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An Explicit Formula of the Dirichlet-to-Neumann Map for a Radial Potential in Dimension 3

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Abstract

In this paper, we provide an explicit expression for the full Dirichlet-to-Neumann map corresponding to a radial potential for the Schrödinger equation in 3-dimensional. We numerically implement the coefficients of the explicit formulas. In this work, Lipschitz type stability is established near the edge of the domain with giving estimation constant. That is necessary for the reconstruction of the potential from Dirichlet-to-Neuman map.

Keywords

Calderón's Problem, Schrödinger Operator, Potential, Inverse Potential Problem, Dirichlet-to-Neuman Map, Numerical Simulations, Lipschitz Stability

1. Introduction

Let us consider a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, the boundary value problem for the Schrödinger equation in Ω is given as follows

$$\begin{cases} (-\Delta + q)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The boundary value is assumed to be in $H^{1/2}(\partial\Omega)$, and the potential q is real-valued function satisfying $q \in L^\infty(\Omega)$.

The inverse problem for the Schrödinger equation is related to Calderón's problem (see [1] [2] [3]). The inverse problem for the Schrödinger equation is to determine the potential function q from the measurements of $\dfrac{\partial u}{\partial \nu}$ for all possible functions f on the boundary of Ω. That is the knowledge of the map, called also

Dirichlet-to-Neumann map for the Schrödinger equation that associates to any $f \in H^{1/2}(\partial\Omega)$ the normal derivative, $\frac{\partial u}{\partial \nu} \in H^{-1/2}(\partial\Omega)$, of the unique solution of 1.

In the literature, the spectral geometry of the Dirichlet-to-Neumann map is a new and rapidly developing branch of spectral theory (see [4] and references therein). The eigenvalue problem for the Dirichlet-to-Neumann map, called the Steklov problems, was first introduced by V. A. Steklov more than a century ago. The geometric properties of Steklov eigenvalues and eigenfunctions have only recently begun to be explored. In [5], A. Rüland presented her recent work on Carleman estimates and quantitative unique continuation for solutions of fractional Schrödinger equations. As an application, she has obtained an upper bound on the vanishing order of the eigenfunctions and on the size of the nodal set. Her work is closely related to the results of Bellova and Lin in [6], Zelditch in [7] and Zhu in [8], since the Dirichlet-to-Neumann map could be viewed as a special case of a fractional Schrödinger operator. In [9], Akhmetgaliyev *et al.* were interested by computational method for extremal Steklov eigenvalue problems and applied it to study the problem of maximizing the p^{th} Steklov eigenvalue as a function of the domain with a volume constraint. Among other things, they reach the conjecture that the domain maximizing the p^{th} Steklov is unique, has the p-fold symmetry, and has at least one axis of symmetry.

Other studies to determine a map related to the Schrödinger operator in quantum mechanics and to estimate the entropy numbers of the function space have also developed. In [10], Dhahbi *et al.* used a new approach, the supersymmetric quantum mechanical (SUSY QM) formalism along with the shape invariance condition, and took into account the recent results from non-Hermitian quantum mechanics, to construct the quantum kinetic energy operator (KEO) within the Schrödinger operator in order to build a new class of exactly solvable models with position varying mass and exhibiting a harmonic-oscillator-like spectrum. In [11], Chen *et al.* discussed the entropy number of diagonal operator. On the one hand, they were interested in the order of entropy number of the finite dimensional diagonal operator. On the other hand, the order of entropy number of a class of infinite dimensional diagonal operators.

Other authors have been interested in Dirichlet-to-Neuman map in order to study the inverse problem of determining the potential for Schrödinger problem.

In [12], Greenleal *et al.* considered the Dirichlet-to-Neuman map associated to the Schrödinger equation with potential in a bounded Lipschitz domain in three or more dimensions, and showed that the integral of the potential over a two-plane is determined by the Cauchy data certain exponentially growing solutions on any neighborhood of the intersection of the two-plane with the boundary. In [13], Imanuvilov *et al.*, in the 2-dimensional case, given a Dirichlet-to-Neuman map on a sub-boundary, proved the uniqueness of the determination of the potential under the assumption of the choice of the function space where the potential belongs. In [14], the authors considered the Schrödinger operator with a one-step radial potential q on a disk and the Λ map which associates to q the corresponding Dirichlet-to-Neuman map Λ_q. They provide some numerical and analytical results on the range of the Λ map and its stability for q. They based their study on a similar case, see [15], where the relationship between piecewise constant radial conductivities and the eigenvalues of the Dirichlet-to-Neuman map is known by an appropriate recurrence formula.

Our contribution in this paper is to determine an explicit formula for the Dirichlet to Neumann map for a piecewise constant radial potential in dimension three in a ball. Lipschitz type stability is established near the edge of the domain by giving estimation constant.

The paper is organized as follows. In Section 2, we define the Dirichlet to Neumann map for the Schrödinger equation, and then present the radial solutions of this equation in Section 3. In Section 4, we give an explicit formula for Dirichlet to Neumann map when the potential is radial, followed by some simulations. In Section 4, we study the stability of the map that associates a Dirichlet to Neumann map to any potential. In Section 5, we present conclusions and perspectives.

Our motivation of this paper is to know the Dirichlet-to-Neumann map Λ_q for a piecewise constant radial potential q in dimension three in a ball from the knowledge of the Cauchy data $\left(f,\frac{\partial u}{\partial \nu}\right)$ in order to be able to solve the inverse Schrödinger problem. Good knowledge of the characteristic properties of the Dirichlet-to-Neumann map Λ_q allows solving the inverse problem which consists to determine the potential q from the knowledge of $\left(f,\Lambda_q\right)$. And the study of this inverse problem also motivated us to study the Lipschitz type stability which will allow us to obtain a result at least at the edge of Ω with an estimate constant to be determined.

2. Definition of the Dirichlet-to-Neuman Map for the Schrödinger Equation

In this section we define the Dirichlet-to-Neuman map Λ_q for the Schrödinger equation formally as

$$\begin{aligned} \Lambda_q : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ f &\mapsto \Lambda_q(f) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}, \end{aligned} \tag{2}$$

where ν is the outer unit normal vector to $\partial\Omega$.

The map $f \mapsto \Lambda_q(f)$ depends linearly on f. Λ_q encodes the measurements of $\frac{\partial u}{\partial \nu}$ for all possible functions f on the boundary of Ω.

We need to assume that 0 is not a Dirichlet eigenvalue of $(-\Delta+q)$ in Ω.

Now, we look more closely at the direct problem with the potential. Let the unit ball B in $\mathbb{R}^3$ be defined by $B=\left\{x\in\mathbb{R}^3 : |x|\le 1\right\}$.

We focus on $q \in L^{\infty}(B)$ with $q(x) = q(|x|)$ is radial, $f \in H^{\frac{1}{2}}(\partial B)$ given and assuming that 0 is not an eigenvalue of

$$\begin{cases} (-\Delta + q)u = 0 & \text{in } |x| < 1, \\ u = f & \text{on } |x| = 1. \end{cases} \tag{3}$$

These choices guarantee the existence of a solution of (3) by the Fourier method and 0 is not an eigenvalue ensuring the uniqueness of the solution.

Then the map $f \mapsto \Lambda_q(f)$ is well defined. To obtain an explicit formula for the Dirichlet-to-Neuman map Λ_q defined in (2), we will consider the following results (see [15]):

1) If $q(x) \neq 0$ then Λ_q is diagonalisable in the sense that the spectrum is discrete, $\{\lambda_k[q], k \in \mathbb{N}_0\}$.

In this case, if $\mathcal{N}_k$ is the subspace of spherical harmonics of degree *k*, then

$$\Lambda_q\big|_{\mathcal{N}_k} = \lambda_k[q] I_{\mathcal{N}_k}.$$

2) If $q(x) = 0$ and $\phi_k \in \mathcal{N}_k$ then $\Lambda_0(\phi_k) = k\phi_k, k = 0,1,2,\cdots$

3) $\lambda_k[q] - k \to 0$ if $k \to \infty$.

Then in the following, we give a recurrence relation for the explicit calculation of the spectrum in the case where $q(x)$ is a step potential, to give an approximation of the spectrum of a general potential. For this, we need to introduce some properties that will be useful.

3. Radial Solutions

In this section, we present some results obtained from writing problem (3) in polar coordinates $r > 0, \theta \in \mathbb{S}^2$. We want to obtain "complex geometrical optics" solutions or solutions of Faddeev types, see [16].

Lemma 3.1: *If u is the solution of* (3) *and* $v(r,\theta) = u(r\theta)$ *in terms of spherical harmonics, then the function v satisfies the problem*

$$\begin{cases} r^2 v'' + 2rv' + \Delta_S v - q(r) r^2 v = 0, \\ \lim\limits_{r>0, r\to 0} v(r,\theta) < \infty, \quad v(1,\theta) = f(\theta). \end{cases} \tag{4}$$

where $-\Delta_S Y_\ell^k = \ell(\ell+1) Y_\ell^k$, $Y_\ell^k \in \mathcal{N}_\ell$.

Proof Lemma 3.1: In spherical coordinates, the Laplacien is given by

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\alpha}\frac{\partial}{\partial \alpha}\left(\sin\alpha \frac{\partial}{\partial \alpha}\right) + \frac{1}{r^2 \sin^2\alpha}\frac{\partial^2}{\partial \phi^2},$$

Or

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S,$$

where

$$\Delta_S = \frac{1}{\sin\alpha}\frac{\partial}{\partial \alpha}\left(\sin\alpha \frac{\partial}{\partial \alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial \phi^2}.$$

Then

$$(-\Delta+q)v=0$$

is equivalent to

$$-\frac{\partial^2 v}{\partial r^2}-\frac{2}{r}\frac{\partial v}{\partial r}-\frac{1}{r^2}\Delta_S v+q(r)v=0.$$

Or

$$r^2\frac{\partial^2 v}{\partial r^2}+2r\frac{\partial v}{\partial r}+\Delta_S v-q(r)vr^2=0.$$

As the solution must be finite at the origin then we have the following condition

$$\lim_{r>0,r\to 0} v(r,\theta)<\infty,$$

In addition, we have $r=1$ in $\mathbb{S}^2$. Then the Dirichlet condition is equivalent to $v(1,\theta)=f(\theta)$ where f depends only of $\theta\in\mathbb{S}^2$. □

Lemma 3.2: *If* $f(\theta)=\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} f_{\ell k}Y_\ell^k(\theta)$ *in* $H^{1/2}(\mathbb{S}^2)$, *then the equation* (3) *admits a unique solution of the form*

$$v(r,\theta)=\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} v_{\ell k}(r)Y_\ell^k(\theta), \tag{5}$$

where $v_{\ell k}$ *satisfies the problem*

$$\begin{cases} r^2 v''_{\ell k}+2rv'_{\ell k}-\ell(\ell+1)v_{\ell k}-q(r)r^2 v_{\ell k}=0, \quad r\in(0,1), \\ \lim_{r>0,r\to 0} v_{\ell k}(r)<\infty, \quad v_{\ell k}(1)=f_{\ell k}. \end{cases} \tag{6}$$

Proof Lemma 3.2: We know, that problem (3) admits a unique solution $v(r,\theta)$ with $r>0$ and $\theta\in\mathbb{S}^2$, following Section 2.

If $f(\theta)=\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} f_{\ell k}Y_\ell^k(\theta)$ then we can write $v(1,\theta)=\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} f_{\ell k}Y_\ell^k(\theta)$.

Using the separation of variables $v(r,\theta)=v(r)g(\theta)$, the completeness property of the spherical harmonics implies that any well-behaved function g of $\theta\in\mathbb{S}^2$, that is which is single valued, continuous and finite, can be written as $g(\theta)=\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} a_{\ell k}Y_\ell^k(\theta)$, $\theta\in\mathbb{S}^2$ for some choice of coefficients $a_{\ell k}$.

Choosing $a_{\ell k}=1$, then we have $g(\theta)=\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} Y_\ell^k(\theta)$, $\theta\in\mathbb{S}^2$. Taking $r=1$ in $v(r,\theta)=v(r)g(\theta)$, we obtain $v(1,\theta)=\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} v_{\ell k}(1)Y_\ell^k(\theta)$ and then, $v_{\ell k}(1)=f_{\ell k}$ in $\mathbb{S}^2$.

From $v\in H^1\left(\overline{B(0,1)}\right)$, we obtain $v(r,\theta)=\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} v_{\ell k}(r)Y_\ell^k(\theta)$. Since $v(r,\theta)$ satisfies (3) then, we have

$$\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} r^2\frac{\partial^2 v_{\ell k}(r)}{\partial r^2}Y_\ell^k(\theta)+\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} 2r\frac{\partial v_{\ell k}(r)}{\partial r}Y_\ell^k(\theta)$$
$$+\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} v_{\ell k}(r)\Delta_S Y_\ell^k(\theta)-\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} q(r)r^2 v_{\ell k}(r)Y_\ell^k(\theta)=0.$$

with $\Delta_S = \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{\partial}{\partial\alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\alpha^2}{\partial\phi^2}$, where $\theta = (\alpha,\phi) \in \mathbb{S}^2$.

Since $-\Delta_S Y_\ell^k = \ell(\ell+1)Y_\ell^k$, we have

$$\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell}\left(r^2\frac{\partial^2 v_{\ell k}(r)}{\partial r^2} + 2r\frac{\partial v_{\ell k}(r)}{\partial r} - \ell(\ell+1)v_{\ell k}(r) - q(r)r^2 v_{\ell k}(r)\right)Y_\ell^k(\theta) = 0.$$

We know $\{Y_\ell^k\}_{\ell=0,1,2,\cdots;k=-\ell,-\ell+1,\cdots,\ell-1,\ell}$ is an orthonormal basis, then $v_{\ell k}(r)$ verifies (6).

Lemma 3.3: *If* $v_{\ell k}$ *is the solution of* (6), *then we have*

$$\Lambda_q(f) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} v'_{\ell k}(1)Y_\ell^k(\theta). \tag{7}$$

Proof Lemma 3.3: We know for all $f \in H^{1/2}(\mathbb{S}^2)$, $\Lambda_q(f) = \left.\frac{\partial v}{\partial r}\right|_{\mathbb{S}^2}$ where v is the unique solution of (3).

Using lemma (3.2), we have

$$\frac{\partial v}{\partial r} = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell}\frac{\partial v_{\ell k}}{\partial r}Y_\ell^k(\theta) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} v'_{\ell k}Y_\ell^k(\theta),$$

for all $r > 0$, $\theta \in \mathbb{S}^2$

Then $\left.\frac{\partial v}{\partial r}\right|_{\mathbb{S}^2} = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} v'_{\ell k}(1)Y_\ell^k(\theta)$, $r = 1$ and $\theta \in \mathbb{S}^2$. □

We note that the differential equation in (6) does not depend on *k*, so we will eliminate the dependence on *k*. Then

Lemma 3.4: *If in* (6) *we take* $f = Y_\ell$, *that is, the spherical harmonic of degree* ℓ, *it follows that*

$$\Lambda_q(Y_\ell)(\theta) = v'_\ell(1)Y_\ell(\theta). \tag{8}$$

Then $v'_l(1)$ *is an eigenvalue of* Λ_q *with multiplicity* $2\ell+1$ *and its eigenfunctions are* $\{Y_\ell^k\}_{k=-\ell,-\ell+1,\cdots,\ell-1,\ell}$.

Proof Lemma 3.4: If $f = Y_\ell$, then from (3.2) $v(r,\theta) = v_\ell(r)Y_\ell$.

It follows that from (3.3), $\left.\frac{\partial v}{\partial r}\right|_{\mathbb{S}^2} = v'_\ell(1)Y_\ell(\theta)$ and $\Lambda_q(Y_\ell)(\theta) = v'_\ell(1)Y_\ell(\theta)$.

According to [15], $v'_\ell(1)$ is an eigenvalue of Λ_q with multiplicity $2\ell+1$ and its eigenfunctions are $\{Y_\ell^k\}_{k=-\ell,-\ell+1,\cdots,\ell-1,\ell}$.

In the next section, we will use these results to give an explicit formula for the DN map when the potential is a radial function.

4. Explicit Formula for the Dirichlet to Neumann Map

4.1. The Case Where *q* Is a Piecewise Constant Radial Potential

Let us introduce the theorem where the expression of the Dirichlet-to-Neuman map is presented when q is a piecewise constant radial potential, based on the results of the previous section.

In the following, for all $\ell \in \mathbb{N}$, $p_\ell^m(r)$ denotes the Bessel function of the first type $j_\ell\left(\sqrt{|\gamma_m|}r\right)$ or the Bessel function of the second type $i_\ell\left(\sqrt{|\gamma_m|}r\right)$, and $q_\ell^m(r)$ denotes the modified Bessel function of the first type $y_\ell\left(\sqrt{|\gamma_m|}r\right)$ or the modified Bessel function of the second type $(-1)^{\ell+1} k_\ell\left(\sqrt{|\gamma_m|}r\right)$.

Theorem 4.1: *Let the unit ball B in* $\mathbb{R}^3$ *and the scaled potential* $q \in L^\infty(B)$ *with*

$$q(r) = \sum_{m=1}^{n} \gamma_m \chi_{(r_{m-1}, r_m)}, \quad r = |x|, \tag{9}$$

where $n \geq 1$, $\gamma_m, r_m \in \mathbb{R}$, *with* $m = 1, 2, \cdots, n$ *and* $0 = r_0 < r_1 < \cdots < r_{n-1} < r_n = 1$, *such that the Dirichlet problem for* $-\Delta + q$ *is well-posed.*

Then there is an explicit formula for the Dirichlet-to-Neumann map defined as follows:

$$\begin{aligned} &\Lambda_q\left(Y_\ell^k(\theta)\right) \\ &= \left[C\left(k_n p_{\ell-1}^n(1) - \frac{k_n p_\ell^n(1)}{q_\ell^n(1)} q_{\ell-1}^n(1)\right) + \frac{k_n q_{\ell-1}^n(1) - \ell q_\ell^n(1)}{q_\ell^n(1)}\right] Y_\ell^k(\theta), \quad \ell = 1, 2, \cdots \end{aligned} \tag{10}$$

with C depending on n and ℓ.

Remark 4.1. *We assume that* $\gamma_m \neq 0$ *to simplify the calculations. If we want to consider this case in the simulations, we approximate it by* $\gamma = -0.01$.

Proof of Theorem 4.1. q is a piecewise constant radial function, $q(r) = q(|x|)$, defined by

$$q(r) = \sum_{m=1}^{n} \gamma_m \chi_{(r_{m-1}, r_m)}, \quad r = |x|,$$

with $0 = r_0 < r_1 < \cdots < r_{n-1} < r_n = 1$, $\gamma_m \in \mathbb{R}$, and there is no case where $\gamma_m = \gamma_{m+1}$ for all $m \in \{1, 2, \cdots, n-1\}$.

We solve the Schrödinger equation with $f = Y_\ell^k$ for a fixed ℓ. Thus in equation (6) we have $f_{\ell k} = 1$.

We look for a solution y of (6), of type

$$y(r) = \sum_{m=1}^{n} y_m(r), \tag{11}$$

where y_1 is the solution of

$$\begin{cases} r^2 y'' + 2ry' - \ell(\ell+1)y - \gamma_1 r^2 y = 0, & r \in (0, r_1), \\ \lim_{r>0, r\to 0} y(r) < \infty, \end{cases} \tag{12}$$

For $m = 2, 3, \cdots, n-1$, we have a y_m which satisfies

$$r^2 y'' + 2ry' - \ell(\ell+1)y - \gamma_m r^2 y = 0, \quad r \in (r_{m-1}, r_m), \tag{13}$$

and y_n in this equation

$$\begin{cases} r^2 y'' + 2ry' - \ell(\ell+1)y - \gamma_n r^2 y = 0, & r \in (r_{n-1}, 1), \\ y(1) = 1, \end{cases} \tag{14}$$

and the following compatibility conditions are satisfied

$$\begin{cases} y_1(r_1) = y_2(r_1) \\ y_1'(r_1) = y_2'(r_1) \\ y_2(r_2) = y_3(r_2) \\ y_2'(r_2) = y_3'(r_2) \\ \qquad \vdots \\ y_{n-2}(r_{n-2}) = y_{n-1}(r_{n-2}) \\ y_{n-2}'(r_{n-2}) = y_{n-1}'(r_{n-2}) \\ y_{n-1}(r_{n-1}) = y_n(r_{n-1}) \\ y_{n-1}'(r_{n-1}) = y_n'(r_{n-1}) \end{cases} \tag{15}$$

The general solution of the equation

$$r^2 y'' + 2ry' - \ell(\ell+1)y - \gamma_m r^2 y = 0, \quad r \in (r_{m-1}, r_m), \ m = 1, 2, \cdots, n,$$

is

$$y_m(r) = A_m^\ell j_\ell\left(\sqrt{|\gamma_m|}r\right) + B_m^\ell y_\ell\left(\sqrt{|\gamma_m|}r\right), \quad \text{if } \gamma_m < 0, \ A_m^\ell, B_m^\ell \in \mathbb{R},$$

where j_ℓ and y_ℓ are the Bessel functions of the first and second type, respectively,

$$y_m(r) = A_m^\ell r^\ell + B_m^\ell r^{-(\ell+1)}, \quad \text{if } \gamma_m = 0, \ A_m^\ell, B_m^\ell \in \mathbb{R},$$

and

$$y_m(r) = A_m^\ell i_\ell\left(\sqrt{|\gamma_m|}r\right) + B_m^\ell (-1)^{\ell+1} k_\ell\left(\sqrt{|\gamma_m|}r\right), \quad \text{if } \gamma_m > 0, \ A_m^\ell, B_m^\ell \in \mathbb{R},$$

where i_ℓ and k_ℓ are the modified Bessel functions of the first and second type, respectively,

For $m = 1, 2, \cdots, n$, let us introduce the functions $z_\ell^m, p_\ell^m, w_\ell^m$, and q_ℓ^m such that

- $z_\ell^m(r)$ and $p_\ell^m(r)$ will be denoted by $j_\ell\left(\sqrt{|\gamma_m|}r\right)$ or $i_\ell\left(\sqrt{|\gamma_m|}r\right)$ depending on whether $\gamma_m < 0$ or $\gamma_m > 0$,
- $w_\ell^m(r)$ and $q_\ell^m(r)$ will be denoted by $y_\ell\left(\sqrt{|\gamma_m|}r\right)$ or $(-1)^{\ell+1} k_\ell\left(\sqrt{|\gamma_m|}r\right)$ depending on whether $\gamma_m < 0$ or $\gamma_m > 0$.

Let pose $k_m = \sqrt{|\gamma_m|}, m = 1, 2, \cdots, n$.

As the functions $y_\ell\left(\sqrt{|\gamma_1|}r\right)$ or $(-1)^{\ell+1} k_\ell\left(\sqrt{|\gamma_1|}r\right)$ go to $-\infty$ when $r \to 0$, we have

$$y_1(r) = A_1^\ell z_\ell^1(r), \text{ or } A_1^\ell p_\ell^1(r).$$

For $m = 1, 2, \cdots, n-1$, we have

$$y_m(r) = A_m^\ell z_\ell^m(r) + B_m^\ell w_\ell^m(r) \quad \left(\text{or } A_m^\ell p_\ell^m(r) + B_m^\ell q_\ell^m(r)\right),$$

and

$$y_n(r) = A_n^\ell z_\ell^n(r) + B_n^\ell w_\ell^n(r) \quad \left(\text{or } A_n^\ell p_\ell^n(r) + B_n^\ell q_\ell^n(r)\right), \quad \text{with } A_n^\ell + B_n^\ell = 1.$$

We will need the following derivative formulas.

If f_ℓ^m denotes j_ℓ^m, y_ℓ^m, i_ℓ^m, or $(-1)^{\ell+1} k_\ell^m$ with $f_\ell^m(r) = f_\ell(k_m r)$, then

$\left(f_\ell^m\right)'(r) = k_m f_\ell'(k_m r)$, where f_ℓ' satisfies

$$f_\ell'(z) = f_{\ell-1}(z) - \frac{\ell+1}{z} f_\ell(z). \tag{16}$$

From 15 and 16, we have the following system of $(2n-2)\times(2n-2)$ equations

$$\begin{cases}
(S1)\begin{cases} A_1^\ell z_\ell^1(r_1) = A_2^\ell p_\ell^2(r_1) + B_2^\ell q_\ell^2(r_1) \\ A_1^\ell k_1 z_\ell'(k_1 r_1) = A_2^\ell k_2 p_\ell'(k_2 r_1) + B_2^\ell k_2 q_\ell'(k_2 r_1) \end{cases} \\
(S2)\begin{cases} A_2^\ell z_\ell^2(r_2) + B_2^\ell w_\ell^2(r_2) = A_3^\ell p_\ell^3(r_2) + B_3^\ell q_\ell^3(r_2) \\ A_2^\ell k_2 z_\ell'(k_2 r_2) + B_2^\ell k_2 w_\ell'(k_2 r_2) = A_3^\ell k_3 p_\ell'(k_3 r_2) + B_3^\ell k_3 q_\ell'(k_3 r_2) \end{cases} \\
\qquad\vdots \\
(Sm)\begin{cases} A_m^\ell z_\ell^m(r_m) + B_m^\ell w_\ell^m(r_m) = A_{m+1}^\ell p_\ell^{m+1}(r_m) + B_{m+1}^\ell q_\ell^{m+1}(r_m) \\ A_m^\ell k_m z_\ell'(k_m r_m) + B_m^\ell k_m w_\ell'(k_m r_m) = A_{m+1}^\ell k_{m+1} p_\ell'(k_{m+1} r_m) + B_{m+1}^\ell k_{m+1} q_\ell'(k_{m+1} r_m) \end{cases} \\
\qquad\vdots \\
\begin{cases} A_{n-2}^\ell z_\ell^{n-2}(r_{n-2}) + B_{n-2}^\ell w_\ell^{n-2}(r_{n-2}) = A_{n-1}^\ell p_\ell^{n-1}(r_{n-2}) + B_{n-1}^\ell q_\ell^{n-1}(r_{n-2}) \\ A_{n-2}^\ell k_{n-2} z_\ell'(k_{n-2} r_{n-2}) + B_{n-2}^\ell k_{n-2} w_\ell'(k_{n-2} r_{n-2}) = A_{n-1}^\ell k_{n-1} p_\ell'(k_{n-1} r_{n-2}) + B_{n-1}^\ell k_{n-1} q_\ell'(k_{n-1} r_{n-2}) \end{cases} \\
(S(n-1))\begin{cases} A_{n-1}^\ell z_\ell^{n-1}(r_{n-1}) + B_{n-1}^\ell w_\ell^{n-1}(r_{n-1}) = A_n^\ell p_\ell^n(r_{n-1}) + B_n^\ell q_\ell^n(r_{n-1}) \\ A_{n-1}^\ell k_{n-1} z_\ell'(k_{n-1} r_{n-1}) + B_{n-1}^\ell k_{n-1} w_\ell'(k_{n-1} r_{n-1}) = A_n^\ell k_n p_\ell'(k_n r_{n-1}) + B_n^\ell k_n q_\ell'(k_n r_{n-1}) \end{cases}
\end{cases} \tag{17}$$

where A_n^ℓ and B_n^ℓ are related by

$$A_n^\ell p_\ell^n(1) + B_n^\ell q_\ell^n(1) = 1. \tag{18}$$

We recall (see 8) that our aim is to calculate $y'(1)$ or $y_n'(1)$. By condition 18, we are only interesting in finding the unknown A_n^ℓ of the system 17.

Our strategy will be to find the unknowns A_2^ℓ and B_2^ℓ in terms of A_1^ℓ by solving $(S1)$. And for $m = 2,3,\cdots,n-2$, solve $S(m)$ to obtain A_{m+1}^ℓ and B_{m+1}^ℓ in terms of A_1^ℓ. Then transform the system $S(n-1)$ into a system of two unknowns A_1^ℓ and A_n^ℓ and two equations. At this point we solve $S(n-1)$.

For this purpose, we introduce

$$F\left(r_m, k_m, h_\ell^m, g_\ell^{m+1}\right) = k_{m+1} h_\ell^m(r_m) g_{\ell-1}^{m+1}(r_m) - k_m h_{\ell-1}^m(r_m) g_\ell^{m+1}(r_m). \tag{19}$$

where h_ℓ^m denotes z_ℓ^m or w_ℓ^m, and g_ℓ^m denotes p_ℓ^m or q_ℓ^m.

And we will need the following formulas of the Wronskians W, see [17].

$$\begin{aligned} W\{j_\ell(z), y_\ell(z)\} &= z^{-2}, \\ W\left\{i_\ell(z), (-1)^{\ell+1} k_\ell(z)\right\} &= \frac{(-1)^\ell \pi}{2} z^{-2}. \end{aligned} \tag{20}$$

The problem with including the $\gamma_m = 0$ case is that the functions r^ℓ and $r^{-(\ell+1)}$ do not satisfy 19. Perhaps a linear combination of different types of functions would make it easy to take into account the case $\gamma_m = 0$ into the general scheme.

We start by solving (S1) according to A_1^ℓ.

$$\begin{cases} A_1^\ell z_\ell^1(r_1) = A_2^\ell p_\ell^2(r_1) + B_2^\ell q_\ell^2(r_1) \\ A_1^\ell k_1 z_\ell'(k_1 r_1) = A_2^\ell k_2 p_\ell'(k_2 r_1) + B_2^\ell k_2 q_\ell'(k_2 r_1) \end{cases} \tag{21}$$

$$A_2^\ell = \frac{A_1^\ell}{k_2 W\left\{p_\ell^2(r_1), q_\ell^2(r_1)\right\}}\left[k_2 z_\ell^1(r_1) q_\ell'(k_2 r_1) - k_1 q_\ell^2(r_1) z_\ell'(k_1 r_1)\right]$$

$$A_2^\ell = \frac{A_1^\ell}{k_2 W\left\{p_\ell^2(r_1), q_\ell^2(r_1)\right\}}\left[k_2 z_\ell^1(r_1) q_{\ell-1}^2(r_1) - k_1 q_\ell^2(r_1) z_{\ell-1}^1(r_1)\right],$$

$$\boxed{A_2^\ell = \frac{A_1^\ell}{k_2 W\left\{p_\ell^2(r_1), q_\ell^2(r_1)\right\}} F\left(r_1, k_1, z_\ell^1, q_\ell^2\right)} \tag{22}$$

$$B_2^\ell = \frac{A_1^\ell}{k_2 W\left\{p_\ell^2, q_\ell^2\right\}(r_1)}\left[k_1 p_\ell^2(r_1) z_\ell'(k_1 r_1) - k_2 z_\ell^1(r_1) p_\ell'(k_2 r_1)\right]$$

$$B_2^\ell = \frac{A_1^\ell}{k_2 W\left\{p_\ell^2(r_1), q_\ell^2(r_1)\right\}}\left[k_1 p_\ell^2(r_1) z_{\ell-1}^1(r_1) - k_2 z_\ell^1(r_1) p_{\ell-1}^2(r_1)\right],$$

$$\boxed{B_2^\ell = -\frac{A_1^\ell}{k_2 W\left\{p_\ell^2(r_1), q_\ell^2(r_1)\right\}} F\left(r_1, k_1, z_\ell^1, p_\ell^2\right)} \tag{23}$$

We now solve (Sm) in terms of A_m^ℓ and B_m^ℓ for $m = 2, 3, \cdots, n-2$.

$$\begin{cases} A_m^\ell z_\ell^m(r_m) + B_m^\ell w_\ell^m(r_m) = A_{m+1}^\ell p_\ell^{m+1}(r_m) + B_{m+1}^\ell q_\ell^{m+1}(r_m) \\ A_m^\ell k_m z_\ell'(k_m r_m) + B_m^\ell k_m w_\ell'(k_m r_m) = A_{m+1}^\ell k_{m+1} p_\ell'(k_{m+1} r_m) + B_{m+1}^\ell k_{m+1} q_\ell'(k_{m+1} r_m) \end{cases} \tag{24}$$

We call

$F_m^\ell(r_m) = A_m^\ell z_\ell^m(r_m) + B_m^\ell w_\ell^m(r_m)$ and $G_m^\ell(r_m) = A_m^\ell k_m z_\ell'(k_m r_m) + B_m^\ell k_m w_\ell'(k_m r_m)$.

$$\begin{aligned} A_{m+1}^\ell &= \frac{1}{k_{m+1} W\left(p_\ell^{m+1}(r_m), q_\ell^{m+1}(r_m)\right)}\left[k_{m+1} q_\ell'(k_{m+1} r_m) F_m^\ell(r_m) - q_\ell^{m+1}(r_m) G_m^\ell(r_m)\right] \\ &= \frac{1}{k_{m+1} W\left(p_\ell^{m+1}(r_m), q_\ell^{m+1}(r_m)\right)}\left[k_{m+1} A_m^\ell z_\ell^m(r_m) q_\ell'(k_{m+1} r_m) + k_{m+1} B_m^\ell w_\ell^m(r_m) q_\ell'(k_{m+1} r_m)\right. \\ &\quad \left. - k_m A_m^\ell q_\ell^{m+1}(r_m) z_\ell'(k_m r_m) - k_m B_m^\ell q_\ell^{m+1}(r_m) w_\ell'(k_m r_m)\right], \end{aligned}$$

Using (16), we have

$$\begin{aligned} A_{m+1}^\ell = \frac{1}{k_{m+1} W\left(p_\ell^{m+1}(r_m), q_\ell^{m+1}(r_m)\right)}&\left[A_m^\ell\left(k_{m+1} z_\ell^m(r_m) q_{\ell-1}^{m+1}(r_m) - k_m q_\ell^{m+1}(r_m) z_{\ell-1}^m(r_m)\right)\right. \\ &\left. + B_m^\ell\left(k_{m+1} w_\ell^m(r_m) q_{\ell-1}^{m+1}(r_m) - k_m q_\ell^{m+1}(r_m) w_{\ell-1}^m(r_m)\right)\right], \end{aligned}$$

$$\boxed{A_{m+1}^\ell = \frac{1}{k_{m+1} W\left(p_\ell^{m+1}(r_m), q_\ell^{m+1}(r_m)\right)}\left[A_m^\ell F\left(r_m, k_m, z_\ell^m, q_\ell^m\right) + B_m^\ell F\left(r_m, k_m, w_\ell^m, q_\ell^m\right)\right]} \tag{25}$$

$$\begin{aligned} B_{m+1}^\ell &= \frac{1}{k_{m+1} W\left(p_\ell^{m+1}(r_m), q_\ell^{m+1}(r_m)\right)}\left[p_\ell^{m+1}(r_m) G_m^\ell(r_m) - k_{m+1} p_\ell'(k_{m+1} r_m) F_m^\ell(r_m)\right] \\ &= \frac{1}{k_{m+1} W\left(p_\ell^{m+1}(r_m), q_\ell^{m+1}(r_m)\right)}\left[A_m^\ell k_m p_\ell^{m+1}(r_m) z_\ell'(k_m r_m) + B_m^\ell k_m p_\ell^{m+1}(r_m) w_\ell'(k_m r_m)\right. \\ &\quad \left. - A_m^\ell k_{m+1} z_\ell^m(r_m) p_\ell'(k_{m+1} r_m) - B_m^\ell k_{m+1} w_\ell^m(r_m) p_\ell'(k_{m+1} r_m)\right], \end{aligned}$$

$$B_{m+1}^{\ell}=\frac{1}{k_{m+1}W\left(p_{\ell}^{m+1}(r_m),q_{\ell}^{m+1}(r_m)\right)}\Big[A_m^{\ell}\left(k_m p_{\ell}^{m+1}(r_m)z_{\ell-1}^{m}(r_m)-k_{m+1}z_{\ell}^{m}(r_m)p_{\ell-1}^{m+1}(r_m)\right)$$
$$+B_m^{\ell}\left(k_m p_{\ell}^{m+1}(r_m)w_{\ell-1}^{m}(r_m)-k_{m+1}w_{\ell}^{m}(r_m)p_{\ell-1}^{m+1}(r_m)\right)\Big],$$

$$\boxed{B_{m+1}^{\ell}=-\frac{1}{k_{m+1}W\left(p_{\ell}^{m+1}(r_m),q_{\ell}^{m+1}(r_m)\right)}\Big[A_m^{\ell}F\left(r_m,k_m,z_{\ell}^{m},p_{\ell}^{m}\right)+B_m^{\ell}F\left(r_m,k_m,w_{\ell}^{m},p_{\ell}^{m}\right)\Big]} \quad (26)$$

We define for $m=2,3,\cdots,n-2$

$$I_m(\ell)=\begin{pmatrix} I_m^{11}(\ell) & I_m^{12}(\ell) \\ I_m^{21}(\ell) & I_m^{22}(\ell)\end{pmatrix}=\begin{pmatrix} F\left(r_m,k_m,z_{\ell}^{m},p_{\ell}^{m}\right) & F\left(r_m,k_m,z_{\ell}^{m},q_{\ell}^{m}\right) \\ F\left(r_m,k_m,w_{\ell}^{m},p_{\ell}^{m}\right) & F\left(r_m,k_m,w_{\ell}^{m},q_{\ell}^{m}\right)\end{pmatrix}. \quad (27)$$

$$\begin{cases} C_1(\ell)=\dfrac{1}{k_2W\left\{p_{\ell}^{2}(r_1),q_{\ell}^{2}(r_1)\right\}}I_1^{1,2}(\ell), \quad D_1(\ell)=-\dfrac{1}{k_2W\left\{p_{\ell}^{2}(r_1),q_{\ell}^{2}(r_1)\right\}}I_1^{1,1}(\ell), \\ C_m(\ell)=\dfrac{1}{k_{m+1}W\left\{p_{\ell}^{m+1}(r_m),q_{\ell}^{m+1}(r_m)\right\}}\left[C_{m-1}(\ell)I_m^{1,2}(\ell)+D_{m-1}(\ell)I_m^{2,2}(\ell)\right], \\ D_m(\ell)=-\dfrac{1}{k_{m+1}W\left\{p_{\ell}^{m+1}(r_m),q_{\ell}^{m+1}(r_m)\right\}}\left[C_{m-1}(\ell)I_m^{1,1}(\ell)+D_{m-1}(\ell)I_m^{2,1}(\ell)\right], \end{cases} \quad (28)$$

Therefore we have for $m=2,3,\cdots,n-2$,

$$\begin{aligned} &A_2^{\ell}=C_1(\ell)A_1^{\ell}, \quad B_2^{\ell}=D_1(\ell)A_1^{\ell}, \\ &A_{m+1}^{\ell}=C_m(\ell)A_1^{\ell}, \quad B_{m+1}^{\ell}=D_m(\ell)A_1^{\ell}, \end{aligned} \quad (29)$$

Then, we have in particular

$$A_{n-1}^{\ell}=A_1^{\ell}C_{n-2}(\ell), \text{ and } B_{n-1}^{\ell}=A_1^{\ell}D_{n-2}(\ell).$$

Finally, we solve the ($\mathcal{S}(n-1)$) system

$$\begin{cases} A_{n-1}^{\ell}z_{\ell}^{n-1}(r_{n-1})+B_{n-1}^{\ell}w_{\ell}^{n-1}(r_{n-1})=A_n^{\ell}p_{\ell}^{n}(r_{n-1})+B_n^{\ell}q_{\ell}^{n}(r_{n-1}) \\ A_{n-1}^{\ell}k_{n-1}z_{\ell}'(k_{n-1}r_{n-1})+B_{n-1}^{\ell}k_{n-1}w_{\ell}'(k_{n-1}r_{n-1})=A_n^{\ell}k_n p_{\ell}'(k_n r_{n-1})+B_n^{\ell}k_n q_{\ell}'(k_n r_{n-1}) \end{cases}$$

Now we use that

$$A_n^{\ell}p_{\ell}^{n}(1)+B_n^{\ell}q_{\ell}^{n}(1)=1 \Leftrightarrow B_n^{\ell}=\frac{1}{q_{\ell}^{n}(1)}-\frac{p_{\ell}^{n}(1)}{q_{\ell}^{n}(1)}A_n^{\ell},$$

to obtain

$$\begin{cases} A_1^{\ell}\left[C_{n-2}(\ell)z_{\ell}^{n-1}(r_{n-1})+D_{n-2}(\ell)w_{\ell}^{n-1}(r_{n-1})\right]-A_n^{\ell}\left[p_{\ell}^{n}(r_{n-1})-\dfrac{p_{\ell}^{n}(1)}{q_{\ell}^{n}(1)}q_{\ell}^{n}(r_{n-1})\right]=\dfrac{q_{\ell}^{n}(r_{n-1})}{q_{\ell}^{n}(1)} \\ A_1^{\ell}\left[C_{n-2}(\ell)k_{n-1}z_{\ell}'(k_{n-1}r_{n-1})+D_{n-2}(\ell)k_{n-1}w_{\ell}'(k_{n-1}r_{n-1})\right]-A_n^{\ell}\left[k_n p_{\ell}'(k_n r_{n-1})-\dfrac{p_{\ell}^{n}(1)}{q_{\ell}^{n}(1)}k_n q_{\ell}'(k_n r_{n-1})\right]=\dfrac{k_n q_{\ell}'(k_n r_{n-1})}{q_{\ell}^{n}(1)} \end{cases}$$

This system is equivalent to

$$\begin{cases} M_{11}A_1^{\ell}+M_{12}A_n^{\ell}=\dfrac{q_{\ell}^{n}(r_{n-1})}{q_{\ell}^{n}(1)} \\ M_{21}A_1^{\ell}+M_{22}A_n^{\ell}=\dfrac{k_n q_{\ell}'(k_n r_{n-1})}{q_{\ell}^{n}(1)} \end{cases} \quad (30)$$

where

$$\begin{cases} M_{11} = C_{n-2}(\ell) z_\ell^{n-1}(r_{n-1}) + D_{n-2}(\ell) w_\ell^{n-1}(r_{n-1}), \\ M_{12} = -\left(p_\ell^n(r_{n-1}) - \dfrac{p_\ell^n(1)}{q_\ell^n(1)} q_\ell^n(r_{n-1}) \right), \\ M_{21} = C_{n-2}(\ell) k_{n-1} z_\ell'(k_{n-1} r_{n-1}) + D_{n-2}(\ell) k_{n-1} w_\ell'(k_{n-1} r_{n-1}), \\ M_{22} = -\left(k_n p_\ell'(k_n r_{n-1}) - \dfrac{p_\ell^n(1)}{q_\ell^n(1)} k_n q_\ell'(k_n r_{n-1}) \right). \end{cases} \tag{31}$$

If the solution of this system is $\begin{pmatrix} X(1,1) \\ X(2,1) \end{pmatrix}$, the $\ell - 1$ eigenvalue is

$$\lambda_{\ell-1} = X(2,1)\left(k_n p_\ell'(k_n) - \frac{p_\ell^n(1)}{q_\ell^n(1)} k_n q_\ell'(k_n) \right) + \frac{k_n q_\ell'(k_n)}{q_\ell^n(1)}, \text{ with } \ell = 1, 2, \cdots$$

Or

$$\lambda_{\ell-1} = X(2,1)\left(k_n p_{\ell-1}^n(1) - \frac{k_n p_\ell^n(1)}{q_\ell^n(1)} q_{\ell-1}^n(1) \right) + \frac{k_n q_{\ell-1}^n(1) - \ell q_\ell^n(1)}{q_\ell^n(1)}, \quad \ell = 1, 2, \cdots \tag{32}$$

where $X(2,1)$ depends on n and ℓ.

Taking $C = X(2,1)$, we have the result. □

Finally, we have obtained an explicit (although rather long and complicated) expression of the Dirichlet-to-Neuman map.

We will illustrate that $\lambda_{\ell-1}, \ell \geq 1$ in (32) verify the proprieties 1 and 3 in section 2 with various examples. We will do some numerical simulations for this.

4.2. The Case Where the Potential q Is a Continuous Radial Function

In this section, we assume that the potential q is a continuous function with $q(r) > 0$ or $q(r) < 0$ in the interval $[0,1]$.

Let introduce the theorems which gives us the expression of the Dirichlet-to-Neuman map when the potential q is a continuous function, based on the results of a piecewise constant radial potential.

For all $\ell \in \mathbb{N}$, $p_\ell^m(r)$ denotes the Bessel function of the first type $j_\ell\left(\sqrt{|\gamma_m|} r\right)$ or the Bessel function of the second type $i_\ell\left(\sqrt{|\gamma_m|} r\right)$, and $q_\ell^m(r)$ denotes the modified Bessel function of the first type $y_\ell\left(\sqrt{|\gamma_m|} r\right)$ or the modified Bessel function of the second type $(-1)^{\ell+1} k_\ell\left(\sqrt{|\gamma_m|} r\right)$.

Theorem 4.2: *Let the unit ball B in $\mathbb{R}^3$ and a continuous radial potential function $q \in L^\infty(B)$ with $q(r) = q(|x|)$, where $q(r) > 0$ or $q(r) < 0$, such that the Dirichlet problem for $-\Delta + q$ is well posed.*

Let n be a large integer number such that $[0,1] = \bigcup_1^n [r_{m-1}, r_m]$ with $m = 1, \cdots, n$ and where $r_0 = 0$, $r_n = 1$ and $r_m - r_{m-1} = \dfrac{1}{n}$.

Let a denote $k_m = \sqrt{|q(r_m)|}$.

There is, for n enough large integer numbers and $\ell = 1, 2, \cdots$, an explicit for-

mula for the Dirichlet-to-Neumann map defined as follows:

$$\Lambda_q\left(Y_\ell^k(\theta)\right) = \left[\tilde{C}\left(\sqrt{|q(1)|}p_{\ell-1}^*(1) - \frac{\sqrt{|q(1)|}p_\ell^*(1)}{q_\ell^*(1)}q_{\ell-1}^*(1)\right) + \frac{\sqrt{|q(1)|}q_{\ell-1}^*(1) - \ell q_\ell^*(1)}{q_\ell^*(1)}\right]Y_\ell^k(\theta). \quad (33)$$

with $\tilde{C}$ *depending* ℓ, $p_\ell^*(1) = p_\ell\left(\sqrt{|q(1)|}\right)$, $p_{\ell-1}^*(1) = p_{\ell-1}\left(\sqrt{|q(1)|}\right)$, $q_\ell^*(1) = q_\ell\left(\sqrt{|q(1)|}\right)$ *and* $q_{\ell-1}^*(1) = q_{\ell-1}\left(\sqrt{|q(1)|}\right)$.

Proof of theorem 4.2. Let introduce $q_i, i = 1,2$ be piecewise constant radial functions, $q_i(r) = q(|x|)$, defined by

$$q_1(r) = \sum_{m=1}^{n} q(r_{m-1})\chi_{(r_{m-1},r_m)}, \quad q_2(r) = \sum_{m=1}^{n} q(r_m)\chi_{(r_{m-1},r_m)}, \quad r = |x|.$$

Then from theorem 4.1,

$$\Lambda_{q_1}\left(Y_\ell^k(\theta)\right) = \left[C_1\left(k_{n-1}^1 p_{\ell-1}^n(1) - \frac{k_{n-1}^1 p_\ell^n(1)}{q_\ell^n(1)}q_{\ell-1}^n(1)\right) + \frac{k_{n-1}^1 q_{\ell-1}^n(1) - \ell q_\ell^n(1)}{q_\ell^n(1)}\right]Y_\ell^k(\theta),$$

And

$$\Lambda_{q_2}\left(Y_\ell^k(\theta)\right) = \left[C_2\left(k_n^2 p_{\ell-1}^n(1) - \frac{k_n^2 p_\ell^n(1)}{q_\ell^n(1)}q_{\ell-1}^n(1)\right) + \frac{k_n^2 q_{\ell-1}^n(1) - \ell q_\ell^n(1)}{q_\ell^n(1)}\right]Y_\ell^k(\theta),$$

for all $\ell = 1,2,\cdots$ with C_1 and C_2 depending on n and ℓ.

If q is increasing, then $q_1(r) \le q(r) \le q_2(r)$; if not, $q_1(r) \le q(r) \le q_2(r)$.

We have $\lim_{n\to\infty} q_2(r) = \lim_{n\to\infty} q_1(r) = q(r)$, and then there is $\tilde{C}(\ell)$ such that $\lim_{n\to\infty} C_1(n,\ell) = \lim_{n\to\infty} C_2(n,\ell) = \tilde{C}(\ell)$.

We know that $p_\ell^n(1) = p_\ell(k_n)$, $p_{\ell-1}^n(1) = p_{\ell-1}(k_n)$, $q_\ell^n = q_\ell(k_n)$, $q_{\ell-1}^n = q_{\ell-1}(k_n)$, and $k_n = \sqrt{|q(1)|}$, then $p_\ell^*(1) = p_\ell\left(\sqrt{|q(1)|}\right)$, $p_{\ell-1}^*(1) = p_{\ell-1}\left(\sqrt{|q(1)|}\right)$, $q_\ell^*(1) = q_\ell\left(\sqrt{|q(1)|}\right)$ and $q_{\ell-1}^*(1) = q_{\ell-1}\left(\sqrt{|q(1)|}\right)$.

Taking n goes to ∞ in Λ_{q_1} and Λ_{q_2} and using the theorem 4.1, we have the result 33. □

4.3. Numerical Simulations

In this section, we denote $k = \ell - 1$, and then $k = 0,1,2,\cdots$ when $\ell = 1,2,\cdots$, and then we write λ_k in the simulations. We will numerically compute the potential q, λ_k, $k - \lambda_k$, and $\log\left(|k - \lambda_k|\right)$, $k = 0,1,2,\cdots$. We will check numerically if the eigenvalues found in theorems (4.1) and (4.2) verify the properties 1 to 3 introduced in section 2. We will use the Matlab trial version [2021*b*] for it.

We consider the case where the radial potential is defined by a piecewise constant function

$$q(r) = \sum_{m=1}^{n} \gamma_m \chi_{(r_{m-1},r_m)}, \quad r = |x|,$$

where $n \ge 1$, $\gamma_m, r_m \in \mathbb{R}$, with $m = 1,2,\cdots,n$, $0 = r_0 < r_1 < \cdots < r_{n-1} < r_n = 1$,

And the case where the radial potential is defined by a continuous function

$$q(r), \quad r = |x|,$$

with $[0,1]=\bigcup_1^n[r_{m-1},r_m]$, $m=1,\cdots,n$ where n is a large integer number, $r_0=0$, $r_n=1$ and $r_m-r_{m-1}=\frac{1}{n}$, such that the Dirichlet problem for $-\Delta+q$ is well-posed.

At the first, we consider two examples of piecewise constant radial potential functions where the length of interval $[r_{m-1},r_m]$ is arbitrary. We denote Case 1 the case where the potential value at each interval is a random value between −2 and 0, and Case 2 where the potential value at each interval is a random value between −2 and 2.

Secondly, we consider an example of radial continuous potential function in $[0,1]=\bigcup_1^n[r_{m-1},r_m]$, $m=1,\cdots,n$ where the length of intervals $[r_{m-1},r_m]$ is constant and equal to $\frac{1}{n}$.

We denote this example Case 3 taking $q(r)=0.05+r^2$. We approximate it by two piecewise constant radial potential functions $q_1(r)$ and $q_2(r)$ such that $q_1(r)\le q(r)\le q_2(r)$. Using the results of the above section for these cases, we obtain the following results.

4.3.1. Case 1

In this case, in **Figure 1** there is an example of the potential q and in **Figure 2** we see the corresponding eigenvalues. As excepted, we confirm in **Figure 3** and **Figure 4**. These figures show that the eigenvalues defining the Dirichlet-to-Neuman map in theorem (4.1) verify the 1 to 3 properties considered in Section 2.

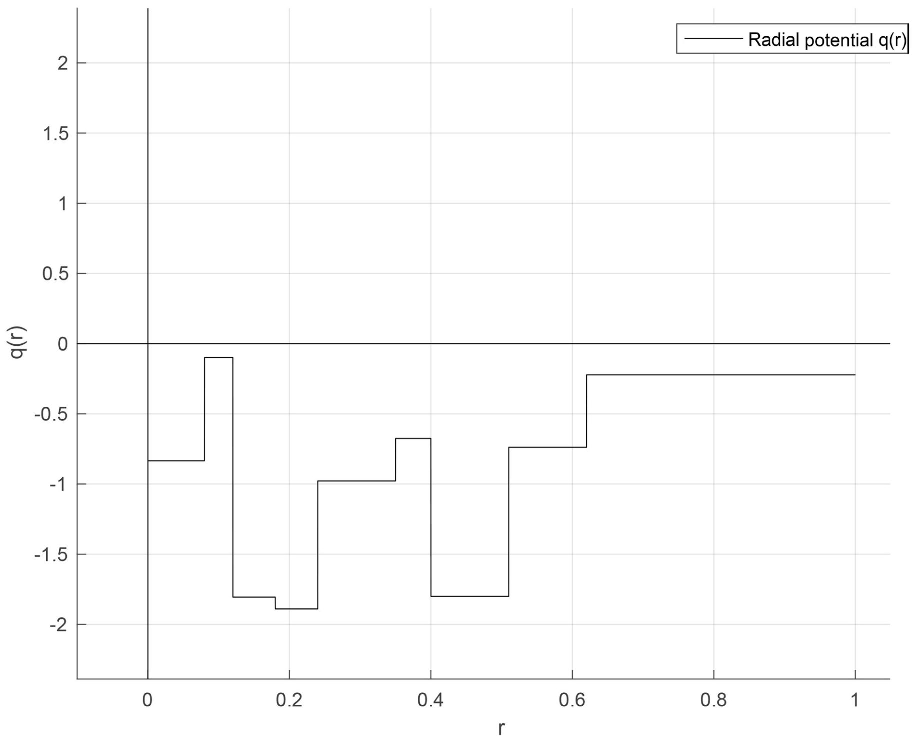

Figure 1. Radial potential q(r) in Case 1.

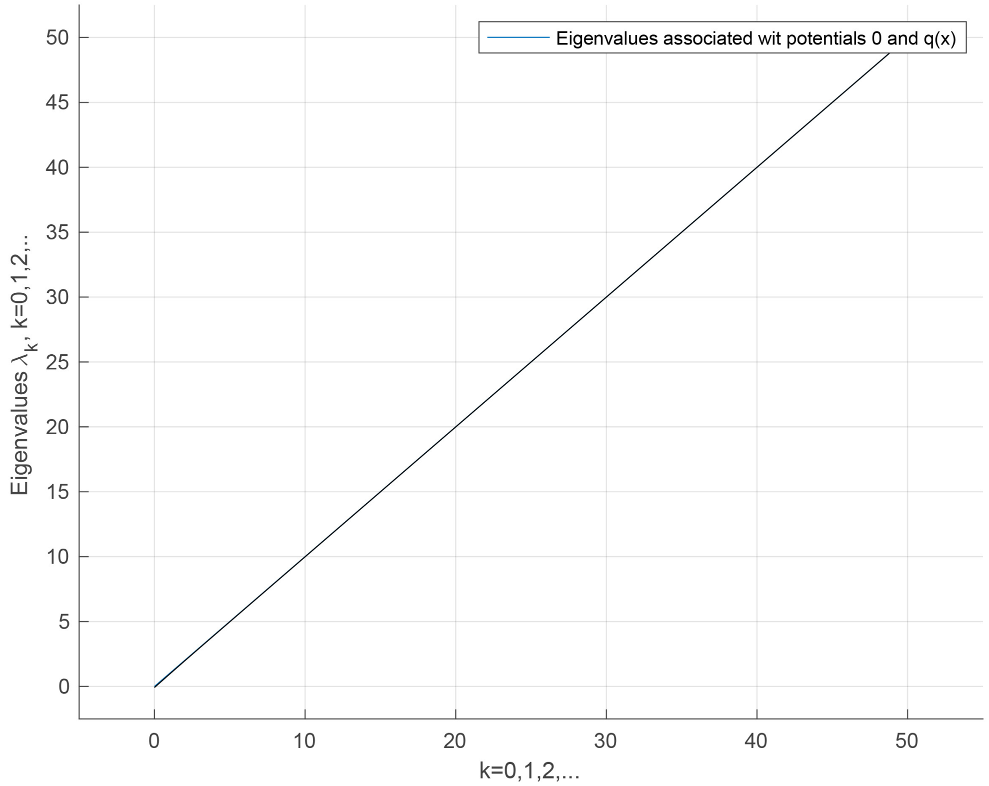

Figure 2. Eigenvalues associated with potential in Case 1.

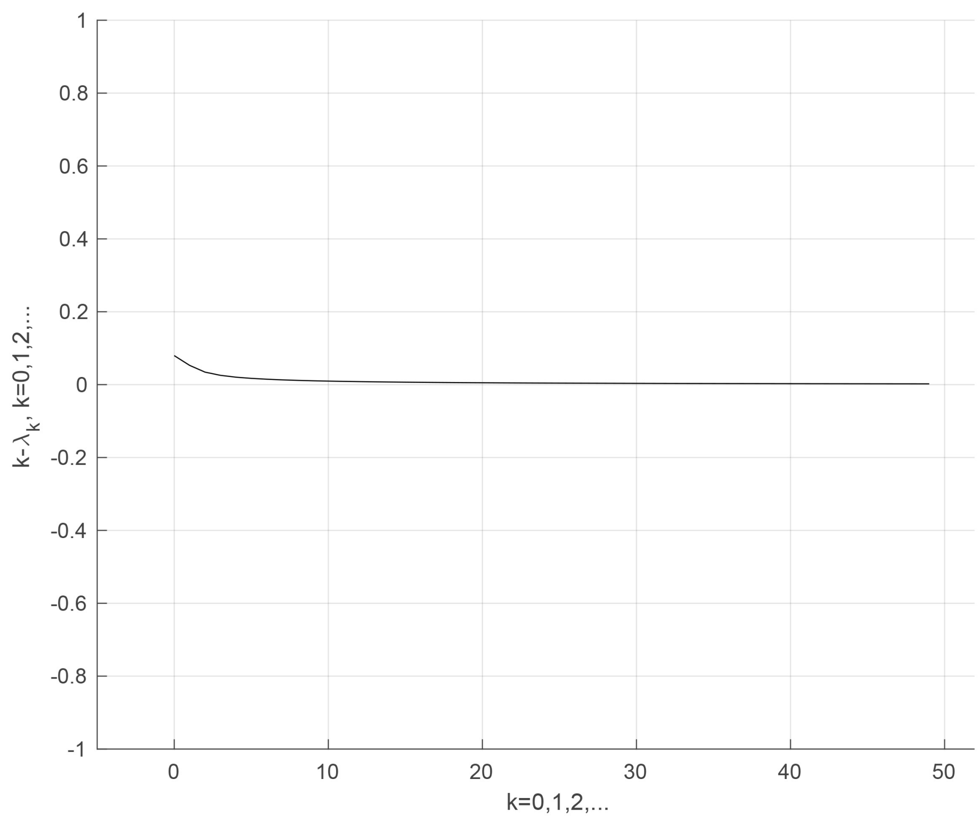

Figure 3. (Eigenvalues-order)-limit in Case 1.

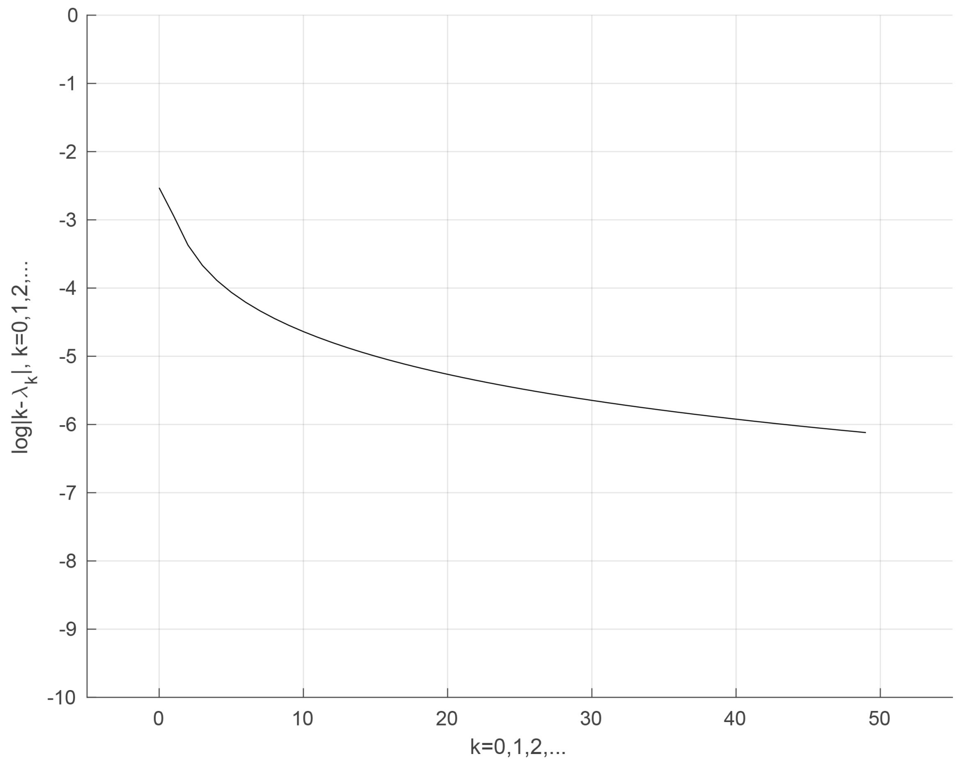

Figure 4. Confirmation eigenvalues limit in Case 1.

4.3.2. Case 2

In this case, in **Figure 5** there is also an example of the potential q and in **Figure 6** we see the corresponding eigenvalues. As excepted, we confirm in **Figure 7** and **Figure 8**. These figures show that the eigenvalues defining the Dirichlet-to-Neuman map in the theorem (4.1) verify also the 1 to 3 properties considered in Section 2.

We could do the test for potential positives only and we would have the same results. To do this, we only need to take the absolute values of γ_m in Case 2. We could also do this test for fixed size intervals; the results will show that the coefficients of the Dirichlet-to-Neuman map in theorem (4.1) are eigenvalues that verify the properties 1 to 3.

4.3.3. Case 3

In this case, in **Figure 9** we have the potential curve $q(r) = 0.05 + r^2$ in red and this with its approximation by a piecewise constant radial potential in black. In **Figure 10** we see the corresponding eigenvalues. As excepted, we confirm in **Figure 11** and **Figure 12**. These figures show that the eigenvalues defining the Dirichlet-to-Neuman map in theorem (4.2) verify the 1 to 3 properties considered in Section 2.

Remark 4.2. *Theorems are essential tools to determine the explicit expression of the DN map when f, defined in* $\mathbb{S}^2$, *is usually written as Fourier series*

$$f(\theta) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} \hat{f}_{\ell k} Y_\ell^k(\theta).$$

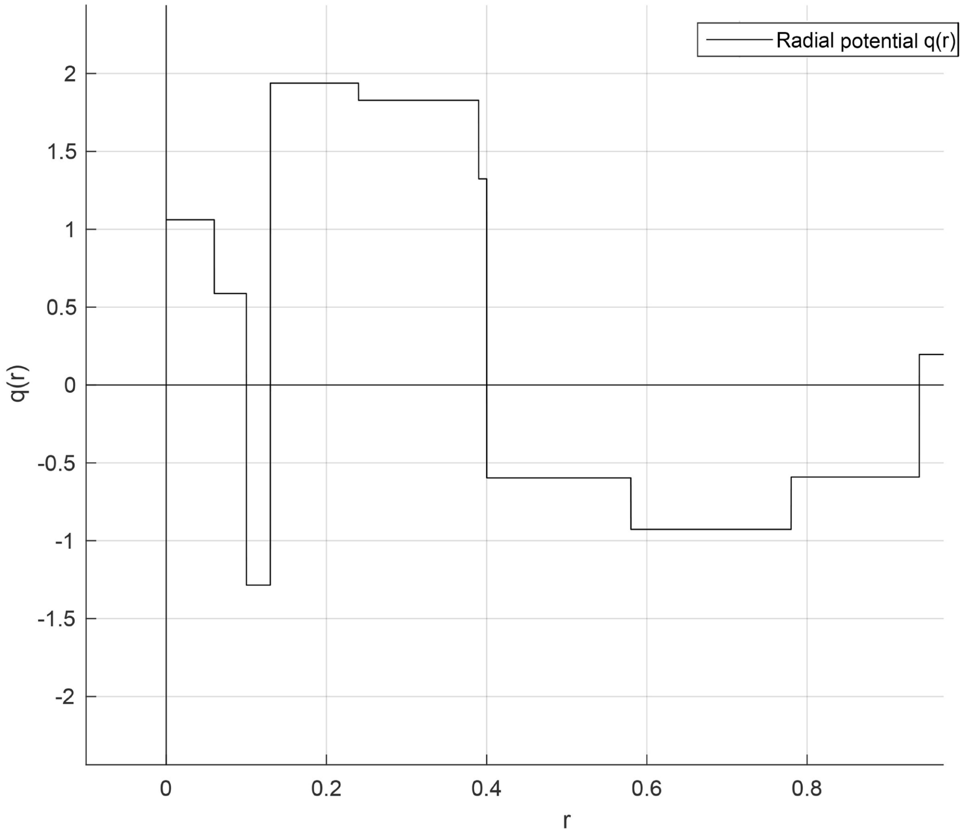

Figure 5. Radial potential q(r) in Case 2.

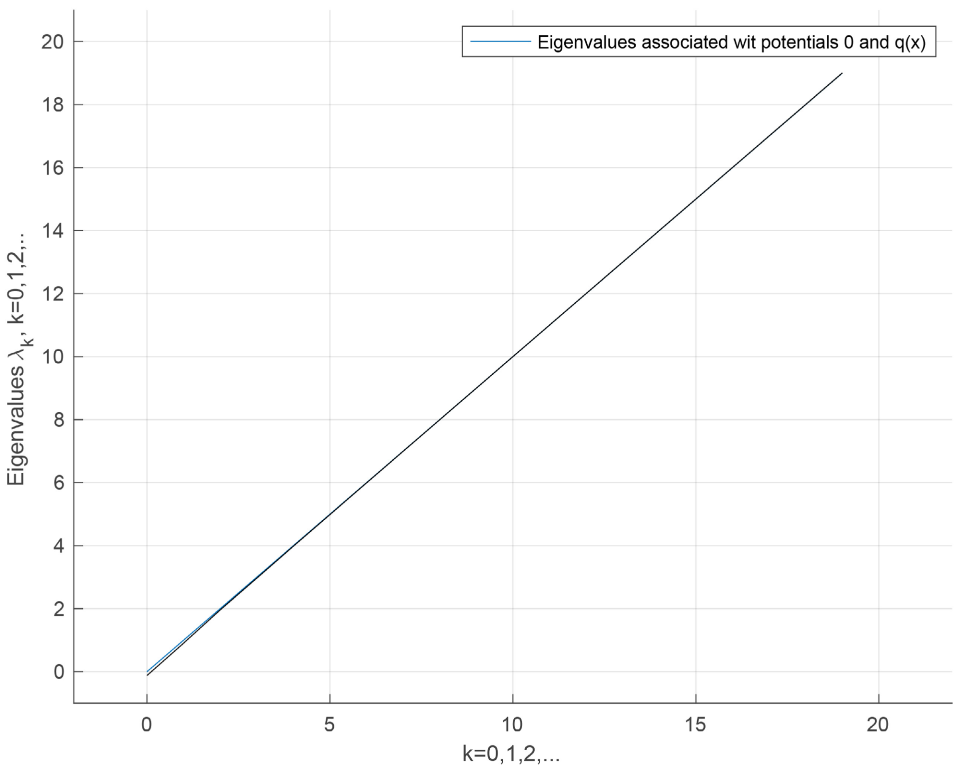

Figure 6. Eigenvalues associated with potentials in Case 2.

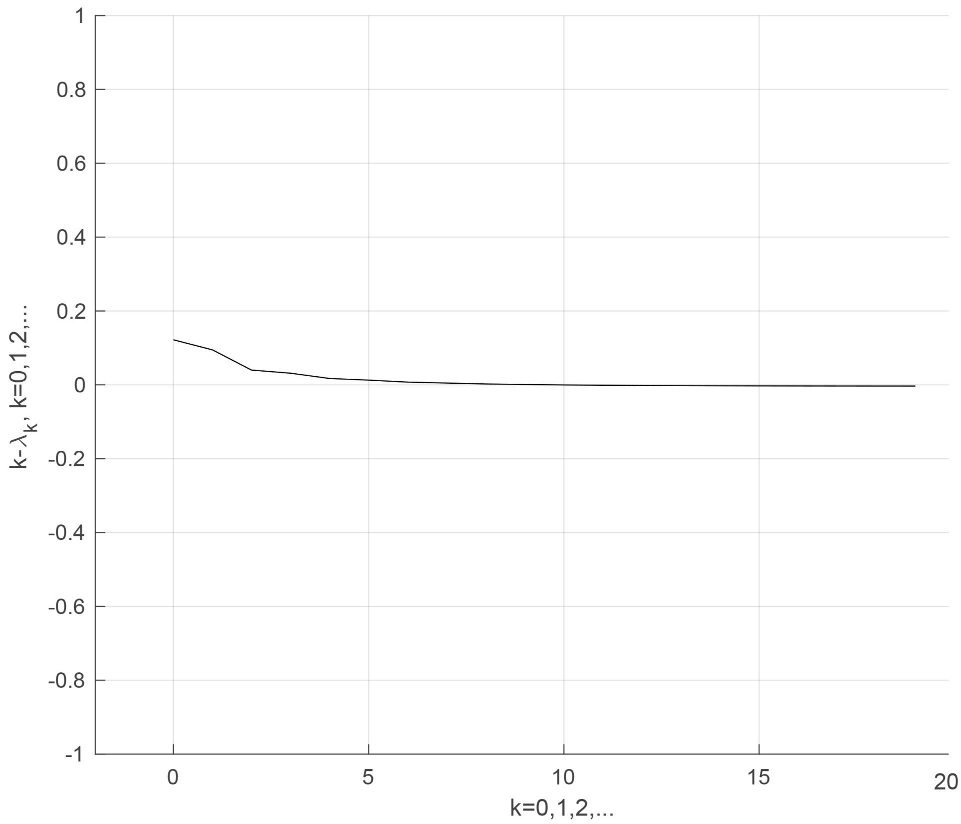

Figure 7. (Eigenvalues-order)-limit in Case 2.

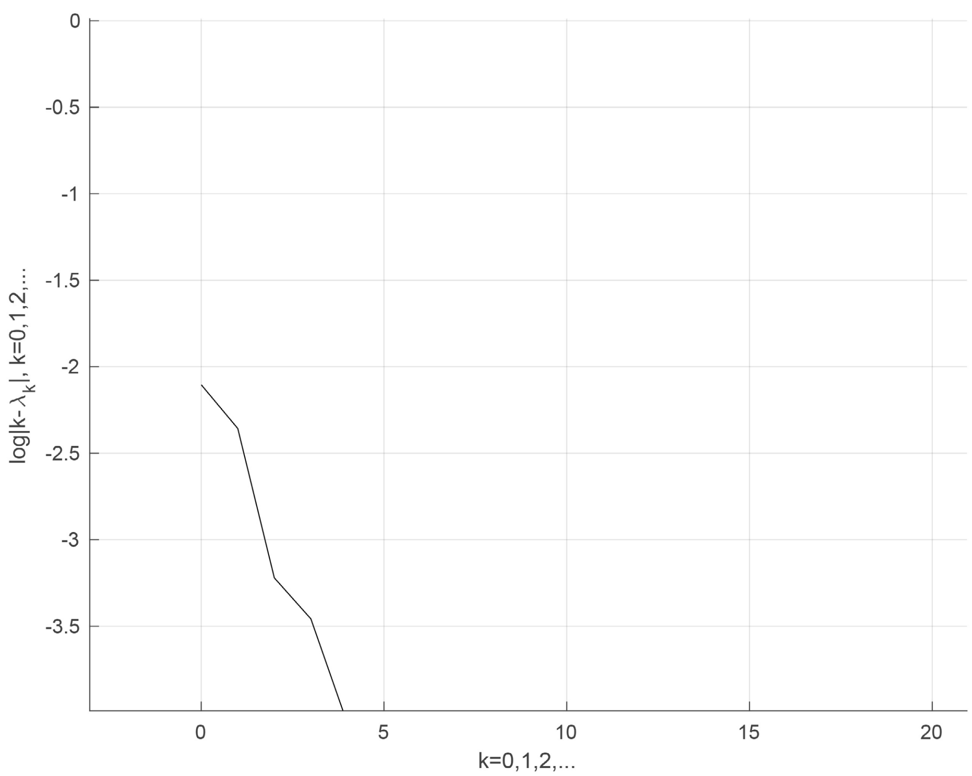

Figure 8. Confirmation eigenvalues limit in Case 2.

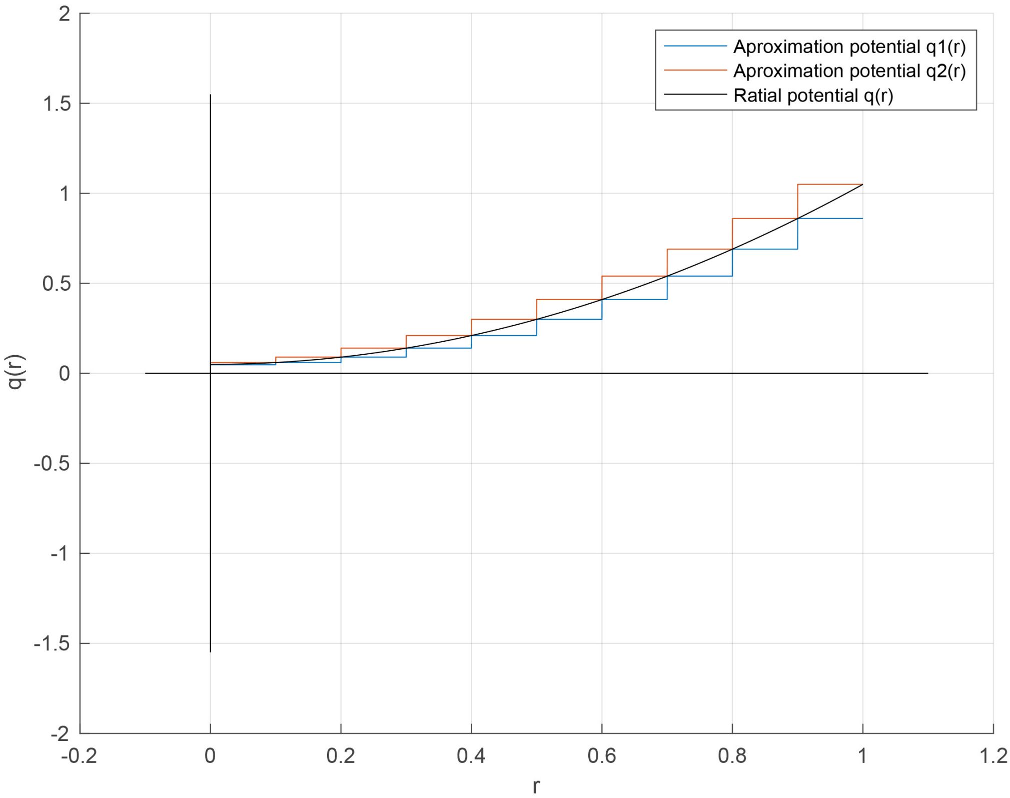

Figure 9. Continuous radial potential in Case 3.

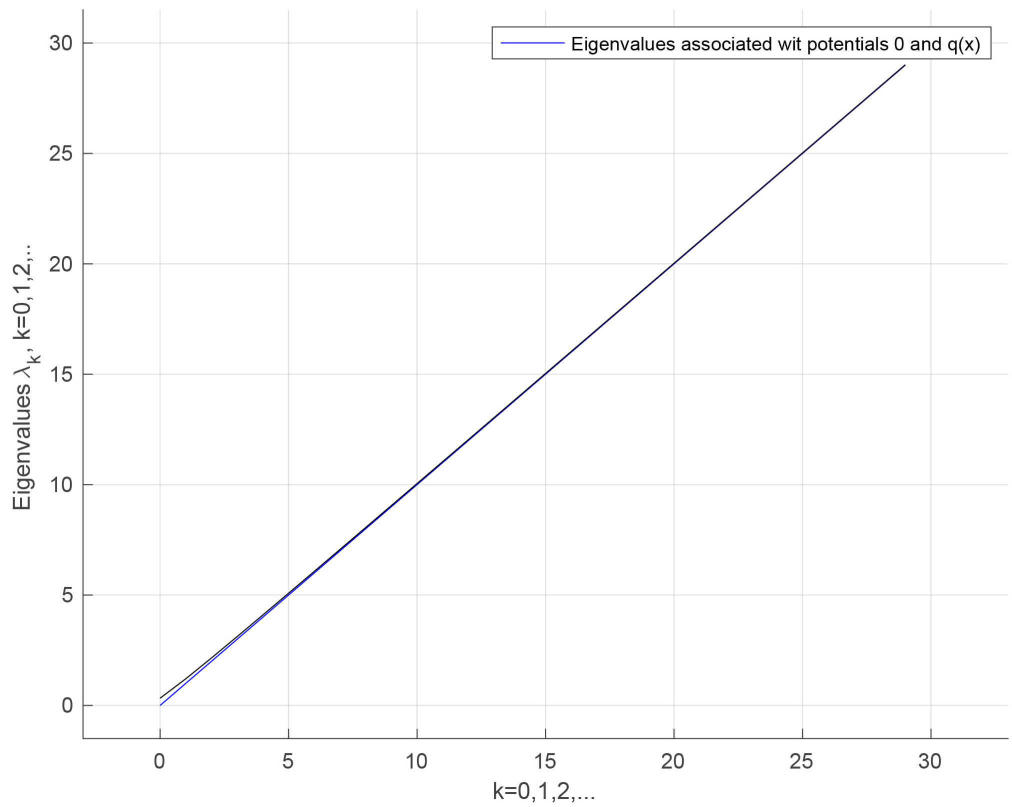

Figure 10. Eigenvalues associated with Continuous radial potential in Case 3.

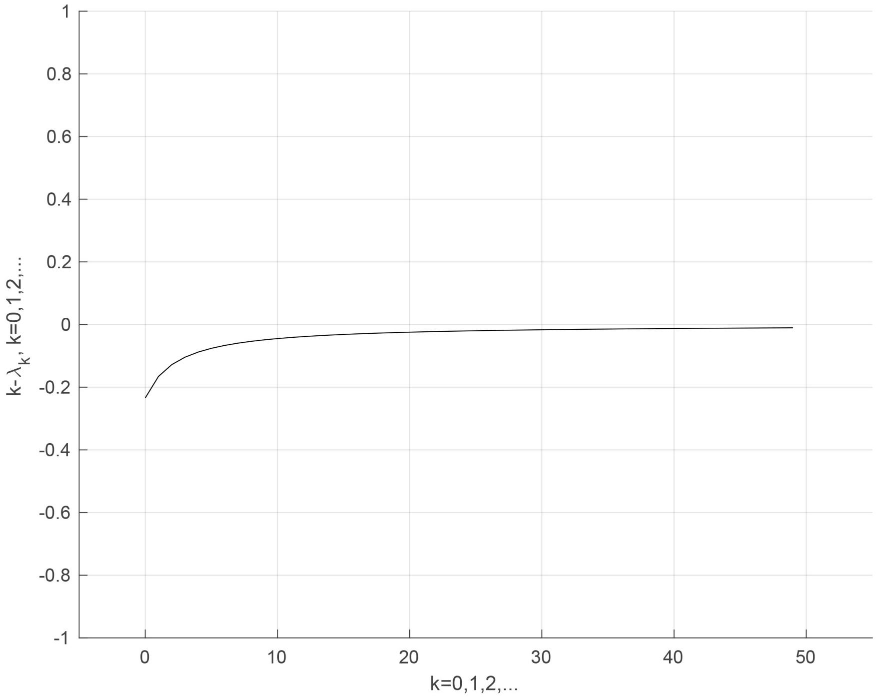

Figure 11. (Eigenvalues-order)-limit in Case 3.

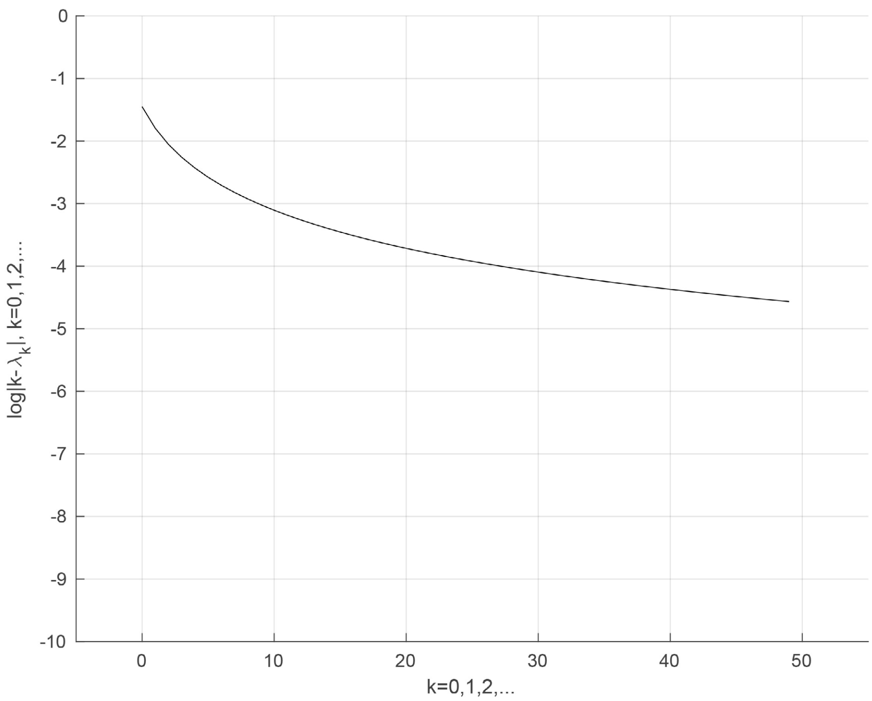

Figure 12. Confirmation eigenvalues limit in Case 3.

These results are very important for studying the inverse problem for our Schrödinger equation. We are interested in the stability of the map that associates a Dirichlet-to-Neumann map to any potential. That is the purpose of the following section.

4.4. Stability

In this section, we are interested in the map

$$\begin{aligned} \Lambda : L^{\infty}\left(\mathbb{S}^2\right) &\to \mathcal{L}\left(H^{1/2}\left(\mathbb{S}^2\right), H^{-1/2}\left(\mathbb{S}^2\right)\right) \\ q &\mapsto \Lambda_q, \end{aligned} \quad (34)$$

where the Dirichlet-to-Neumann map Λ_q is defined in theorem (4.1). This is an important role in the inverse potential problem, which consists to study its inversion. In the mathematical literature, the Dirichlet to Neumann map is invertible in its range. Taking into account how the measurements for the inverse problem for our Schrödinger equation are made at the $\mathbb{S}^2$, we know that there may be some noise in the measured Dirichlet-to-Neumann map and that the noisy version of the real Dirichlet-to-Neumann map may not be a Dirichlet-to-Neumann map corresponding to piecewise constant potentials. Therefore, the stability analysis of Λ, possibly including a regularization strategy useful for the numerical algorithm, would be interesting.

Let us consider the following map $\Lambda : q \mapsto \Lambda_q$. We are interested in a quantification of the difference of two potentials in the L^{∞} topology in terms of the distance of their associated Dirichlet-to-Neumann maps. This stability is necessary for all reconstruction algorithms to recover the potential from the Dirichlet-to-Neumann map, see [18] [19]. Then we would like to estimate $q_1 - q_2$ in a certain norm defined by

$$\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2} \to H^{-1/2}} = \sup_{f \in H^{1/2}\left(\mathbb{S}^2\right),\, f \neq 0} \frac{\left\|\left(\Lambda_{q_1} - \Lambda_{q_2}\right) f\right\|_{H^{-1/2}\left(\mathbb{S}^2\right)}}{\|f\|_{H^{1/2}\left(\mathbb{S}^2\right)}}$$

There are stability results when the potential q has some smoothness.

In [20], Joel *et al.* estimate the difference $q_1 - q_2$ in a lower norm in terms of the difference of the Dirichlet-to-Neumann data maps for $\frac{d}{2} < s \in \mathbb{N}$, $d \geq 3$ and $\|q_l\|_{s,\Omega} \leq M$, with d the space dimension.

In [21], for any $d \geq 3$ and $m > 0$, Mandache proved that there is $\alpha > 0$ such that for every $M > 0$ there is $C(M) > 0$, so that $\|q_i\|_{C^m} \leq M, i = 1, 2$ implies

$$\|q_1 - q_2\|_{L^{\infty}(\Omega)} \leq C(M)\left(\log\left(1 + \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2} \to H^{-1/2}}^{-1}\right)\right)^{-\alpha}, \quad (35)$$

He shows that (35) is optimal, in the sense that it cannot hold with $\alpha > m(2d-1)/d$.

According to [21], for arbitrary potentials q, the Lipschitz stability cannot be held.

In [18], M. Salo proved for $q_i \in L^{\infty}(\Omega)$ that a log-stability estimate holds when $q_1 - q_2 \in H^{-1}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with $\mathcal{C}^{\infty}$ boundary, and dimension $d \geq 3$.

We work in the case of piecewise constant arbitrary potentials q. Let us introduce for $n \geq 1$ and finite, $m = 1,2,\cdots,n$ and $0 = r_0 < r_1 < \cdots < r_{n-1} < r_n = 1$, the space

$$\mathcal{Q} = \left\{ q \in \mathrm{L}^{\infty}(B) : q(r) = \sum_{m=1}^{n} \gamma_m \chi_{(r_{m-1},r_m)}, r = |x|, r_m \in [0,1], \gamma_m \in \mathbb{R} \right\}$$

In the case where $\gamma_m = 0, m = 1,2,\cdots,n$, we approximate it by -0.01.

Here, we establish Lipschitz stability by giving a constant, which depends on n and ℓ in the dimension n of the potential space.

Our method follows the ideas in [22] [23], where Alessandrini *et al.* considered special classes of piecewise constant conductivities, and the method of Bereta *et al.* in [24], for $n \geq 2$.

The Lipschitz stability of an inverse boundary value problem for a Schrödinger type equation is proved by Bereta *et al.* in [24], for $n \geq 2$.

Here, we study the Lipschitz stability of the map that associates a Dirichlet to Neumann map to any piecewise constant potential q.

Theorem 4.3. *Let the unit ball B in* $\mathbb{R}^3$ *and the scaled potential* $q_i, i = 1,2$ *verifes*

$$q_i(r) = \sum_{m=1}^{n} \gamma_m^i \chi_{(r_{m-1},r_m)}, i = 1,2, \quad r = |x|,$$

where $n \geq 1$, $\gamma_m^i, r_m \in \mathbb{R}$, *with* $m = 1,2,\cdots,n$ *and* $0 = r_0 < r_1 < \cdots < r_{n-1} < r_n = 1$, *and* $k_m^i = \sqrt{|\gamma_m^i|}$, *such that the Dirichlet problems for* $-\Delta + q_i$ *is well-posed. Assume that* $\gamma_n^1 \times \gamma_n^2 > 0$ *and there is a positive constant M such that*

$$\|q_i\|_{L^{\infty}(B)} \leq M.$$

Then there is a constant $C = C(n, M, \ell)$, *such that*:

$$|\gamma_n^1 - \gamma_n^2| \leq C\left(\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}}\right). \tag{36}$$

The result gives us the Lipschitz stability near to the edge $\mathbb{S}^2$.

Proof of theorem 4.3. $q_i \in \mathcal{Q}, i = 1,2$, then we can write

$$q_1(r) = \sum_{m=1}^{n} \gamma_m^1 \chi_{(r_{m-1}^1, r_m^1)} \quad \text{and} \quad q_2(r) = \sum_{m=1}^{n} \gamma_m^2 \chi_{(r_{m-1}^2, r_m^2)}, \quad r = |x|, \quad \text{for} \quad n \geq 1,$$

$m = 1,2,\cdots,n$, $\gamma_m^1, r_m^1, \gamma_m^2, r_m^2 \in \mathbb{R}$, $0 = r_0^1 < r_1^1 < \cdots < r_{n-1}^1 < r_n^1 = 1$ and $0 = r_0^2 < r_1^2 < \cdots < r_{n-1}^2 < r_n^2 = 1$. We assume that $r_m^1 = r_m^2 = r_m$ for all $m = 0,1,2,\cdots$

We have for all $Y_\ell^k \in H^{1/2}(\mathbb{S}^2)$, $\ell \geq 1$,

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}} = \sup_{Y_\ell^k \in H^{1/2}(\mathbb{S}^2), \ell \geq 1} \|(\Lambda_{q_1} - \Lambda_{q_2}) Y_\ell^k\|_{H^{-1/2}(\mathbb{S}^2)}$$

From theorem (4.1), we obtain

$$\|(\Lambda_{q_1} - \Lambda_{q_2}) Y_\ell^k\|_{H^{-1/2}(\mathbb{S}^2)}^2 = |(\lambda_{\ell-1}^1 - \lambda_{\ell-1}^2)|^2 \|Y_\ell^k\|^2 \quad \text{for all } \ell \geq 1$$

where $\lambda_{\ell-1}^1, \lambda_{\ell-1}^2$ verify the relation (32) for all $n \geq 1$ and finite, $\ell \geq 1$. Then

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}} = \sup_{\ell \geq 1} |(\lambda_{\ell-1}^1 - \lambda_{\ell-1}^2)|$$

Let denote $P_{n,\ell}^1[q_1] = k_n^1 \left(X^1(2,1)\left(p_\ell'(k_n^1) - \frac{p_\ell^n(1)}{q_\ell^n(1)} q_\ell'(k_n^1) \right) + \frac{q_\ell'(k_n^1)}{q_\ell^n(1)} \right)$ where

$\begin{pmatrix} X^1(1,1) \\ X^1(2,1) \end{pmatrix}$ is the solution of 30 associated to q_1 and $P^1_{n,\ell}[q_1]$ the $\ell-1$ eingenvalue associated to q_1.

And $P^2_{n,\ell}[q_2] = k_n^2 \left(X^2(2,1)\left(p'_\ell(k_n^2) - \frac{p^n_\ell(1)}{q^n_\ell(1)} q'_\ell(k_n^2) \right) + \frac{q'_\ell(k_n^2)}{q^n_\ell(1)} \right)$ where $\begin{pmatrix} X^2(1,1) \\ X^2(2,1) \end{pmatrix}$ is the solution of 30 associated to q_2 and $P^2_{n,\ell}[q_2]$ the $\ell-1$ eigenvalue associated to q_2.

We have

$$\lambda^1_{\ell-1} - \lambda^2_{\ell-1} = P^1_{n,\ell}[q_1] - P^2_{n,\ell}[q_2]$$

Then

$$\left\| \Lambda_{q_1} - \Lambda_{q_2} \right\|_{H^{1/2} \to H^{-1/2}} \geq \left| P^1_{n,\ell}[q_1] - P^2_{n,\ell}[q_2] \right|$$

Let denote $A = X^1(2,1)\left(p'_\ell(k_n^1) - \frac{p^n_\ell(1)}{q^n_\ell(1)} q'_\ell(k_n^1) \right) + \frac{q'_\ell(k_n^1)}{q^n_\ell(1)}$,

$B = X^2(2,1)\left(p'_\ell(k_n^2) - \frac{p^n_\ell(1)}{q^n_\ell(1)} q'_\ell(k_n^2) \right) + \frac{q'_\ell(k_n^2)}{q^n_\ell(1)}$ and $D = \inf(A, B)$.

We have $\left| P^1_{n,\ell}[q_1] - P^2_{n,\ell}[q_2] \right| \geq \left| k_n^1 - k_n^2 \right| D \geq \frac{\left\| \gamma_n^1 \right| - \left| \gamma_n^2 \right\|}{2\sqrt{M}} D$ for all $n \geq 1$ finite, $\ell \geq 1$ and $M > 0$.

We have D is positive real depending on n, ℓ and $\gamma_n^1 \times \gamma_n^2 > 0$. Then for all $M > 0$

$$\left\| \Lambda_{q_1} - \Lambda_{q_2} \right\|_{H^{1/2} \to H^{-1/2}} \geq \frac{D(\ell,n)}{2\sqrt{M}} \left| \gamma_n^1 - \gamma_n^2 \right|.$$

If we take $C(\ell, n, M) = \frac{2\sqrt{M}}{D(\ell,n)}$, then we have the result. □

Remark 4.3. *The study of stability for a continuous radial potential function would follow from the study of stability in the case where the potential is a piecewise radial function. It is sufficient to approximate this continuous function by two piecewise radial functions.*

5. Conclusion

We can conclude that when we consider that the potential $q(r)$ is radial function for the Schrödinger equation defined in the unit ball which has no zero on the interval $(0,1)$, there exists an explicit formula for the Dirichlet-to-Neumann map given in theorem (4.1) for all piecewise constant radial potential function, and in theorem (4.2) for all continuous radial potential function. We have established a Lipschitz stability result near the edge of the domain with a constant depending on the dimension of the potential space and the order of the eigenvalues. The Lipschitz stability result of the map that associates a Dirichlet to Neumann map to any radial potential q is essential for the study of its inversion. This explicit formula of the Dirichlet-to-Neumann map $\Lambda_q(f)$ in dimension 3 is a first in the literature; it will open the way to the development of important research on inverse problems. In this perspective, we will consider, among other things, the numerical study of the Dirichlet to Neumann map in the unit ball in $\mathbb{R}^3$, the reconstructing of the potential from the Dirichlet-to-Neumann map both theoretically and numerically, and then the analytical study of the Dirichlet to Neumann map in the case where the potential has one or more zeros on the interval $(0,1)$. In addition, Lipschitz type stability in the depth of the domain will be studied by giving an estimation constant.

Data Availability Statement

The data used to support the findings of this study are included in the article.

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Conflicts of Interest

The author declares that there is no conflict of interest.

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Appendix

We consider the spherical Bessel functions

$$j_\ell(r)=\sqrt{\frac{\pi}{2r}}J_{\ell+\frac{1}{2}}(r),\quad y_\ell(r)=\sqrt{\frac{\pi}{2r}}Y_{\ell+\frac{1}{2}}(r),\tag{37}$$

that satisfies the equation

$$r^2y''+2ry'+\left(r^2-\ell(\ell+1)\right)y=0.$$

The modified spherical Bessel functions

$$i_\ell(r)=\sqrt{\frac{\pi}{2r}}I_{\ell+\frac{1}{2}}(r),\quad k_\ell(r)=\sqrt{\frac{\pi}{2r}}K_{\ell+\frac{1}{2}}(r),\tag{38}$$

that satisfies the equation.

If $f_\ell=j_\ell,y_\ell,i_\ell,(-1)^{\ell+1}k_\ell$ then

$$f_\ell'(r)=f_{\ell-1}-\frac{\ell+1}{r}f_\ell(r).\tag{39}$$