that two graphs with the same number of vertices belong to the same Clifford class if the centers of their Clifford algebras have the same dimension. Their Clifford algebras are isomorphic [15].

To assimilate the purpose of this, let a Clifford graph algebras have dimension 2^n over $\mathbb{C}$, and has a basis of monomials e_α, where every element $c = \sum a_\alpha e_\alpha$ of the center form a subalgebra and also the dimension of the center is always a power of 2. For instance, as we can notice in **Figure 1** if the generators of the Clifford graph algebra of the complete graph are e_i and the generators of the Clifford of path graph are e_i'. Then, it has been verified that the Clifford algebras of these two graphs are isomorphic by giving an isomorphism as

$$\begin{cases} e_1' = e_1 \\ e_2' = e_1 e_2 \\ e_i' = e_2 e_i & \text{for } i > 2 \end{cases} \tag{2.3}$$

These graphs with the same dimension of Clifford algebras belong to the same Clifford class and can be used to graphically describe and present the physics of n-qubit systems.

3. Toric Geometry and Clifford Graph Algebra

Toric geometry has been considered as a potent tool in context of study of complex Calabi-Yau manifolds used in the string theory compactification and associated subjects [16].

Firstly, $n/2$-complex dimensional toric manifold, which we denote as $M_\Delta^{n/2}$, is obtained by considering the $\left(\frac{n}{2}+r\right)$-dimensional complex spaces $C^{\frac{n}{2}+r}$ parameterized by homogeneous coordinates $x = \left(x_1; x_2, x_3; \cdots; x_{\frac{n}{2}} + r\right)$, and r toric transformations $\hat{T}_a$ acting on the x_i's as follows

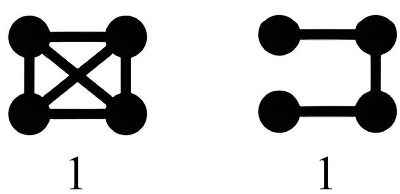

Figure 1. Isomorphism between Clifford graph algebras of complete and path graph.

$$T_a : x_i \to x_i \left(\lambda_a^{q_i^a} \right). \tag{3.1}$$

Here, we consider the λ_a as r non-vanishing complex parameters. Then, the q_i^a are integers called Mori vectors for each a which encode several geometrical information on the manifold and its applications to modern physics. Thus these toric manifolds can be identified with the coset space $C^{\frac{n}{2}+r} \Big/ C^{*r}$. In fact, toric graphic realization is generally represented by an integral polytope Δ, namely a toric diagram which is spanned by $\left(\frac{n}{2}+r\right)$ vertices v_i of the standard lattice Z^n. The toric data $\{v_i; q_{ia}\}$ must satisfy the following r relations

$$\sum_{i=0}^{\frac{n}{2}+r-1} q_i^a v_i = 0, \quad a = 1, \cdots, r. \tag{3.2}$$

These equations encode geometric data of $M_\Delta^{n/2}$. The q_i^a integers are interpreted in lower dimensional field theory, namely in the $N = 2$ gauge linear sigma model language, as the $U(1)^r$ gauge charges of $N = 2$ chiral multiples. Furthermore, they have also a geometric perception in terms of the intersections of complex curves C_a and divisors D_i of $M_\Delta^{n/2}$ [17] [18]. This exceptional connection has been explored in many fields in modern physics. In particular, it has been used to construct type IIA local geometry.

The basic example in toric geometry is CP^1 which can play an essential role in the building block of higher dimensional toric varieties. It is defined by $r = 1$ and the Mori vector charge takes the values $q_i = (1,1)$. This geometry has an $U(1)$ toric action CP^1 acting as follows

$$z = \exp^{i\theta} z, \tag{3.3}$$

where $z = x_1 = x_2$, with two fixed points v_0 and v_1 placed on the real line. These later are the North and south poles respectively which describe such a geometry, considered as the (real) two-sphere $S^2 \sim CP^1$, satisfying the following constraint toric equation

$$v_0 + v_1 = 0. \tag{3.4}$$

Then, we consider a class of toric varieties that we are interested in by giving a trivial product of one-dimensional projective spaces CP^1's admitting a similar description. We will show later that this class can be used to elaborate a special mapping between quantum information systems and graph theory. We treat the cases of $CP^1 \times CP^1$ and $CP^1 \times CP^1 \times CP^1$ for simplicity reason and in the case of higher dimensional geometries $\otimes_{i=1}^{\frac{n}{2}} CP_i^1$, the toric descriptions can be given by a similar way. In fact, the $\frac{n}{2}$-dimensional toric manifolds present $U(1)^{\frac{n}{2}}$ toric actions. A close inspection shows that there is a similarity between toric graphs of such manifolds and qubit systems using a link with an interesting construction of Clifford graph algebras [15]. To assimilate this mapping, we present

G as a graph with n vertices and no multiple edges and no loops. This later is associated with algebra A_G over $\mathbb{C}$, with n generators $e_1, e_2, \cdots, e_n$ corresponding to the vertices, thus the Clifford algebra associated with a given graph it's called a Clifford graph algebra, precisely we are interested in the number of graphs with n vertices such that the center of their Clifford graph algebras has dimension 2^k being denoted by

$$Cliff_{2^k}(n), \tag{3.5}$$

where k is an integer that fixes the dimension of the center of Clifford algebras.

The connection that we are after is given by considering a special class of toric manifolds associated with $\otimes_{i=1}^{\frac{n}{2}} = CP_i^1$ and with $U(1)^{\frac{n}{2}}$ toric actions exhibiting 2^n fixed points. Then, roughly speaking, we can propose a correspondence connecting two different subjects apparently separate which are Clifford graph and toric geometry

$$2^n \text{ fixed points} \leftrightarrow n \text{ generators.}$$

Furthermore, in toric geometry language, the manifolds are represented by 2^n vertices belonging to the Z^n lattice satisfying n toric equations. Then these graphs share a strong resemblance with Clifford graph algebra formed by n nodes connected with a precise number of edges which are equal or less than $\frac{n(n-1)}{2}$. Precisely when we fix the number of vertices $n = k$ and the number of edges equal to $n-1$. Thus, we can propose the following notation

$$Cliff_{2^n}(2^n, 2^n - 1), \tag{3.6}$$

which can be associated with the number of toric actions as following: number of toric actions $\leftrightarrow$ $Cliff_{2^n}(2^n, 2^n - 1)$.

We believe that this type of toric manifolds can be used to describe graphically the physics of quantum systems by the help of Clifford graph algebras.

4. Qubits Systems as Clifford Graphs Algebra

Motivated from combinatorial computations in quantum physics, we explore toric geometry to deal with qubit information systems [9] [11]. Precisely, we elaborate a toric description in terms of a trivial fibration of one dimensional projective space CP^1's. First, it is recalled that the qubit is a two-level system which can be realized, for instance, by a 1/2 spin particle. The one qubit is in a superposition of two states generally written in Dirac notation as

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle, \tag{4.1}$$

where a_i are complex numbers satisfying the normalization condition $|a_0|^2 + |a_1|^2 = 1$.

It has been revealed that this constraint can be interpreted geometrically in terms of the Bloch sphere, identified with the quotient group $SU(2)/U(1)$ [16] [18]. The analysis can be extended to the case of bipartite systems which can be entangled states playing a fundamental role in the applications of quantum information. Using the usual notation $|i_1 i_2\rangle = |i_1\rangle |i_2\rangle$, the corresponding state in superposition can be expressed as follows

$$|\psi\rangle = \sum_{i_1 i_2} a_{i_1 i_2} |i_1 i_2\rangle = a_{00}|00\rangle + a_{10}|10\rangle + a_{01}|01\rangle + a_{11}|11\rangle, \tag{4.2}$$

where a_{ij} are complex numbers verifying the normalization condition

$$|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1, \tag{4.3}$$

describing the CP^3 projective space.

For n-qubits, the general state has the following form

$$|\psi\rangle = \sum_{i_1 \cdots i_n = 0,1} a_{i_1 \cdots i_n} |i_1 \cdots i_n\rangle, \tag{4.4}$$

where a_{ij} satisfy the normalization condition

$$\sum_{i_1 \cdots i_n = 0,1} |a_{i_1 \cdots i_n}|^2 = 1. \tag{4.5}$$

In fact, this condition defines the $CP^{2^n - 1}$ projective space generalizing the Bloch sphere associated with $n = 1$.

Roughly, the qubit systems can be described by toric manifolds $M_\Delta^{n/2}$ having a strong resemblance with Clifford graph algebras. A close inspection in Clifford graph algebras and toric geometry shows that when we fix the number of qubit $n = k$ and the number of edges is $2^n - 1$, we can propose the following correspondence connecting quantum systems and toric geometry via Clifford graph algebras (**Table 1**).

To see how this works in practice, we present at first the usual toric geometry notation. Inspired by combinatorial formalism used in quantum information theory, the toric data can be written as follows

$$\sum_{i_1 \cdots i_n} q^a_{i_1 \cdots i_n} v_{i_1 \cdots i_n} = 0, \quad a = 1, \cdots, r, \tag{4.6}$$

where the vertex subscripts indicate the corresponding quantum states. To illustrate this notation, we present the model associated with $CP^1 \times CP^1$ toric variety where $\frac{n}{2} = 2$. This model is related to $n = 4$ classification of Clifford graph algebras. Here, the combinatorial Mori vectors can take the following form

Table 1. This table presents correspondence between toric geometry, and quantum systems via Clifford graphs algebras.

Qubits systems	Toric Geometry	Clifford Graph algebra
Basis state	Fixed point	Vertices such as the dim of the center of corresponding Clifford algebra is 2^n
Number of qubits	Number of toric actions	$Cliff_{2^n}(2^n, 2^n - 1) \equiv$ number of graphs with 2^n vertices such that the center of their Clifford graph algebras has dimension 2^n and $2^n - 1$ edges.

$$\begin{aligned} q^1_{i_1 i_2} &= \left(q^1_{00}, q^1_{01}, q^1_{10}, q^1_{11}\right) = (1,0,0,1), \\ q^2_{i_1 i_2} &= \left(q^1_{00}, q^1_{01}, q^1_{10}, q^1_{11}\right) = (0,1,1,0). \end{aligned} \tag{4.7}$$

The corresponding manifold to this toric equations is

$$\sum_{i_1 i_2} q^a_{i_1 i_2} v_{i_1 i_2} = 0, \quad a = 1,2. \tag{4.8}$$

The toric data require the following vertices

$$\begin{aligned} v_{00} &= (-1,0), \ v_{01} = (0,1), \\ v_{01} &= (0,1), \quad v_{11} = (1,0), \end{aligned} \tag{4.9}$$

which can be connected to the Clifford graph algebra denoted as $Cliff_{2^n=4}\left(2^n = 4, 2^{n-1} = 3\right)$.

These data can be encoded in Clifford graph algebras describing the case of two qubits illustrated below in **Figure 2**. These two graphs have been chosen among eleven graphs in the case of $n = 4$ nodes of Clifford graph algebras, and it has belonged to the same Clifford class so their Clifford algebras are isomorphic [15].

Due to this representation, we can link the basis of n-qubit with the toric geometry. The mapping is given by

$$v_{i_1 \cdots i_n} \to \left| i_1 \cdots i_n \right\rangle, \tag{4.10}$$

and the number of qubits is related as follow in Equation (4.11) to the number of Clifford graph algebras with 2^n vertices and $2^n - 1$ edges such the dimension of their center is none other than 2^n,

$$n \to Cliff_{2^n}\left(2^n, 2^n - 1\right). \tag{4.11}$$

In the case of 3-qubit $n = 3$, it is remarked that the geometry can be identified with the blow-up of $CP^1 \times CP^1 \times CP^1$ toric manifold. In toric geometry language, this manifold is described by the following equations

$$\sum_{i_1 i_2 i_3} q^a_{i_1 i_2 i_3} v_{i_1 i_2 i_3} = 0, \quad a = 1, \cdots, 5, \tag{4.12}$$

where 2^3 vertices $v_{i_1 i_2 i_3}$ belong to Z^3.

These combinatorial equations can be solved by the following Mori vectors

$$\begin{aligned} q^1_{i_1 i_2 i_3} &= (1,0,0,1,0,0,0,0), \\ q^2_{i_1 i_2 i_3} &= (0,1,0,0,1,0,0,0), \\ q^3_{i_1 i_2 i_3} &= (0,0,1,0,0,1,0,0), \\ q^4_{i_1 i_2 i_3} &= (1,-1,0,0,0,0,1,0), \\ q^5_{i_1 i_2 i_3} &= (0,0,1,1,0,0,0,1). \end{aligned} \tag{4.13}$$

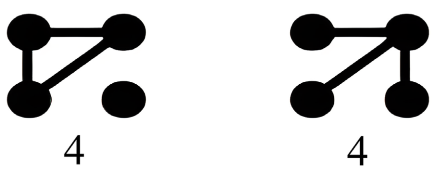

Figure 2. Clifford graph representation of two qubits $n = 2$.

The corresponding vertices $v_{i_1 i_2 i_3}$ are given by

$$\begin{aligned} v_{000} &= (1,0,0), \quad v_{100} = (0,1,0), \\ v_{010} &= (1,0,0), \quad v_{001} = (-1,0,0), \\ v_{110} &= (1,0,0), \quad v_{101} = (0,0,-1), \\ v_{110} &= (1,0,0), \quad v_{111} = (0,-1,-1), \end{aligned} \tag{4.14}$$

which can be connected to the Clifford graphs algebra denoted as $Cliff_{2^n=8}(2^n = 8, 2^n - 1 = 7)$.

The study above is limited to the case of pure states, however, the study of mixed states is very important in modern physics and has a strong utility in many applications of quantum information. To deal with it we consider the case of two qubits described in **Figure 2**. It is denoted that in the graph of the left side, we have four 0-simplex, three 1-simplex and one 2-simplex. Moreover, the graph on the right side of **Figure 2** is composed of four 0-simplex and three 1-simplex. Then it has been revealed that this composition of simplex forms the space of mixed states $\mathcal{M}$, and the space of pure states $\mathcal{P}$ is nothing but the vertices [19] [20].

We can remark that the number of 0-simplex in both graphs of **Figure 2** is related to the number of vertices of Clifford graphs algebra which is obvious, also the number of 1-simplex corresponds to the number of edges in these later graphs.

$$\mathcal{P} \equiv 0\text{-simplex}.$$

$$\mathcal{M} \equiv \text{simplex}.$$

$$E \equiv 1\text{-simplex}.$$

Another link is between the dimension of complex projective space of qubits systems CP^{2^n-1} and the number of 1-simplex in corresponding Clifford graphs algebras.

To push further the desired link, we make contact with quantum computational using the representation of a quantum n-gate as unitary $2^n \times 2^n$ matrix which acts on n-qubits in $2n$-dimensional complex vector space [21]. The isomorphism exists between a complex Clifford algebra with 2^n generators and the algebra of all $2^n \times 2^n$ matrices

$$Cl(2n, \mathbb{C}) \cong \mathbb{C}(2^n \times 2^n) \tag{4.15}$$

Precisely, the action of $Cl(2n, \mathbb{C})$ on n-qubit systems is none other than the action of $2^n \times 2^n$ matrix on complex vector in 2^n-dimensional complex space. To assimilate this, we take a basis of 2^n vector space

$$e_L = e_{l_1} \otimes e_{l_2} \otimes \cdots \otimes e_{l_n}, \quad l_k = 0,1 \tag{4.16}$$

where one used in quantum information science

$$|i\rangle = |i_1 i_2 \cdots i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle, \tag{4.17}$$

and where one has

$$|0\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad |1\rangle=\begin{pmatrix}0\\1\end{pmatrix}. \tag{4.18}$$

4^n-dimensional basis of Clifford algebra is given by

$$g_I=\sigma_{I_1}\otimes\sigma_{I_2}\otimes\cdots\otimes\sigma_{I_n},\quad \sigma_{I_k}\in 1,\sigma_x,\sigma_y,\sigma_z. \tag{4.19}$$

It has been revealed that the action is defined for an element on 2^n vector space by

$$g_I=\left(\sigma_{I_1}\left|i_1\right\rangle\right)\otimes\left(\sigma_{I_2}\left|i_2\right\rangle\right)\otimes\cdots\otimes\left(\sigma_{I_n}\left|i_n\right\rangle\right), \tag{4.20}$$

where $\sigma_{I_k}\left|i_k\right\rangle$ is the action of Pauli 2×2 matrix on 2D complex vector $|0\rangle$ or $|1\rangle$.

Thus, any linear transformation of 2^n-dimensional space can be represented by elements of Clifford algebra with $2n$ generators

$$Cl(2n,\mathbb{C})\cong\mathbb{C}\left(2^n\times 2^n\right):\mathbb{C}^{2n}\to\mathbb{C}^{2n}. \tag{4.21}$$

Here, we can represent the edges as the actions of $Cl(2n,\mathbb{C})$.

This mapping can be explored to study many related applications of quantum information including quantum computing. We expect that this analysis can be pushed further to deal with other toric manifolds having essential utility in modern physics.

5. Conclusions

Employing toric geometry and Clifford graph algebras correspondence, we have discussed qubit information systems. Precisely, we have presented a one-to-one correspondence between three apparently separate subjects namely toric geometry, Clifford graph algebras and quantum information theory. We think that this work may be explored to deal with qubit system problems using geometry considered as a strong tool to understand modern physics. In particular, we have considered in some details the cases of two and three qubits, and we find that they are associated with $CP^1\times CP^1$ and $CP^1\times CP^1\times CP^1$ toric manifolds, respectively. Developing a geometric procedure, we have revealed that the qubit physics can be converted into a scenario turning toric data of such varieties with the help of the construction of Clifford graph algebras. Pure and mixed states are dealt with.

In this work, we have completed partial results which come up with many open questions. An obvious one is to examine toric Calabi-Yau supermanifolds. We believe that in this issue, we can deal with superqubit systems. Furthermore, an important question is to investigate the entanglement states in the context of toric geometry and its application including mirror symmetry. We hope to address such questions in the future.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Nielsen, M.A. and Chuang, I.L. (2010) Quantum Computation and Quantum Information. Cambridge University Press, Cambridge.

[2] Schuld, M., Sinayskiy, I. and Petruccione, F. (2015) Simulating a Perceptron on a Quantum Computer. *Physics Letters A*, **379**, 660-663. https://doi.org/10.1016/j.physleta.2014.11.061

[3] Aganagic, M., Jafferis, D. and Saulina, N. (2006) Branes, Black Holes and Topological Strings on Toric Calabi-Yau Manifolds. *Journal of High Energy Physics*, **12**, 18. https://doi.org/10.1088/1126-6708/2006/12/018

[4] Belhaj, A., *et al.* (2007) On Black Objects in Type IIA Superstring Theory on Calabi-Yau Manifolds. *Nuclear Physics B*, **776**, 287. https://doi.org/10.1016/j.nuclphysb.2007.03.047

[5] Duff, M.J. and Ferrara, S. (2008) Black Hole Entropy and Quantum Information. *Lecture Notes in Physics*, **755**, 93-114. https://doi.org/10.1007/978-3-540-79523-0_2

[6] Borsten, L., Duff, M.J., Marrani, A. and Rubens, W. (2011) On the Black-Hole/Qubit Correspondence. *The European Physical Journal Plus*, **126**, Article No. 37. https://doi.org/10.1140/epjp/i2011-11037-5

[7] Aadel, Y., Belhaj, A., Bensed, M., *et al.* (2017) Qubit Systems from Colored Toric Geometry and Hypercube Graph Theory. *Communications in Theoretical Physics*, **68**, Article No. 285. https://doi.org/10.1088/0253-6102/68/3/285

[8] Belhaj, A., Sedra, M.B. and Segui, A. (2015) Graph Theory and Qubit Information Systems of Extremal Black Branes. *Journal of Physics A*, **48**, Article ID: 045401. https://doi.org/10.1088/1751-8113/48/4/045401

[9] Levay, P. (2001) Qubits from Extra Dimensions. *Physical Review D*, **84**, Article ID: 125020. https://doi.org/10.1103/PhysRevD.84.125020

[10] Belhaj, A., Benslimane, Z., Sedra, M.B. and Segui, A. (2016) Qubits from Black Holes in M-Theory on K3 Surface. *International Journal of Geometric Methods in Modern Physics*, **13**, Article ID: 1650075. https://doi.org/10.1142/S0219887816500754

[11] Belfakir, A., Belhaj, A., Maadi, Y.E., Ennadifi, S.E., Hassouni, Y. and Segui, A. (2019) Stringy Dyonic Solutions and Clifford Structures. *International Journal of Geometric Methods in Modern Physics*, **16**, Article ID: 1950138. https://doi.org/10.1142/S021988781950138X

[12] Belhaj, A., Bensed, M., Benslimane, Z., Sedra, M.B. and Segui, A. (2017) Four-Qubit Systems and Dyonic Black Hole-Black Branes in Superstring Theory. *International Journal of Geometric Methods in Modern Physics*, **15**, Article ID: 1850065. https://doi.org/10.1142/S0219887818500652

[13] Gregoric, M. and Mankoc Borstnik, N.S. (2009) Quantum Gates and Quantum Algorithms with Clifford Algebra Technique. *International Journal of Theoretical Physics*, **48**, 507-515. https://doi.org/10.1007/s10773-008-9826-7

[14] Somaroo, S.S., Cory, D.G. and Havel, T.F. (1998) Expressing the Operations of

Quantum Computing in Multiparticle Geometric Algebra. *Physics Letters A*, **240**, 1-7. https://doi.org/10.1016/S0375-9601(98)00010-3

[15] Khovanova, T. (2011) Clifford Algebras and Graphs. *Physical Review D*, **83**, Article ID: 046007.

[16] Leung, N.C. and Vafa, C. (1998) Branes and Toric Geometry. *Advances in Theoretical and Mathematical Physics*, **2**, 91-118. https://doi.org/10.4310/ATMP.1998.v2.n1.a4

[17] Katz, S., Mayr, P. and Vafa, C. (1998) Mirror Symmetry and Exact Solution of 4D N = 2 Gauge Theories I. *Advances in Theoretical and Mathematical Physics*, **1**, 53-114. https://doi.org/10.4310/ATMP.1997.v1.n1.a2

[18] Belhaj, A., Diaz, P., Garcia del Moral, M.P. and Segui, A. (2012) On Chern-Simons Quivers and Toric Geometry. *International Journal of Geometric Methods in Modern Physics*, **9**, Article ID: 1250050. https://doi.org/10.1142/S0219887812500508

[19] Mielnik, B. (1968) Geometry of Quantum States. *Communications in Mathematical Physics*, **9**, 55-80. https://doi.org/10.1007/BF01654032

[20] Mielnik, B. (1974) Generalized Quantum Mechanics. *Communications in Mathematical Physics*, **37**, 221-256. https://doi.org/10.1007/BF01646346

[21] Vlasov, A.Y. (1999) Quantum Gates and Clifford Algebras. Poster on TMR Network School on Quantum Computation and Quantum Information Theory, Torino.