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New Exact Traveling Wave Solutions of (2 + 1)-Dimensional Time-Fractional Zoomeron Equation

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Abstract

In this paper, the new mapping approach and the new extended auxiliary equation approach were used to investigate the exact traveling wave solutions of (2 + 1)-dimensional time-fractional Zoomeron equation with the conformable fractional derivative. As a result, the singular soliton solutions, kink and anti-kink soliton solutions, periodic function soliton solutions, Jacobi elliptic function solutions and hyperbolic function solutions of (2 + 1)-dimensional time-fractional Zoomeron equation were obtained. Finally, the 3D and 2D graphs of some solutions were drawn by setting the suitable values of parameters with Maple, and analyze the dynamic behaviors of the solutions.

Keywords

Exact Traveling Wave Solutions, (2 + 1)-Dimensional Time-Fractional Zoomeron Equation, The New Mapping Approach, The New Extended Auxiliary Equation Approach

1. Introduction

Fractional partial differential equations (FPDEs) have a wide of applications in different fields, such as biology, physics, signal processing, fluid mechanics, and electromagnetic, and so on. In recent decades, many effective methods have been presented to obtain the exact traveling wave solutions of FPDEs, for example, (G'/G) expansion method [1] [2] [3], $(1/G')$ expansion method [4] [5] [6] [7], the $\exp(-\Phi(\xi))$ function method [8] [9], the F-expansion method [10] [11], sine-cosine method [12] [13] and others [14] [15] [16]. There are many important definitions of fractional derivative, such as Riemann-Liouville, Caputo, Atangana' s-conformable and the conformable fractional derivative, etc. [17]

[18] [19] [20] [21].

In this paper, we use the complex traveling wave transformation to deduce (2 + 1)-dimensional conformable time-fractional Zoomeron equation into ordinary differential equation. Furthermore, inspired by the reference [22], we introduce the new mapping approach and the new extended auxiliary equation approach [23] [24] [25] to investigate the exact solutions of (2 + 1)-dimensional time-fractional Zoomeron equation [20]:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}\left[\frac{u_{xy}}{u}\right]-\frac{\partial^{2}u}{\partial x^{2}}\left[\frac{u_{xy}}{u}\right]+2\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\left[u^{2}\right]_{x}=0,0<\alpha\leq 1. \tag{1}$$

when $\alpha=1$, Equation (1) reduces to the (2 + 1)-dimensional Zoomeron equation [26]. Aksoy E. [27] obtained two types of exact analytical solutions including hyperbolic function solutions and trigonometric function solutions by using sub-equation and generalized Kudryashov methods in Equation (1). Hosseini K. [28] obtained several new wave form solutions of Equation (1) such as kink, singular kink, and periodic wave solutions using $\exp(-\Phi(\varepsilon))$ expansion approach and modified Kudryashov method. Akbulut A. [20] obtained analytical solutions of Equation (1) with auxiliary equation method. Based on the study of Akbulut A., Topsakal M. [21] obtained new exact solutions of Equation (1) by using the auxiliary equation method. These methods are effective in investigation of the solutions of Equation (1), the aim of this investigation is to establish more general solutions and some new solutions using the two methods mentioned above.

The organization of this paper is as follows: In Section 2, we introduce the conformable fractional derivative. In Section 3, we introduce the new mapping approach and the new extended auxiliary equation approach to investigate the solutions of (2 + 1)-dimensional time-fractional Zoomeron equation, and analyze the dynamic behaviors of the solutions in Section 4. Finally, we give some conclusions in Section 5.

2. The Conformable Fractional Derivative

In this section, we introduce the conformable fractional derivative [20] [21].

Definition 2.1. [20] Suppose a function $f:[0,\infty)\to R$. Then, the conformable fractional derivative of *f* of order α, which is defined by

$$(\mathrm{T}_{\alpha}f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f\left(t\right)}{\varepsilon} \tag{2}$$

for all $t>0,0<\alpha\leq 1$.

Properties. [20] [21] Let $\alpha\in(0,1)$ and f,g be α-differentiable at a point $t>0$, then some properties of the conformable fractional derivative are by follows:

1) Linearity: $\mathrm{T}_{\alpha}(af+bg)=a(\mathrm{T}_{\alpha}f)+b(\mathrm{T}_{\alpha}g)$, for all $a,b\in R$.

2) Leibniz rule: $\mathrm{T}_{\alpha}(fg)=f\mathrm{T}_{\alpha}(g)+g\mathrm{T}_{\alpha}(f)$.

3) $\mathrm{T}_{\alpha}(t^{p})=pt^{p-\alpha}$, for all $p\in R$.

4) $T_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$.

5) $T_\alpha(f/g)=\dfrac{g(T_\alpha f)-f(T_\alpha g)}{g^2}$.

6) Additionally, if f is differentiable, then

$$T_\alpha(f)(t)=t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}(t).$$

Theorem 2.1 (Chain rule). [20] [21] Assume function $f,g:[0,\infty)\to R$ be α-differentiable, then the following rule is obtained

$$T_\alpha(f\circ g)(t)=t^{1-\alpha}g(t)f'(g(t)) \tag{3}$$

where $0<\alpha\le 1$.

Definition 2.2 (Conformable fractional integral). [21] Let $0<\alpha\le 1$ and $0\le a<b$. A function $f:[a,b]\to R$ is α-ractional integrable on $[a,b]$ if the integral

$$I^\alpha f(x)=\int_a^b f(x)\mathrm{d}_\alpha x=\int_a^b f(x)x^{\alpha-1}\mathrm{d}x \tag{4}$$

exist and is finite.

Theorem 2.2. [29] Let $f\in C[a,b]$ and $0<\alpha\le 1$. Then

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}I^\alpha f(x)=f(x). \tag{5}$$

3. Description of the Methods

Suppose that a nonlinear fractional differential equation with the conformable time-fractional derivative:

$$H\left(u,\frac{\partial^\alpha u}{\partial t^\alpha},\frac{\partial u}{\partial x},\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}},\frac{\partial^2 u}{\partial x^2},\cdots\right)=0 \tag{6}$$

where H is a polynomial of $u(x,t)$ and its partial conformable derivatives including the highest order derivative and the nonlinear term.

We use the complex traveling wave transformation

$$u(x,y,t)=u(\xi),\xi=kx+hy-l\frac{t^\alpha}{\alpha} \tag{7}$$

where k,h,l are non-zero arbitrary constants. Equation (1) converts into a nonlinear ordinary differential equation:

$$P(u,u',u'',\cdots)=0 \tag{8}$$

where P is a polynomial of $u(x,t)$ and its partial derivatives, $'=\dfrac{\mathrm{d}}{\mathrm{d}\xi}$.

3.1. The New Mapping Approach

We suppose the solution of Equation (8) as follow:

$$u(\xi)=\sum_{i=0}^{2N}a_i\varphi^i(\xi) \tag{9}$$

where $a_i\left(i=0,1,\cdots,2N\right)$ are constants, the positive integer N can be determined by balancing the highest order derivative and the nonlinear term in Equation (8). $\varphi(\xi)$ satisfies the following equation:

$$\left(\varphi'\right)^2\left(\xi\right)=r+p\varphi^2+\frac{1}{2}q\varphi^4+\frac{1}{3}s\varphi^6 \tag{10}$$

where r,p,q and s are arbitrary constants, the solutions of Equation (10) given by reference [23] with $s\neq 0$.

3.2. The New Extended Auxiliary Equation Approach

We suppose the solution of Equation (8) as follow:

$$u\left(\xi\right)=\sum_{i=0}^{2N}a_iF^i\left(\xi\right) \tag{11}$$

where $a_i\left(i=0,1,\cdots,2N\right)$ are constants and the positive integer N can be determined by balancing the highest order derivative and the nonlinear term in Equation (8). $F\left(\xi\right)$ satisfies the following equation:

$$\left(F'\right)^2\left(\xi\right)=c_0+c_2F^2\left(\xi\right)+c_4F^4\left(\xi\right)+c_6F^6\left(\xi\right) \tag{12}$$

where $c_j\left(j=0,2,4,6\right)$ are constants and $c_6\neq 0$. Equation (12) has the following solutions:

$$F\left(\xi\right)=\frac{1}{2}\left[\frac{-c_4}{c_6}\left(1\pm f_i\left(\xi\right)\right)\right]^{\frac{1}{2}} \tag{13}$$

where the function $f_i\left(\xi\right)\left(i=1,2,\cdots,12\right)$ is the Jacobi elliptic function $sn\left(\xi\right)$, $cn\left(\xi\right)$, $dn\left(\xi\right)$, while $0<m<1$ is the modulus of the Jacobi elliptic functions. When $m\to 1$ or $m\to 0$, the Jacobi elliptic function solutions degenerate to hyperbolic functions and trigonometric functions [24] [25].

4. Applications

We substitute Equation (7) into Equation (1), which deduce the nonlinear ordinary differential equation:

$$khl^2\left(\frac{u''}{u}\right)''-k^3h\left(\frac{u''}{u}\right)''-2kl\left(u^2\right)''=0 \tag{14}$$

We integrate Equation (14) twice, then we have

$$kh\left(l^2-\kappa^2\right)u''-2klu^3-\beta u=0 \tag{15}$$

where the prime denotes the derivative with respect to ξ, the second constant of integration is zero. Balancing the highest order derivative term u'' and the highest order nonlinear term u^3, we get $N+2=3N$, hence $N=1$.

4.1. Application of the New Mapping Approach

We assume that the solution of Equation (9) as follow:

$$u(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi) \tag{16}$$

where a_0, a_1, a_2 are constants.

Substituting Equation (16) and its derivatives and Equation (10) into Equation (15), yields a system of equations of $\varphi^i(\xi)$, then setting the coefficients of $\varphi^i(\xi)(i = 0,1,2,\cdots)$ to zero, we can deduce the following set of algebraic polynomials with the respect a_0, a_1, a_2:

$$\begin{aligned}
&\varphi^0(\xi): 2rkh(l^2 - k^2)a_2 - 2kla_0^3 - \beta a_0 = 0 \\
&\varphi^1(\xi): pkh(l^2 - k^2)a_1 - 6kla_0^2 a_1 - \beta a_1 = 0 \\
&\varphi^2(\xi): 4pkh(l^2 - k^2)a_2 - 6kla_0(a_1^2 + a_0 a_2) - \beta a_2 = 0 \\
&\varphi^3(\xi): qkh(l^2 - k^2)a_1 - 2kla_1(a_1^2 + 6a_0 a_2) = 0 \\
&\varphi^4(\xi): 3qkh(l^2 - \kappa^2)a_2 - 6kla_2(a_1^2 + a_0 a_2) = 0 \\
&\varphi^5(\xi): skh(l^2 - \kappa^2)a_1 - 6kla_1 a_2^2 = 0 \\
&\varphi^6(\xi): \frac{8}{3}skh(l^2 - \kappa^2)a_2 - 2kla_2^3 = 0
\end{aligned} \tag{17}$$

Solving the above algebraic equations, we obtain the following two results:

Type 1. Substituting $s = \frac{3q^2}{16p}, r = 0$ into Equation (17), we have

$$a_0 = \pm\sqrt{-\frac{\beta}{2kl}}, a_1 = 0, a_2 = \pm\frac{qkh(l^2 - k^2)}{\beta}\sqrt{-\frac{\beta}{2kl}}, p = -\frac{\beta}{2kh(l^2 - k^2)}, q = q \tag{18}$$

Substituting Equation (18) and the solutions in reference [23] into Equation (16), we get

$$u_1(\xi) = \pm\sqrt{-\frac{\beta}{2kl}}\left(2 + \tanh\left(\varepsilon\sqrt{-\frac{\beta}{2kh(l^2 - k^2)}}\xi\right)\right) \tag{19}$$

$$u_2(\xi) = \pm\sqrt{-\frac{\beta}{2kl}}\left(2 + \coth\left(\varepsilon\sqrt{-\frac{\beta}{2kh(l^2 - k^2)}}\xi\right)\right) \tag{20}$$

where $\varepsilon = \pm 1, \beta > 0, h > 0, k < 0$.

Type 2. Substituting $r = 0$ into Equation (17), we have

$$a_0 = a_0, a_1 = 0, a_2 = \frac{qh(l^2 - k^2)}{2la_0}, p = \frac{la_0^2}{h(l^2 - k^2)}, q = q, s = \frac{3q^2 h(l^2 - k^2)}{16la_0^2} \tag{21}$$

Substituting Equation (21) and the solutions in reference [23] into Equation (16), we have

$$u_3(\xi) = a_0\left[1 - \frac{\operatorname{sech}^2\left(\sqrt{\frac{la_0^2}{h(l^2 - k^2)}}\xi\right)}{1 - \frac{1}{4}\left(1 + \varepsilon\tanh\sqrt{\frac{la_0^2}{h(l^2 - k^2)}}\xi\right)^2}\right] \tag{22}$$

$$u_4(\xi)=a_0\left[1+\frac{\operatorname{csch}^2\left(\sqrt{\frac{la_0^2}{h(l^2-k^2)}}\xi\right)}{1-\frac{1}{4}\left(1+\varepsilon\coth\sqrt{\frac{la_0^2}{h(l^2-k^2)}}\xi\right)^2}\right] \tag{23}$$

$$u_5(\xi)=a_0\left[1-\frac{\operatorname{sech}^2\left(\sqrt{\frac{la_0^2}{h(l^2-k^2)}}\xi\right)}{1+\varepsilon\tanh\sqrt{\frac{la_0^2}{h(l^2-k^2)}}\xi}\right] \tag{24}$$

$$u_6(\xi)=a_0\left[1+\frac{\operatorname{csch}^2\left(\sqrt{\frac{la_0^2}{h(l^2-k^2)}}\xi\right)}{1+\varepsilon\coth\sqrt{\frac{la_0^2}{h(l^2-k^2)}}\xi}\right] \tag{25}$$

where $\varepsilon=\pm1, l>0, h>0, \xi=kx+hy-l\frac{t^\alpha}{\alpha}$.

4.2. Application of the New Extended Auxiliary Equation Approach

We assume the solution of Equation (11) as follow:

$$u(\xi)=a_0+a_1F(\xi)+a_2F^2(\xi) \tag{26}$$

where a_0,a_1,a_2 are constants.

Substituting Equation (26) and its derivatives and Equation (12) into Equation (15), yields a system of equations of $F^i(\xi)$, then setting the coefficients of $F^i(\xi)(i=0,1,2,\cdots)$ to zero, we can deduce the following set of algebraic polynomials with the respect a_0,a_1,a_2:

$$\begin{aligned}
&F^0(\xi): 2c_0kh(l^2-k^2)a_2-2kla_0^3-\beta a_0=0\\
&F^1(\xi): c_2kh(l^2-k^2)a_1-6kla_0^2a_1-\beta a_1=0\\
&F^2(\xi): 4c_2kh(l^2-k^2)a_2-6kla_0(a_1^2+a_0a_2)-\beta a_2=0\\
&F^3(\xi): 2c_4kh(l^2-k^2)a_1-2kla_1(a_1^2+6a_0a_2)=0\\
&F^4(\xi): 6c_4kh(l^2-k^2)a_2-6kla_2(a_1^2+a_0a_2)=0\\
&F^5(\xi): 3c_6kh(l^2-k^2)a_1-6kla_1a_2^2=0\\
&F^6(\xi): 8c_6kh(l^2-k^2)a_2-2kla_2^3=0
\end{aligned} \tag{27}$$

Solving the above algebraic equations, we obtain the following results:

$$\begin{aligned}
&a_0=a_0, a_1=0, a_2=a_2\\
&c_0=\frac{2kla_0^3+\beta a_0}{2kh(l^2-k^2)a_2}, c_2=\frac{6kla_0^2+\beta}{4kh(l^2-k^2)}, c_4=\frac{la_0a_2}{h(l^2-k^2)}, c_6=\frac{la_2^2}{4h(l^2-k^2)}
\end{aligned} \tag{28}$$

Substituting Equation (28) into (13), we have

$$F(\xi)=\frac{1}{2}\left[-\frac{4a_0}{a_2}\left(1\pm f_i(\xi)\right)\right]^{\frac{1}{2}} \tag{29}$$

Substituting Equation (28) and (29) into (26), we get the solution

$$u(\xi)=\mp a_0 f_i(\xi) \tag{30}$$

where $f_i(\xi)(i=1,2,\cdots,12)$ given by reference [24]. Insetting them into Equation (30), we obtain the following Jacobi elliptic function solutions of Equation (1):

1) If $c_0=\frac{c_4^3\left(m^2-1\right)}{32c_6^2m^2}, c_2=\frac{c_4^2\left(5m^2-1\right)}{16c_6m^2}, c_6>0$, then

$$u_1(\xi)=\mp a_0 sn\left(\sqrt{\frac{la_0^2}{m^2h\left(l^2-k^2\right)}}\xi, m\right) \tag{31}$$

$$u_2(\xi)=\frac{\mp a_0}{msn\left(\sqrt{\frac{la_0^2}{m^2h\left(l^2-k^2\right)}}\xi, m\right)} \tag{32}$$

where $\xi=kx+hy+\frac{m^2\beta t^\alpha}{\alpha ka_0^2\left(m^2+1\right)}$.

If $m\to 1$, then $sn(\xi)\to\tanh(\xi)$, we get the hyperbolic function solutions:

$$u_1'(\xi)=\mp a_0\tanh\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)}}\xi\right) \tag{33}$$

$$u_2'(\xi)=\mp a_0\coth\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)}}\xi\right) \tag{34}$$

where $\xi=kx+hy+\frac{\beta t^\alpha}{2\alpha ka_0^2}$.

2) If $c_0=\frac{c_4^3\left(1-m^2\right)}{32c_6^2}, c_2=\frac{c_4^2\left(5-m^2\right)}{16c_6}, c_6>0$, then

$$u_3(\xi)=\mp a_0 msn\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)}}\xi, m\right) \tag{35}$$

$$u_4(\xi)=\frac{\mp a_0}{sn\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)}}\xi, m\right)} \tag{36}$$

where sn is elliptic sine, $\xi=kx+hy+\frac{\beta t^\alpha}{\alpha ka_0^2\left(m^2+1\right)}$.

If $m\to 1$, then we get the same solutions with Equation (33)-(34).

If $m \to 0$, then $sn(\xi) \to \sin(\xi)$, we get the periodic function solution:

$$u_4'(\xi) = \mp a_0 \csc\left(\sqrt{\frac{la_0^2}{h(l^2-k^2)}}\xi\right) \tag{37}$$

where $\xi = kx + hy + \dfrac{\beta t^\alpha}{\alpha \kappa a_0^2}$.

3) If $c_0 = \dfrac{c_4^3}{32c_6^2 m^2}$, $c_2 = \dfrac{c_4^2(4m^2+1)}{16c_6 m^2}$, $c_6 < 0$, then

$$u_5(\xi) = \mp a_0 cn\left(\sqrt{\frac{-la_0^2}{m^2 h(l^2-k^2)}}\xi, m\right) \tag{38}$$

$$u_6(\xi) = \frac{\mp a_0\sqrt{1-m^2}\, sn\left(\sqrt{\dfrac{-la_0^2}{m^2 h(l^2-k^2)}}\xi, m\right)}{dn\left(\sqrt{\dfrac{-la_0^2}{m^2 h(l^2-k^2)}}\xi, m\right)} \tag{39}$$

where $\xi = kx + hy - \dfrac{m^2\beta t^\alpha}{\alpha k a_0^2(1-2m^2)}$.

If $m \to 1$, then $cn(\xi) \to \operatorname{sech}(\xi)$, we get the hyperbolic function solution:

$$u_5'(\xi) = \mp a_0 \operatorname{sech}\left(\sqrt{\frac{-la_0^2}{h(l^2-k^2)}}\xi\right) \tag{40}$$

where $\xi = kx + hy + \dfrac{\beta t^\alpha}{\alpha k a_0^2}$.

4) If $c_0 = \dfrac{c_4^3 m^2}{32c_6^2(m^2-1)}$, $c_2 = \dfrac{c_4^2(5m^2-4)}{16c_6(m^2-1)}$, $c_6 < 0$, then

$$u_7(\xi) = \frac{\mp a_0 dn\left(\sqrt{\dfrac{-la_0^2}{h(l^2-k^2)(1-m^2)}}\xi, m\right)}{\sqrt{1-m^2}} \tag{41}$$

$$u_8(\xi) = \frac{\mp a_0}{dn\left(\sqrt{\dfrac{-la_0^2}{h(l^2-k^2)(1-m^2)}}\xi, m\right)} \tag{42}$$

where $\xi = kx + hy - \dfrac{(m^2-1)\beta t^\alpha}{\alpha k a_0^2(2-m^2)}$.

If $m \to 0$, then $dn(\xi) \to 1$, we get the solutions:

$$u_7'(\xi) = u_8'(\xi) = \mp a_0 \tag{43}$$

where $\xi = kx + hy + \dfrac{\beta t^\alpha}{2\alpha k a_0^2}$.

5) If $c_0 = \frac{c_4^3}{32c_6^2\left(1-m^2\right)}, c_2 = \frac{c_4^2\left(4m^2-5\right)}{16c_6\left(m^2-1\right)}, c_6 > 0$, then

$$u_9(\xi) = \frac{\mp a_0}{cn\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)\left(1-m^2\right)}}\xi, m\right)} \tag{44}$$

$$u_{10}(\xi) = \frac{\mp a_0 dn\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)\left(1-m^2\right)}}\xi, m\right)}{\sqrt{1-m^2}\, sn\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)\left(1-m^2\right)}}\xi, m\right)} \tag{45}$$

where $\xi = kx + hy - \frac{\left(m^2-1\right)\beta t^\alpha}{\alpha k a_0^2\left(1-2m^2\right)}$.

If $m \to 0$, then $cn(\xi) \to \cos(\xi)$, $sn(\xi) \to \sin(\xi)$, $dn(\xi) \to 1$, we get the periodic function solutions:

$$u_9'(\xi) = \mp a_0 \sec\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)}}\right) \tag{46}$$

$$u_{10}'(\xi) = \mp a_0 \csc\left(\sqrt{\frac{la_0^2}{h\left(l^2-k^2\right)}}\right) \tag{47}$$

where $\xi = kx + hy + \frac{\beta t^\alpha}{\alpha k a_0^2}$.

6) If $c_0 = \frac{c_4^3 m^2}{32c_6^2}, c_2 = \frac{c_4^2\left(m^2+4\right)}{16c_6}, c_6 < 0$, then

$$u_{11}(\xi) = \mp a_0 dn\left(\sqrt{\frac{-la_0^2}{h\left(l^2-k^2\right)}}\xi, m\right) \tag{48}$$

$$u_{12}(\xi) = \frac{\mp a_0\sqrt{1-m^2}}{dn\left(\sqrt{\frac{-la_0^2}{h\left(l^2-k^2\right)}}\xi, m\right)} \tag{49}$$

where $\xi = kx + hy + \frac{\beta t^\alpha}{\alpha k a_0^2\left(2-m^2\right)}$.

If $m \to 0$, then $dn(\xi) \to 1$, we have the same solutions with Equation (43).

If $m \to 1$, then $dn(\xi) \to \operatorname{sech}(\xi)$, we get the hyperbolic function solutions:

$$u_{11}'(\xi) = \mp a_0 \operatorname{sech}\left(\sqrt{\frac{-la_0^2}{h\left(l^2-k^2\right)}}\xi\right) \tag{50}$$

where $\xi = kx + hy + \frac{\beta t^\alpha}{\alpha k a_0^2}$.

4.3. Dynamical Behaviors

In this section, we analyze the dynamic behaviors of the solutions in (2 + 1)-dimensional time-fractional Zoomeron equation.

Figure 1 and Figure 2 are the 3D and 2D graphs of the solutions (19) and (20), (22) and (23), where the solutions are kink and anti-kink soliton solutions within the interval $-10 \le x, y \le 10$ with the values of parameters $a_0 = t = \varepsilon = h = q = y = 1$, $\beta = 8$, $\alpha = \frac{1}{2}$, $l = 2$, $k = -1$. And we only give graphs of the solutions with the parameter $a_0 = t = \varepsilon = h = q = y = 1$, $\beta = 8$, $\alpha = \frac{1}{2}$, $l = 2$, $k = -1$.

Figure 3 is the 3D and 2D graphs of the solution (31), where the solution is Jacobi elliptic function solution within the interval $-5 \le x, y \le 5$ with the values of parameters $a_0 = t = h = y = 1$, $\alpha = m = \frac{1}{2}$, $l = 2$, $k = -1$, $\beta = 8$. While 2D graph of the solution (31) is in the interval $-10 \le x \le 10$.

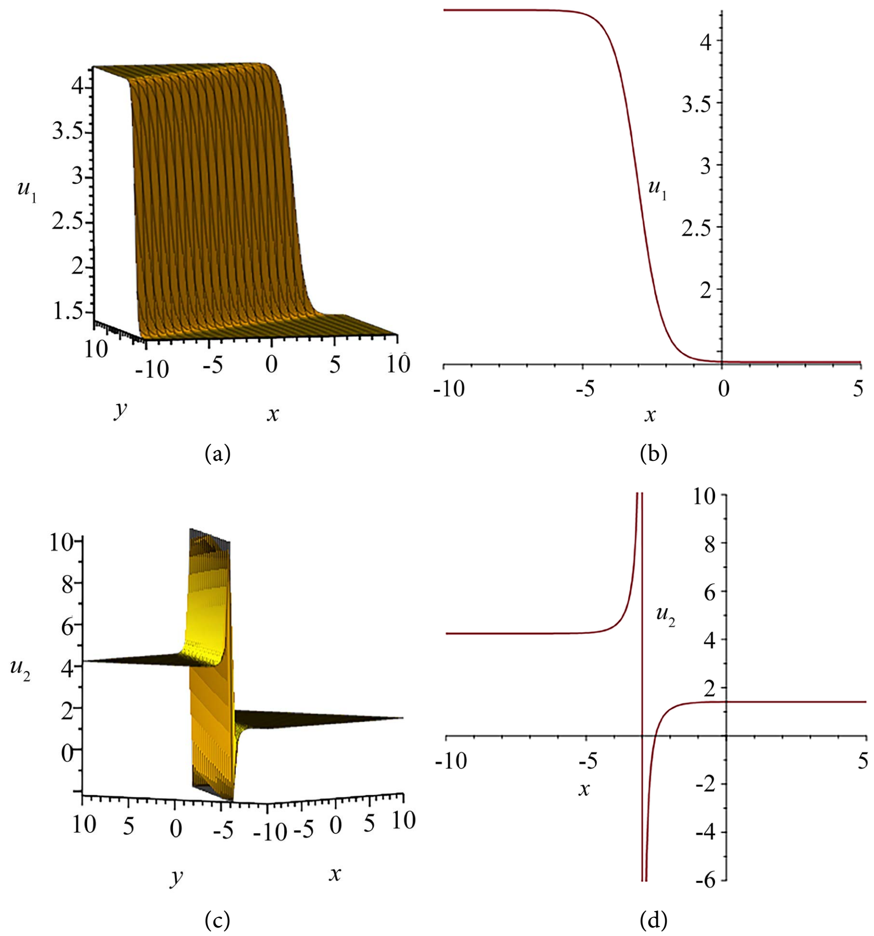

Figure 1. (a), (b) are the 3D and 2D graphs of the solutions (19); (c), (d) are the 3D and 2D graphs of the solutions (20) with the values of parameters $\alpha = \frac{1}{2}$, $l = 2$, $k = -1$, $t = \varepsilon = h = q = y = 1$, $\beta = 8$. (a) 3D graph; (b) 2D graph; (c) 3D graph; (d) 2D graph.

Figure 4 is the 3D and 2D graphs of the solutions (46) and (47), where the solutions are the periodic function solutions within the interval $-5 \le x, y \le 5$ with the values of parameters $a_0 = t = h = y = 1$, $\alpha = \frac{1}{2}$, $l = 2$, $k = -1$, $\beta = 8$. While 2D graphs of the solutions (46) and (47) are in the interval $-10 \le x \le 10$.

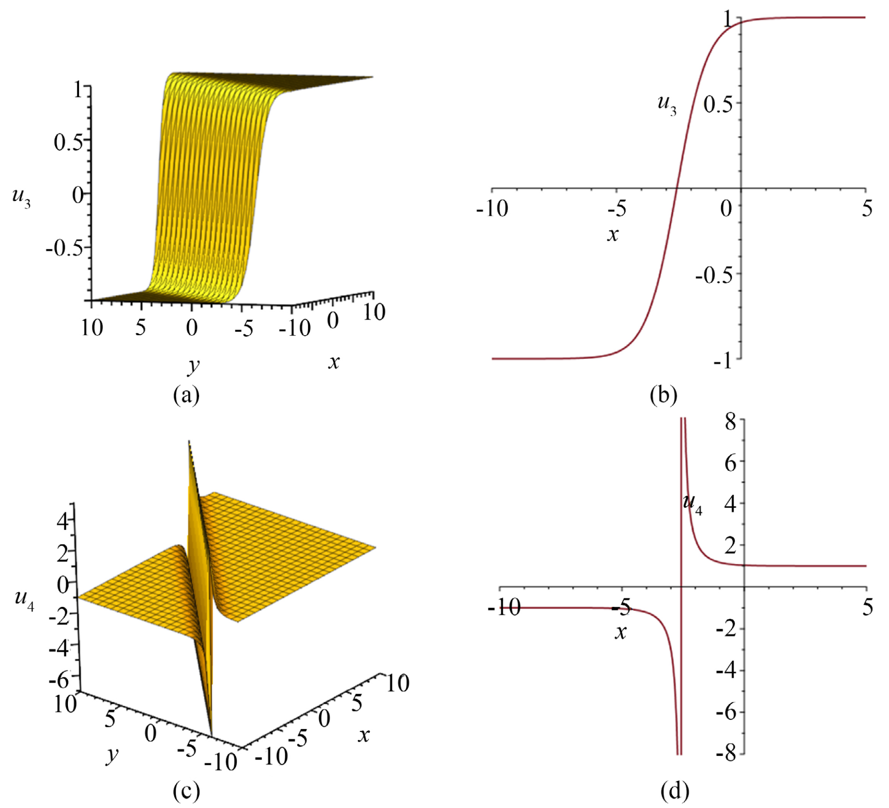

Figure 2. (a), (b) are the 3D and 2D graphs of the solutions (22); (c), (d) are the 3D and 2D graphs of the solutions (23) with the values of parameters $\alpha = \frac{1}{2}$, $l = 2$, $k = -1$, $a_0 = t = \varepsilon = h = q = y = 1$, $\beta = 8$. (a) 3D graph; (b) 2D graph; (c) 3D graph; (d) 2D graph.

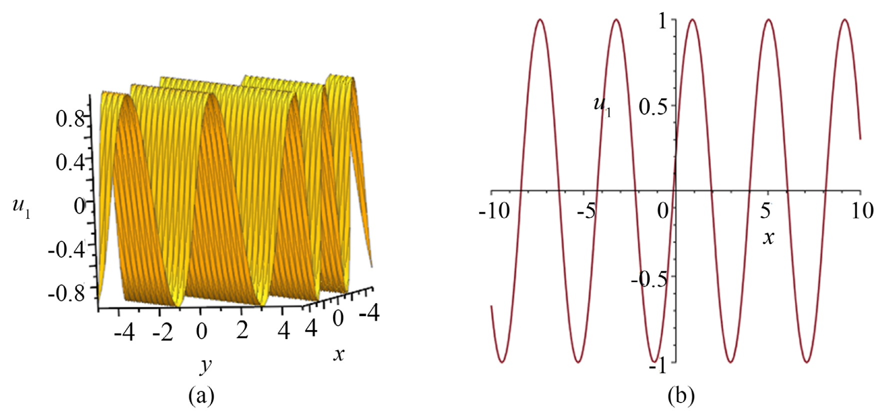

Figure 3. The 3D and 2D graphs of the solution (31) with the values of parameters $\alpha = m = \frac{1}{2}$, $l = 2$, $k = -1$, $a_0 = t = h = y = 1$, $\beta = 8$. (a) 3D graph; (b) 2D graph.

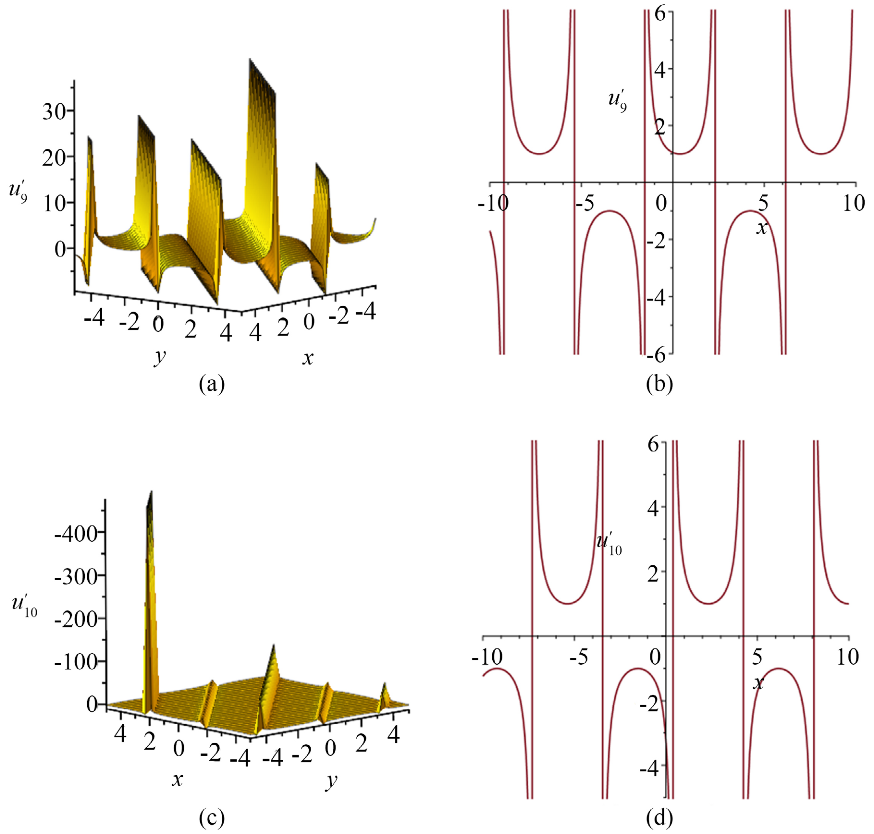

Figure 4. (a), (b) are the 3D and 2D graphs of the solutions (46); (c), (d) are the 3D and 2D graphs of the solutions (47) with the values of parameters $\alpha = \frac{1}{2}$, $l = 2$, $k = -1$, $a_0 = t = h = y = 1$, $\beta = 8$. (a) 3D graph; (b) 2D graph; (c) 3D graph; (d) 2D graph.

5. Conclusion

In conclusion, (2 + 1)-dimensional time-fractional Zoomeron equation has been investigated by the new mapping approach and the new extended auxiliary equation approach. Singular soliton solutions, kink and anti-kink soliton solutions, periodic function soliton solutions, Jacobi elliptic function solutions and hyperbolic function solutions of (2 + 1)-dimensional time-fractional Zoomeron equation have been obtained, where Jacobi elliptic function solutions are new solutions. When $m \to 1$ or $m \to 0$, the Jacobi elliptic function solutions degenerate into the hyperbolic function solutions and the periodic function solutions. Consequently, it is obvious that the application of these two methods is effective to the time-fractional equations.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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