Scientific
Research
Publishing

The Dynamic Behavior of a Class of Kirchhoff Equations with High Order Strong Damping

Guoguang Lin, Chunmeng Zhou*

Department of Mathematics, Yunnan University, Kunming, China
Email: *gglin@ynu.edu.cn, *954137301@qq.com

Abstract

In this paper, we study the long time behavior of a class of Kirchhoff equations with high order strong dissipative terms. On the basis of the proper hypothesis of rigid term and nonlinear term of Kirchhoff equation, firstly, we evaluate the equation via prior estimate in the space E_0 and E_k, and verify the existence and uniqueness of the solution of the equation by using Galerkin's method. Then, we obtain the bounded absorptive set B_{0k} on the basis of the prior estimate. Moreover, by using the Rellich-Kondrachov Compact Embedding theorem, we prove that the solution semigroup $S(t)$ of the equation has the family of the global attractor A_k in space E_k. Finally, we prove that the solution semigroup $S(t)$ is Frechet differentiable on E_k via linearizing the equation. Furthermore, we can obtain the finite Hausdorff dimension and Fractal dimension of the family of the global attractor A_k.

Keywords

Kirchhoff Equation, Prior Estimate, The Existence and Uniqueness of the Solution, Family of Global Attractor, Dimension Estimation

1. Introduction

In this paper, we mainly study the initial boundary value problem of the Kirchhoff equation with high order strong damping:

$$u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + |u|^\rho u = f(x),$$

$$u(x,t) = \frac{\partial^j u}{\partial v^j} = 0, \quad j = 1, 2, \cdots, 2m-1, \quad x \in \partial\Omega, \quad (1.1)$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \partial\Omega, \quad t > 0,$$

where $m > 1, p \geq 2$, $\Omega \subseteq R^n (n \geq 1)$ is a bounded region with a smooth boun-

dary $\partial\Omega$, $Q = \Omega \times [0, \infty)$ stands for the cylinder in $R_x^n \times R_t$, the rigid term $M(s) \in C^1[0, \infty)$ is a general function, $\beta(-\Delta)^{2m} u_t (\beta > 0)$ is the strong dissipative term, $|u|^\rho u$ is the nonlinear term and $\rho \geq -1$, $f(x)$ denotes the external force.

Kirchhoff equation is a kind of important nonlinear wave equation, which is widely used in engineering physics, especially provides a strong support for measuring bridge vibration. Kirchhoff equation originates from a physical model, which is obtained by German physicist Gustav Robert Kirchhoff [1] when he studied the transverse vibration of elastic string:

$$\rho h \frac{\partial^2 u}{\partial t^2} - \left[P_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x \right] \frac{\partial^2 u}{\partial x^2} = f, \quad 0 < x < L, t \geq 0,$$

As $m = 1, p = 2, \beta = 1$, Masanori *et al.* studied Kirchhoff equations with nonlinear dissipative terms in the literature [2]:

$$u_{tt} - M\left(\|\nabla u\|^2 \right) \Delta u + \delta |u|^\rho u + r u_t = f,$$

and they had discussed the existence and attenuation of the global solution of the initial boundary value problem by using Galerkin's method.

Then Igor Chueshov studied the Kirchhoff equation with strong dissipative term for $m = 1, p = 2, \beta = \sigma\left(\|\nabla u\|^2 \right)$ in reference [3]:

$$u_{tt} - \sigma\left(\|\nabla u\|^2 \right) \Delta u_t - \varphi\left(\|\nabla u\|^2 \right) \Delta u + g(u) = f(x),$$

They proved that its weak solution exists and is unique. Particularly, the equation has strong exponential attractor when the nonlinear term $g(u)$ is in a non-supercritical state. Further, Guoguang Lin, Yuhang Chen [4] *et al.* extended the equation to a higher order Kirchhoff-type equation based on the study of Igor Chueshov, and added a structural dissipation term $\Delta^{2m} u$:

$$u_{tt} + N\left(\|\nabla^m u\|_q^q \right) (-\Delta)^m u + a(t)(-\Delta)^m u_t + \Delta^{2m} u = I(x),$$

They discussed the relationship between the order *m* and *q*, made reasonable assumptions about the relevant terms, proved the existence and uniqueness of the comprehensions, and made finite-dimensional estimates of the global attractors. When the coefficient of structural dissipation term is not equal to 1, they can't get the relevant conclusion. More research results can be referred to [5]-[16].

On the basis of previous studies, we improve the order of strong dissipation term further in this paper. At the same time, summing up previous experience, we discuss the difficult problem of the relationship between order *m* and *p* in the rigid term and ρ in the nonlinear term, and get some theoretical results about the long time behavior of the equation.

2. The Basic Assumptions

For the convenience of later narration, the space and sign mentioned in the ar-

ticle are defined as follows:

$H = L^2(\Omega)$, $H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega)$, $E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega)$, A^k is the global attractor family from E_0 to E_k, B_{0k} is a bounded absorption set in E_0, where $k = 1, 2, \cdots, 2m$, $C_i > 0 (i = 0, 1, 2, \cdots)$ are constants. The inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ of H are given by $(u, v) = \int_\Omega u(x)v(x)\mathrm{d}x$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$.

$M(s), \rho$ and p satisfy the following conditions:

(A1) For $\forall s \geq 0$, we have $\varepsilon + 1 \leq \mu_0 \leq M(s) \leq \mu_1$, where μ_0, μ_1 are constants, and

$$\mu = \begin{cases} \mu_0, & \dfrac{\mathrm{d}}{\mathrm{d}t}\|\Delta^m u\|^2 \geq 0 \\ \mu_1, & \dfrac{\mathrm{d}}{\mathrm{d}t}\|\Delta^m u\|^2 \leq 0 \end{cases}$$

(A2) $\begin{cases} 1 \leq \rho \leq \dfrac{8m}{n-4m}, & n > 4m; \\ \rho \geq -1, & n \leq 4m \end{cases}$

(A3) $\begin{cases} 2 \leq p \leq \dfrac{2n}{n-2m}, & n > 2m; \\ p \geq 2, & n \leq 2m \end{cases}$

(A4) $0 \leq \varepsilon \leq \min\left\{\sqrt{1 + \dfrac{\beta\lambda_1^{2m}}{4}} - 1, \dfrac{2\mu}{\beta}, \dfrac{2\mu\lambda_1^{2m}}{\beta\lambda_1^{2m}+1}\right\}$; λ_1 denotes the first eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary on Ω.

3. Existence and Uniqueness of Solutions

Lemma 1. Assume (A1)-(A4) are valid. Let $(u_0, u_1) \in E_0$, $f(x) \in L^2(\Omega)$, then the initial boundary value problem (1.1) has a global solution (u, v) that satisfies $u \in L^\infty(0, +\infty; H_0^{2m}(\Omega))$,

$$v \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2(0, T; H_0^{2m}(\Omega)), \text{ and}$$

$$\|(u, v)\|_{E_0}^2 = \|\Delta^m u\|^2 + \|v\|^2 \leq y_1(0)\mathrm{e}^{-\alpha_1 t} + \dfrac{C_1}{\alpha_1}, \tag{3.1}$$

$$\dfrac{\beta}{2}\int_0^T \|\Delta^m v\|^2 \mathrm{d}t \leq C_1 T + y_1(0), \tag{3.2}$$

where $v = u_t + \varepsilon u$, $\alpha_1 = \min\left\{\dfrac{\beta\lambda_1^{2m}}{2} - 2\varepsilon - \varepsilon^2, 2\varepsilon, \dfrac{2\varepsilon\mu - \beta\varepsilon^2}{\mu}, \varepsilon(\rho+2)\right\}$,

$y_1(0) = \|v_0\|^2 + \varepsilon^2\|u_0\|^2 + \mu\|\Delta^m u_0\|^2 + \dfrac{2}{\rho+2}\|u_0\|_{\rho+2}^{\rho+2}$.

So there exists a nonnegative real number R_1 and $t_1 = t_1(\Omega) > 0$ such that

$$\|(u, v)\|_{E_0}^2 = \|\Delta^m u\|^2 + \|v\|^2 \leq R_1^2 \quad (t > t_1). \tag{3.3}$$

Proof. Let $v = u_t + \varepsilon u$, take the inner product of Equation (1.1) with v in H, and we get that

$$\left(u_{tt} + M\left(\left\| D^m u \right\|_p^p \right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + |u|^\rho u, v \right) = \left(f(x), v \right). \qquad (3.4)$$

We process the terms in Equation (3.4) by using Young's inequality, Holder's inequality, Poincare's inequality and differential mean value theorem, then we obtain that

$$\left(u_{tt}, v \right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \frac{\varepsilon^2}{2}\|u\|^2 - \varepsilon\|v\|^2 + \varepsilon^3\|u\|^2. \qquad (3.5)$$

$$\left(M\left(\left\| D^m u \right\|_p^p \right)(-\Delta)^{2m} u, v \right) \ge \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \Delta^m u \right\|^2 + \varepsilon\mu\left\| \Delta^m u \right\|^2, \qquad (3.6)$$

$$\left(\beta(-\Delta)^{2m} u_t, v \right) \ge \frac{\beta}{4}\left\| \Delta^m v \right\|^2 + \frac{\beta\lambda_1^{2m}}{4}\|v\|^2 - \frac{\beta\varepsilon^2}{2}\left\| \Delta^m u \right\|^2, \qquad (3.7)$$

$$\left(|u|^\rho u, v \right) = \frac{1}{\rho+2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\rho+2}^{\rho+2} + \varepsilon\|u\|_{\rho+2}^{\rho+2}, \qquad (3.8)$$

$$\left(f(x), v \right) \le \frac{1}{2\varepsilon^2}\|f\|^2 + \frac{\varepsilon^2}{2}\|v\|^2. \qquad (3.9)$$

Substitute (3.5)-(3.9) into (3.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 + \varepsilon^2\|u\|^2 + \mu\left\| \Delta^m u \right\|^2 + \frac{2}{\rho+2}\|u\|_{\rho+2}^{\rho+2} \right) + \left(\frac{\beta\lambda_1^{2m}}{2} - 2\varepsilon - \varepsilon^2 \right)\|v\|^2$$

$$+ 2\varepsilon^3\|u\|^2 + \left(2\varepsilon\mu - \beta\varepsilon^2 \right)\left\| \Delta^m u \right\|^2 + 2\varepsilon\|u\|_{\rho+2}^{\rho+2} + \frac{\beta}{2}\left\| \Delta^m v \right\|^2 \le \frac{1}{\varepsilon^2}\|f\|^2 = C_1.$$

Let $\alpha_1 = \min\left\{ \frac{\beta\lambda_1^{2m}}{2} - 2\varepsilon - \varepsilon^2, 2\varepsilon, \frac{2\varepsilon\mu - \beta\varepsilon^2}{\mu}, \varepsilon(\rho+2) \right\}$, then

$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} + \alpha_1 y_1(t) + \frac{\beta}{2}\left\| \Delta^m v \right\|^2 \le C_1,$$

where $y_1(t) = \|v\|^2 + \varepsilon^2\|u\|^2 + \mu\left\| \Delta^m u \right\|^2 + \frac{2}{\rho+2}\|u\|_{\rho+2}^{\rho+2}$.

By using the differential Gronwall's inequality, we obtain

$$y_1(t) \le y_1(0)\mathrm{e}^{-\alpha_1 t} + \frac{C_1}{\alpha_1},$$

$$\frac{\beta}{2}\int_0^T \left\| \Delta^m v \right\|^2 \mathrm{d}t \le C_1 T + y_1(0),$$

thus $\left\| (u,v) \right\|_{E_0}^2 = \left\| \Delta^m u \right\|^2 + \|v\|^2 \le y_1(t)$, moreover

$$\varlimsup_{t\to+\infty} \left\| (u,v) \right\|_{E_0}^2 \le \frac{C_1}{\alpha_1} = R_1^2.$$

Lemma 1 is proved. ∎

Lemma 2. Assume (A1)-(A4) are valid. If $(u_0, u_1) \in E_k, k = 1, 2, \cdots, 2m$, $f(x) \in L^2(\Omega)$, then the initial boundary value problem (1) has a global solution (u,v) that satisfies $u \in L^\infty\left(0, +\infty; H_0^{2m+k}(\Omega) \right)$, $v \in L^\infty\left(0, +\infty; H_0^k(\Omega) \right) \cap L^2\left(0, T; H_0^{2m+k}(\Omega) \right)$, and

$$\left\| (u,v) \right\|_{E_k}^2 = \left\| \nabla^{2m+k} u \right\|^2 + \left\| \nabla^k v \right\|^2 \le y_2(0) e^{-\alpha_2 t} + \frac{C_3}{\alpha_2},$$

$$\frac{\beta}{4} \int_0^T \left\| \nabla^{2m+k} v \right\|^2 dt \le C_3 T + y_2(0), \qquad (3.10)$$

where $v = u_t + \varepsilon u$, $\alpha_2 = \min \left\{ \frac{\beta \lambda_1^{2m}}{4} - 2\varepsilon - \varepsilon^2, 2\varepsilon - \frac{\varepsilon^2}{\lambda_1^{2m} \mu} - \frac{\beta \varepsilon^2}{\mu} \right\}$,

$y_2(0) = \left\| \nabla^k v_0 \right\|^2 + \mu \left\| \nabla^{2m+k} u_0 \right\|^2$.

Thus there exists a nonnegative real number R_k and $t_2 = t_2(\Omega) > 0$ such that

$$\left\| (u,v) \right\|_{E_k}^2 = \left\| \nabla^{2m+k} u \right\|^2 + \left\| \nabla^k v \right\|^2 \le R_k^2, \quad (t > t_2). \qquad (3.11)$$

Proof. Taking the inner product of Equation (1.1) with $(-\Delta)^k v$ in H, and we get that

$$\left(u_{tt} + M \left(\left\| D^m u \right\|_p^p \right) (-\Delta)^{2m} u + \beta (-\Delta)^{2m} u_t + |u|^\rho u, (-\Delta)^k v \right) = \left(f(x), (-\Delta)^k v \right). \quad (3.12)$$

Similar to Lemma 1, each item of Equation (3.12) can be obtained

$$\left(u_{tt}, (-\Delta)^k v \right) \ge \frac{1}{2} \frac{d}{dt} \left\| \nabla^k v \right\|^2 - \frac{2\varepsilon + \varepsilon^2}{2} \left\| \nabla^k v \right\|^2 - \frac{\varepsilon^2}{2\lambda_1^{2m}} \left\| \nabla^{2m+k} u \right\|^2, \qquad (3.13)$$

$$\left(M \left(\left\| D^m u \right\|_p^p \right) (-\Delta)^{2m} u, (-\Delta)^k v \right) \ge \frac{\mu}{2} \frac{d}{dt} \left\| \nabla^{2m+k} u \right\|^2 + \varepsilon \mu \left\| \nabla^{2m+k} u \right\|^2, \qquad (3.14)$$

$$\left(\beta (-\Delta)^{2m} u_t, (-\Delta)^k v \right)$$
$$\ge \frac{\beta}{8} \left\| \nabla^{2m+k} v \right\|^2 - \frac{\beta \varepsilon^2}{2} \left\| \nabla^{2m+k} u \right\|^2 + \frac{\beta \lambda_1^{2m-k}}{4} \left\| (-\Delta)^k v \right\|^2 + \frac{\beta \lambda_1^{2m}}{8} \left\| \nabla^k v \right\|^2, \qquad (3.15)$$

$$\left| \left(|u|^\rho u, (-\Delta)^k v \right) \right| \ge - \left\| u \right\|_\infty^{\rho+1} \left\| (-\Delta)^k v \right\| \ge - \frac{2C_2}{\beta \lambda_1^{2m-k}} - \frac{\beta \lambda_1^{2m-k}}{8} \left\| (-\Delta)^k v \right\|^2, \quad (3.16)$$

$$\left(f(x), (-\Delta)^k v \right) \le \frac{2}{\beta \lambda_1^{2m-k}} \left\| f(x) \right\|^2 + \frac{\beta \lambda_1^{2m-k}}{8} \left\| (-\Delta)^k v \right\|^2. \qquad (3.17)$$

Substitute inequality (3.13)-(3.17) into (3.12), we have

$$\frac{d}{dt} \left(\left\| \nabla^k v \right\|^2 + \mu \left\| \nabla^{2m+k} u \right\|^2 \right) + \left(\frac{\beta \lambda_1^{2m}}{4} - 2\varepsilon - \varepsilon^2 \right) \left\| \nabla^k v \right\|^2$$
$$+ \left(2\varepsilon\mu - \frac{\varepsilon^2}{\lambda_1^{2m}} - \beta \varepsilon^2 \right) \left\| \nabla^{2m+k} u \right\|^2 + \frac{\beta}{4} \left\| \nabla^{2m+k} v \right\|^2 \qquad (18)$$
$$\le \frac{4}{\beta \lambda_1^{2m-k}} \left\| f(x) \right\|^2 + \frac{4C_2}{\beta \lambda_1^{2m-k}} = C_3.$$

Let $\alpha_2 = \min \left\{ \frac{\beta \lambda_1^{2m}}{4} - 2\varepsilon - \varepsilon^2, 2\varepsilon - \frac{\varepsilon^2}{\lambda_1^{2m} \mu} - \frac{\beta \varepsilon^2}{\mu} \right\}$, then

$$\frac{dy_2(x)}{dt} + \alpha_2 y_2(x) + \frac{\beta}{4} \left\| \nabla^{2m+k} v \right\|^2 \le C_3,$$

where $y_2(x) = \left\| \nabla^k v \right\|^2 + \mu \left\| \nabla^{2m+k} u \right\|^2$.

By using the differential Gronwall's inequality, we obtain

$$y_2(x) \leq y_2(0)e^{-\alpha_2 t} + \frac{C_3}{\alpha_2},$$

$$\frac{\beta}{4}\int_0^T \left\|\nabla^{2m+k}v\right\|^2 dt \leq y_2(0) + C_3 T.$$

So $\left\|(u,v)\right\|_{E_k}^2 = \left\|\nabla^{2m+k}u\right\|^2 + \left\|\nabla^k v\right\|^2 \leq y_2(t)$, moreover

$$\overline{\lim_{t \to +\infty}}\left\|(u,v)\right\|_{E_k}^2 \leq \frac{C_3}{\alpha_2} = R_k^2.$$

Lemma 2 is proved. ∎

Lemma 3. (Daprato G. [9]) Let D be the bounded region of $R_x^n \times R_t$, g_μ and g are functions of $L^q(D)(1 < q < \infty)$, $\left\|g_\mu\right\|_{L^q(D)} < C, g_\mu \to g$ a.e. in D, then

$g_\mu \to g$ is weak convergence in $L^q(D)$.

Theorem 1. Assume (A1)-(A4) are valid. If $f \in H, (u_0, u_1) \in E_k$, then the initial boundary value problem (1.1) exists a unique smooth solution $(u,v) \in L^\infty([0,+\infty); E_k)$.

Proof. First of all, we prove the existence of solution.

Let $(-\Delta)^{2m+k}\omega_j = \lambda_j^{2m+k}\omega_j (k = 1,2,\cdots,2m)$, where λ_j represents the eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary on Ω, ω_j represents the eigenfunction determined by the corresponding eigenvalue λ_j, then $\omega_1, \omega_2, \cdots, \omega_h$ constitute the standard orthonormal basis of H. Suppose $u_h = u_h(t) = \sum_{j=1}^h g_{jh}(t)\omega_j$ is the approximate solution of the initial boundary value problem (1.1), where $g_{jh}(t)(1 \leq j \leq h)$ is determined by

$$\left(u_h'' + M\left(\left\|D^m u_h\right\|_p^p\right)(-\Delta)^{2m}u_h + \beta(-\Delta)^{2m}u_h' + |u_h|^\rho u_h, (-\Delta)^k \omega_j\right)$$
$$= \left(f(x), (-\Delta)^k \omega_j\right) \quad (j = 1,2,\cdots,h) \tag{3.19}$$

Equation (3.19) satisfies the initial conditions

$$u_h(0) = u_{0h} = \sum_{j=1}^h \alpha_{jh}(t)\omega_j \to u_0 (h \to +\infty) \text{ in } H_0^{2m+k},$$

$$u_h'(0) = u_{1h} = \sum_{j=1}^h \beta_{jh}(t)\omega_j \to u_1 (h \to +\infty) \text{ in } H_0^k,$$

namely as $h \to +\infty$, $(u_{0h}, u_{1h}) \to (u_0, u_1)$ in E_k. According to the basic theory of solutions of ordinary differential equations, the approximate solution $u_h(t)$ exists on $(0, t_h)$.

Multiply both sides of (3.19) by $g_{jh}' + \varepsilon g_{jh}(t)$ and sum over j. Let $v_h(t) = u_h'(t) + \varepsilon u_h(t)$, according to Lemma 1 and Lemma 2, estimation (3.3), (3.10), (3.11) still hold for $u_h \in H_0^{2m+k}(\Omega), v_h \in H_0^k(\Omega)$, namely we have

$$\left\|(u_h, v_h)\right\|_{E_k}^2 = \left\|\nabla^{2m+k}u_h\right\|^2 + \left\|\nabla^k v_h\right\|^2 \leq R_k^2, \tag{3.20}$$

$$\frac{\beta}{4}\int_0^T \left\|\nabla^{2m+k}v_h\right\|^2 dt \leq y_2(0) + \int_0^T C_3 dt, \tag{3.21}$$

then we obtain $\left(u_h(t), u_h'(t) \right)$ is uniformly bounded on E_k, and (3.20), (3.21) hold the priori estimate of the solution in the Lemma 2. Moreover, we know from (3.20) that (u_h, v_h) is bounded in $L^\infty\left([0, +\infty); E_k \right)$, from (3.21) that v_h is bounded in $L^2\left((0,T); H_0^{2m+k} \right)$.

In space E_k, select the subsequence $\{u_s\}$ from the sequence $\{u_h\}$ such that $(u_s, v_s) \to (u, v)$ is weak * convergence in $L^\infty\left([0, +\infty); E_k \right)$, and we obtain that v_s is bounded in $L^2\left((0,T); H_0^{2m+k} \right)$ via (3.21).

Due to the Rellich-Kohdrachov Compact Embedding theorem, $E_k \hookrightarrow E_0$, $(u_s, v_s) \to (u, v)$ is strong convergence a.e. in E_0.

According to Lemma 3, taking $g_s = |u_s|^\rho u_s$, $q = \dfrac{\rho+2}{\rho+1}$, $g_s \to |u|^\rho u$ a.e. in $L^q(Q)$, where $Q = [0, +\infty) \times \Omega$, then

$$|u_s|^\rho u_s \to |u|^\rho u \text{ is weak }^* \text{ convergence in } L^\infty\left([0, +\infty); L^q(\Omega) \right). \quad (3.22)$$

Then in Equation (3.19), let $h = s$ and take the limit. For fixed j and $s \geq j$, we obtain

$$\left(u_s'' + M\left(\left\| D^m u_s \right\|_p^p \right)(-\Delta)^{2m} u_s + \beta(-\Delta)^{2m} u_s' + |u_s|^\rho u_s, (-\Delta)^k \omega_j \right)$$
$$= \left(f(x), (-\Delta)^k \omega_j \right).$$

Due to $u_h \to u$ is weak * convergence in $L^\infty\left([0, +\infty); H_0^{2m+k}(\Omega) \right)$, we have

$$\left(u_s(t), (-\Delta)^k \omega_j \right) \to \left(u(t), \lambda_j^k \omega_j \right) \text{ is weak }^* \text{ convergence in } L^\infty[0, +\infty),$$

$$\left(u_s'(t), (-\Delta)^k \omega_j \right) \to \left(u'(t), \lambda_j^k \omega_j \right) \text{ is weak }^* \text{ convergence in } L^\infty[0, +\infty).$$

Then, we obtain $\left(u_s''(t), (-\Delta)^k \omega_j \right) = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(u_s'(t), (-\Delta)^k \omega_j \right) \to \left(u''(t), \lambda_j^k \omega_j \right)$ is convergence in $D'[0, +\infty)$, where $D'[0, +\infty)$ is the conjugate space of the infinitely differentiable space $D[0, +\infty)$.

$$\left(M\left(\left\| D^m u_s \right\|_p^p \right)(-\Delta)^{2m} u_s, (-\Delta)^k \omega_j \right) \to \left(M\left(\left\| D^m u \right\|_p^p \right)(-\Delta)^{\frac{2m+k}{2}} u, \lambda_j^{\frac{2m+k}{2}} (-\Delta)^k \omega_j \right)$$

is weak * convergence in $L^\infty[0, +\infty)$.

Due to

$$\left(\beta(-\Delta)^{2m} u_s', (-\Delta)^k \omega_j \right)$$
$$= \left(\beta(-\Delta)^{\frac{k}{2}} v_s, (-\Delta)^{\frac{2m+k}{2}} \omega_j \right) - \beta\varepsilon\left((-\Delta)^{\frac{2m+k}{2}} u_s, (-\Delta)^{\frac{2m+k}{2}} \omega_j \right),$$

$$\left(\beta(-\Delta)^{2m} u_s', (-\Delta)^k \omega_j \right)$$
$$\to \left(\beta(-\Delta)^{\frac{k}{2}} v, \lambda_j (-\Delta)^{\frac{2m+k}{2}} \omega_j \right) - \beta\varepsilon\left((-\Delta)^{\frac{2m+k}{2}} u, \lambda_j^{\frac{2m+k}{2}} \omega_j \right) \text{ is}$$

weak * convergence in $L^\infty[0, +\infty)$.

According to Equation (3.22), we have

$$\left(\left| u_s \right|^\rho u_s, (-\Delta)^k \omega_j \right) \to \left(\left| u \right|^\rho u, \lambda_j^k \omega_j \right) \text{ is weak}^* \text{ convergence in } L^\infty \left[0, +\infty \right).$$

Because ω_j is arbitrary, for $\forall v \in H^k(\Omega) \cap H_0^1(\Omega)$ we have

$$\left(u_{tt} + M \left(\left\| D^m u \right\|_p^p \right) (-\Delta)^{2m} u + \beta (-\Delta)^{2m} u_t + \left| u \right|^\rho u, (-\Delta)^k \omega_j \right)$$
$$= \left(f(x), (-\Delta)^k \omega_j \right). \tag{3.23}$$

Hence, Equation (3.23) is established for all *j*. The existence is proved.

Then we prove the solution of the initial and boundary value problem is unique.

Assume u_1, u_2 are the solution of the initial and boundary value problem (1.1). Let $w = u_1 - u_2$, we have

$$w_{tt} + M \left(\left\| D^m u_1 \right\|_p^p \right) (-\Delta)^{2m} u_1 - M \left(\left\| D^m u_2 \right\|_p^p \right) (-\Delta)^{2m} u_2$$
$$+ \beta (-\Delta)^{2m} w_t + \left| u_1 \right|^\rho u_1 - \left| u_2 \right|^\rho u_2 = 0,$$
$$w(0) = 0, w_t(0) = 0, \tag{3.24}$$
$$w \in L^\infty \left([0, +\infty); H_0^{2m}(\Omega) \right), w_t \in L^\infty \left([0, +\infty); L^2(\Omega) \right).$$

Taking the inner product of Equation (3.24) with w_t in *H*, and we get that

$$\frac{1}{2} \frac{d}{dt} \left\| w_t \right\|^2 + \beta \left\| \Delta^m w_t \right\|^2 + \left(M \left(\left\| \nabla^m u_1 \right\|_p^p \right) (-\Delta)^{2m} u_1 - M \left(\left\| \nabla^m u_2 \right\|_p^p \right) (-\Delta)^{2m} u_2, w_t \right)$$
$$+ \left(\left| u_1 \right|^\rho u_1 - \left| u_2 \right|^\rho u_2, w_t \right) = 0. \tag{3.25}$$

According to Lemma 1 and Lemma 2, the Equation (3.25) is processed as follows

$$\left(M \left(\left\| \nabla^m u_1 \right\|_p^p \right) (-\Delta)^{2m} u_1 - M \left(\left\| \nabla^m u_2 \right\|_p^p \right) (-\Delta)^{2m} u_2, w_t \right)$$
$$\geq \frac{1}{2} M \left(\left\| \nabla^m u_1 \right\|_p^p \right) \frac{d}{dt} \left\| \Delta^m w \right\|^2 - M'(\xi) \left(\left\| \nabla^m u_1 \right\|_p^p - \left\| \nabla^m u_2 \right\|_p^p \right) \left((-\Delta)^{2m} u_2, w_t \right) \tag{3.26}$$
$$\geq \frac{\mu}{2} \frac{d}{dt} \left\| \Delta^m w \right\|^2 - \frac{\beta}{2} \left\| \Delta^m w_t \right\|^2 - \frac{C_4^2}{2\beta \lambda_1^m} \left\| \Delta^m w \right\|^2,$$

$$\left| \left(\left| u_1 \right|^\rho u_1 - \left| u_2 \right|^\rho u_2, w_t \right) \right| \geq -C_5 \left| \int_\Omega \left(\left| u_1 \right|^\rho + \left| u_2 \right|^\rho \right) \left| w \right| \left| w_t \right| dx \right|$$
$$\geq -\frac{C_6}{2\lambda_1^{2m}} \left\| \Delta^m w \right\|^2 - \frac{C_6}{2} \left\| w_t \right\|^2. \tag{3.27}$$

Substituting (3.26)-(4.1) into (3.25), we obtain

$$\frac{d}{dt} \left(\left\| w_t \right\|^2 + \mu \left\| \Delta^m w \right\|^2 \right) \leq \left| \beta \lambda_1^{2m} - C_6 \right| \left\| w_t \right\|^2 + \left(\frac{C_4^2}{\beta \lambda_1^m} + \frac{C_6}{\lambda_1^{2m}} \right) \left\| \Delta^m w \right\|^2.$$

Let $\alpha_3 = \max \left\{ \left| \beta \lambda_1^{2m} - C_6 \right|, \frac{C_4^2}{\beta \mu \lambda_1^m} + \frac{C_6}{\mu \lambda_1^{2m}} \right\}$, by using the integral Gronwall's inequality we receive that

$$\left\| w_t(t) \right\|^2 + \mu \left\| \Delta^m w(t) \right\|^2 \leq \left(\left\| w_t(0) \right\|^2 + \mu \left\| \Delta^m w(0) \right\|^2 \right) e^{\alpha_3 t} = 0.$$

Hence $w_t(t) = 0, \Delta^m w(t) = 0$, a.e. $w(t) = 0, u_1 = u_2$. The uniqueness of solution is proved. ■

4. Global Attractor and Dimension Estimation

Theorem 2. [9] Assume E is a Banach space, and $\{S(t)\}_{t\geq 0}$ are the semigroup operators on E. $S(t): E \to E$, $S(t+s) = S(t)S(s)(\forall t,s \geq 0)$, $S(0) = I$, where I is the unit operator. Suppose $S(t)$ satisfies the following conditions:

1) $S(t)$ is uniformly bounded, *i.e.* $\forall R > 0, \|u\|_E \leq R$, there is a constant $C(R)$, such that

$$\|S(t)u\|_E \leq C(R), \quad (\forall t \in [0,+\infty));$$

2) There is a bounded absorbing set $B_0 \subset E$, *i.e.* $\forall B \subset E$, there exists a constant t_0, so that

$$S(t)B \subset B_0, \quad (\forall t \geq t_0);$$

3) As $t > 0$, $S(t)$ is a completely continuous operator.

It is said that the semigroup operator $S(t)$ exists a compact global attractor A.

If $S(t)$ is the solution semigroup that generated by the initial boundary value problem (1.1), *i.e.* $u(t) = S(t)u_0$, according to Lemma 1 and Lemma 2 we obtain the existence theorem of the following family of global attractors.

Theorem 3. Under the assumption of Theorem 1, let $S(t)$ be the solution semigroup that generated by the initial boundary value problem (1.1), then the initial boundary value problem (1.1) exists a family of global attractors, *i.e.* exists compact set $A_k \subset E_k \subset E_0$, and

$$A_k = \omega(B_{0k}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B_{0k}},$$

where $B_{0k} = \left\{(u,v) \in E_k : \|(u,v)\|_{E_k}^2 = \|u\|_{H_0^{2m+k}(\Omega)}^2 + \|v\|_{H_0^k(\Omega)}^2 \leq R_k^2\right\}$, B_{0k} is the bounded absorbing set in E_k and satisfies:

1) $S(t)A_k = A_k, (t > 0)$;
2) $\lim_{t \to +\infty} dist(S(t)B_{0k}, A_k) = 0$, where for arbitrary bounded set $B_{0k} \subset E_k$, we have

$$dist(S(t)B_{0k}, A_k) = \sup_{x \in B_{0k}} \inf_{y \in A_k} \|S(t)x - y\|_{E_k}.$$

Proof. The proof can be obtained by verifying the three conditions in Theorem 2. Under the conditions in Theorem 1, suppose that the Equation (1.1) has a solution semigroup $S(t): E_k \to E_k$.

(1)According to Lemma 2, for arbitrary $B_{0k} \subset E_k$, it contains the bounded set of the sphere $\{\|(u,v)\|_{E_k} \leq R\}$ such that there exists a constant C, then we have

$$\|S(t)(u_0,v_0)\|_{E_k}^2 = \|u\|_{H_0^{2m+k}(\Omega)}^2 + \|v\|_{H_0^k(\Omega)}^2 \leq R^2 + C,$$

where $t \geq 0$, $(u_0,v_0) \in B_{0k}$, $\{S(t)\}_{t\geq 0}$ is uniformly bounded in E_k.

2) Further, $\forall (u_0,v_0) \in E_k, t \geq \max(t_1,t_2)$, we have

$$\left\| S(t)(u_0,v_0) \right\|_{E_k}^2 = \left\| u \right\|_{H_0^{2m+k}(\Omega)}^2 + \left\| v_0 \right\|_{H_0^k(\Omega)}^2 \le R_k^2 + R_0^2,$$

so B_{0k} is a bounded absorption set of the semigroup $S(t)$.

3) Due to $E_k \hookrightarrow E_0$, we obtain that the bounded set in E_k is the compact set in E_0. Hence, $S(t)$ is completely continuous operator.

Theorem 3 is proved. ∎

In order to estimate the Hausdorff dimension and Fractal dimension of the family of global attractor, we linearize the problem (1.1):

$$\begin{aligned}
&U_{tt} + M\left(\left\| \nabla^m u \right\|_p^p \right)(-\Delta)^{2m} U + M'\left(\left\| \nabla^m u \right\|_p^p \right)\left(\left\| \nabla^m u \right\|_p^p \right)' \nabla^m U (-\Delta)^{2m} u \\
&+ \beta(-\Delta)^{2m} U_t + (\rho+1)|u|^\rho U = 0, \\
&U(x,t) = 0, x \in \partial\Omega, \quad t > 0, \\
&U(x,0) = \xi, U_t(x,0) = \eta,
\end{aligned} \tag{4.1}$$

where $(\xi,\eta) \in E_k$, $(u,u_t) \in S(t)(u_0,u_1)$ is the solution of question (1.1) obtained by $(u_0,u_1) \in A_k$. Then for a given (u_0,u_1), $S(t): E_k \to E_k$, it can be proved that for arbitrary $(\xi,\eta) \in E_k$, there is a unique solution $(U(t),U_t(t)) \in L^\infty([0,+\infty); E_k)$ to the linear problem (4.1).

Lemma 4. For $\forall t > 0, R > 0$, the mapping $S(t): E_k \to E_k$ is differentiable on E_k. Then the derivative of $\varphi = (u_0,u_1)$ is a linear operator of $G(t): (\xi,\eta) \to (U(t),U_t(t))$, where $(U(t),U_t(t))$ is the solution of the linear initial boundary value problem (4.1).

Proof. Let $\varphi_0 = (u_0,v_0) \in E_k$, $\widetilde{\varphi_0} = (u_0+\xi, v_0+\eta) \in E_k$, and $\left\| \varphi_0 \right\|_{E_k} \le R$, $\left\| \widetilde{\varphi_0} \right\|_{E_k} \le R$, define $(u,v) = S(t)\varphi_0$, $(\tilde{u},\tilde{v}) = S(t)\widetilde{\varphi_0}$, then $S(t)$ has Lipschitz property on the bounded set E_k:

$$\left\| S(t)\varphi_0 - S(t)\widetilde{\varphi_0} \right\|_{E_k}^2 \le e^{ct} \left\| (\xi,\eta) \right\|_{E_k}^2. \tag{4.2}$$

Let $\Theta = \tilde{u} - u - U$ is the solution of the linear initial boundary value problem (4.1), then we have

$$\begin{aligned}
&\Theta_{tt} + M\left(\left\| \nabla^m u \right\|_p^p \right)(-\Delta)^{2m} \Theta + \beta(-\Delta)^{2m} \Theta_t = h_1 + h_2, \\
&\Theta(0) = \Theta_t(0) = 0.
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
h_1 &= \left(M\left(\left\| \nabla^m u \right\|_p^p \right) - M\left(\left\| \nabla^m \tilde{u} \right\|_p^p \right) \right)(-\Delta)^{2m} \tilde{u} \\
&\quad + M'\left(\left\| \nabla^m u \right\|_p^p \right)\left(\left\| \nabla^m u \right\|_p^p \right)' \nabla^m U (-\Delta)^{2m} u,
\end{aligned}$$

$$h_2 = |u|^\rho u - |\tilde{u}|^\rho \tilde{u} + (\rho+1)|u|^\rho U,$$

Let $s = \nabla^m u$, $\tilde{s} = \nabla^m \tilde{u}$, $\bar{u} = \tilde{u} - u$, $\xi_1 = (1-\alpha_4)\tilde{s} + \alpha_4 s$,

$\xi_2 = (1-\alpha_5)\tilde{s} + \alpha_5 \xi_1$, $N(\xi) = M'\left(\left\| s \right\|_p^p \right)\left(\left\| s \right\|_p^p \right)'$, $\alpha_4, \alpha_5 \in (0,1)$.

Then

$$h_1 = -N(\xi_1)\nabla^m \overline{u}(-\Delta)^{2m}\tilde{u} + N(s)\nabla^m \overline{u}(-\Delta)^{2m}u - N(s)\nabla^m \Theta(-\Delta)^{2m}u$$

$$= -N(\xi_1)\nabla^m \overline{u}(-\Delta)^{2m}\overline{u} - N'(\xi_2)(1-\alpha_4)(\nabla^m \overline{u})^2(-\Delta)^{2m}u$$

$$- N(s)\nabla^m \Theta(-\Delta)^{2m}u,$$

$$h_2 = -(\rho+1)|\xi_3|^\rho \overline{u} + (\rho+1)|u|^\rho \overline{u} - (\rho+1)|u|^\rho \Theta$$

$$= \rho(\rho+1)|\xi_4|^{\rho-1}\overline{u}^2 - (\rho+1)|u|^\rho \Theta.$$

Taking the inner product of Equation (4.3) with $s = \nabla^m u$, we receive that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^k \Theta_t\right\|^2 + \frac{1}{2}M\left(\left\|\nabla^k u\right\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}\Theta\right\|^2 + \beta\left\|\nabla^{2m+k}\Theta_t\right\|^2 \tag{4.4}$$

$$= \left(h_1 + h_2, (-\Delta)^k \Theta_t\right),$$

$$\left|\left(h_1, (-\Delta)^k \Theta_t\right)\right| \leq \left\|N(\xi_1)\right\|_\infty \left\|\nabla^{2m+k}\overline{u}\right\|\left\|\Delta^m \overline{u}\right\|\left\|\nabla^{m+k}\Theta_t\right\|$$

$$+ C_7\left\|N'(\xi_2)\right\|_\infty \left\|\nabla^{2m+k}\overline{u}\right\|^2\left\|(-\Delta)^2 u\right\|\left\|\Theta_t\right\|$$

$$+ \left\|N(s)\right\|_\infty \left\|(-\Delta)^m u\right\|\left\|\nabla^{m+k}\Theta\right\|\left\|\nabla^k \Theta_t\right\| \tag{4.5}$$

$$\leq \left(\frac{C_8}{2\lambda_1^k} + \frac{C_{10}}{2}\right)\left\|\nabla^k \Theta_t\right\|^2 + \frac{C_{10}}{2\lambda_1^m}\left\|\nabla^{2m+k}\Theta\right\|^2$$

$$+ \frac{\beta}{2}\left\|\nabla^{2m+k}\Theta_t\right\|^2 + \left(\frac{C_8}{2} + \frac{C_9^2}{2\beta\lambda_1^{m+k}}\right)\left\|\nabla^{2m+k}\overline{u}\right\|^4,$$

$$\left|\left(h_2, (-\Delta)^k \Theta_t\right)\right|$$

$$\leq \rho(\rho+1)\left\|\xi_4\right\|_\infty^{\rho-1}\left\|\nabla^k \overline{u}\right\|^2\left\|\Theta_t\right\| + (\rho+1)\left\|u\right\|_\infty^\rho\left\|\nabla^k \Theta\right\|\left\|\nabla^k \Theta_t\right\| \tag{4.6}$$

$$= \frac{C_{11}}{2\lambda_1^{4m}}\left\|\nabla^{2m+k}\overline{u}\right\|^4 + \frac{C_{12}}{2\lambda_1^{2m}}\left\|\nabla^{2m+k}\Theta\right\|^2 + \left(\frac{C_{11}}{2\lambda_1^k} + \frac{C_{12}}{2}\right)\left\|\nabla^k \Theta_t\right\|^2.$$

According to (4.4)-(4.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^k \Theta_t\right\|^2 + \mu\left\|\nabla^{2m+k}\Theta\right\|^2\right) + \beta\left\|\nabla^{2m+k}\Theta_t\right\|^2$$

$$\leq \left(\frac{C_8 + C_{11}}{\lambda_1^k} + C_{10} + C_{12}\right)\left\|\nabla^k \Theta_t\right\|^2 + \left(\frac{C_{10}}{\lambda_1^m} + \frac{C_{12}}{\lambda_1^{2m}}\right)\left\|\nabla^{2m+k}\Theta\right\|^2$$

$$+ \left(C_8 + \frac{C_9^2}{\beta\lambda_1^{m+k}} + \frac{C_{11}}{\lambda_1^{4m}}\right)\left\|\nabla^{2m+k}\overline{u}\right\|^4.$$

Let $\alpha_4 = \max\left\{\frac{C_8 + C_{11}}{\lambda_1^k} + C_{10} + C_{12}, \frac{C_{10}}{\mu\lambda_1^m} + \frac{C_{12}}{\mu\lambda_1^{2m}}\right\}$, we receive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^k \Theta_t\right\|^2 + \mu\left\|\nabla^{2m+k}\Theta\right\|^2\right) \leq \alpha_4\left(\left\|\nabla^k \Theta_t\right\|^2 + \mu\left\|\nabla^{2m+k}\Theta\right\|^2\right) + C_{13}\left\|\nabla^{2m+k}\overline{u}\right\|^4. \tag{4.7}$$

By using the differential Gronwall's inequality, we have

$$\left\|\nabla^k \Theta_t\right\|^2 + \mu\left\|\nabla^{2m+k}\Theta\right\|^2 \leq C_{14}\mathrm{e}^{C_{15}t}\left\|(\xi,\eta)^{\mathrm{T}}\right\|_{E_k}^4.$$

According to the Lipschitz property of $S(t)$, as $\left\|(\xi,\eta)^{\mathrm{T}}\right\|_{E_k}^2 \to 0$, we obtain

$$\frac{\left\|S(t)\widetilde{\varphi_0} - S(t)\varphi_0 - G(t)(\xi,\eta)\right\|_{E_k}^2}{\left\|(\xi,\eta)^{\mathrm{T}}\right\|_{E_k}^2} \le C_{14}\mathrm{e}^{C_{15}t}\left\|(\xi,\eta)^{\mathrm{T}}\right\|_{E_k}^2 \to 0.$$

That means $S(t)$ is uniformly differentiable on E_k. ■

Theorem 4. Under the conditions of Theorem 3, the family of global attractor A_k of the initial boundary value problem (1.1) have the finite Hausdorff dimension and Fractal dimension, and $d_H(A_k) < \dfrac{N}{2}, d_F(A_k) < \dfrac{3N}{2}$.

Proof. In order to estimate the dimension of the family of global attractor A_k, we rewrite the initial boundary value problem as a first-order evolution equation

$$\Psi_t + \Lambda_\varepsilon \Psi + g(\Psi) = F(\Psi),$$

where $\Psi = (u,v)^{\mathrm{T}} = (u, u_t + \varepsilon u)^{\mathrm{T}}$, $g(\Psi) = (0, |u|^\rho u)^{\mathrm{T}}$, $F(\Psi) = (0, f(x))^{\mathrm{T}}$,

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ \left(M\left(\|\nabla^m u\|_p^p\right) - \varepsilon\beta\right)(-\Delta)^{2m} + \varepsilon^2 I & \beta(-\Delta)^{2m} - \varepsilon I \end{pmatrix}.$$

Further, let $L(\Psi) = \Psi_t = F(\Psi) - \Lambda_\varepsilon \Psi - g(\Psi)$, where $L : E_k \to E_k$ is Frechet differential.

Similarly, we rewrite the linear Equation (4.1) as

$$P_t + \overline{\Lambda_\varepsilon} P + g'(\phi)P = 0, \tag{4.8}$$

where $P = (U, U_t + \varepsilon U)^{\mathrm{T}}$, $P_t = L_t(\Psi)$, $g'(\phi)P = (0, (\rho+1)|u|^\rho U)^{\mathrm{T}}$, U is the solution of Equation (4.1),

$$\overline{\Lambda_\varepsilon} = \begin{pmatrix} \varepsilon I & -I \\ \left(M\left(\|\nabla^m u\|_p^p\right) - \varepsilon\beta\right)(-\Delta)^{2m} + \varepsilon^2 I + D & \beta(-\Delta)^{2m} - \varepsilon I \end{pmatrix},$$

$$D \cdot U = M'\left(\|\nabla^m u\|_p^p\right)\left(\|\nabla^m u\|_p^p\right)' \nabla^m U(-\Delta)^{2m} u.$$

For every fixed $(u_0, v_0) \in E_k$, assume that $U_1(t), U_2(t), \cdots, U_N(t)$ are N solutions to Equation (4.8). The initial value $U_1(0) = \xi_1, U_2(0) = \xi_2, \cdots, U_N(0) = \xi_N$, where $\xi_i \in E_k \ (i = 1, 2, \cdots, N)$.

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|U_1(t) \wedge U_2(t) \wedge \cdots \wedge U_N(t)\right\|_{\wedge E_k}^2$$
$$- 2tr\left(L_t(\Psi) \cdot Q_N(t)\right)\left\|\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_N\right\|_{\wedge E_k}^2 = 0,$$

where $\wedge$ represents the cross product, *tr* represents the trace of the operator, Q_N represents the orthogonal projection from E_k to $span\{U_1(t), U_2(t), \cdots, U_N(t)\}$.

According to uniform Gronwall's inequality, we obtain

$$\left\| U_1(t) \wedge U_2(t) \wedge \cdots \wedge U_N(t) \right\|^2_{\wedge E_k}$$
$$\leq \left\| \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_N \right\|^2_{\wedge E_k} \exp\left(\int_0^t trL_t(\Psi(\tau)) \cdot Q_N(\tau) d\tau \right). \tag{4.9}$$

For arbitrarily given time τ, let $\omega_j(\tau) = \left(\xi_j(\tau), \eta_j(\tau) \right)^T$ $(j = 1, 2, \cdots, N)$ are the orthogonal projection of $span\{ U_1(t), U_2(t), \cdots, U_N(t) \}$. Then, we have

$$\left\| \nabla^{2m+k} \xi_j \right\|^2 + \left\| \nabla^k \eta_j \right\|^2 = 1. \tag{4.10}$$

Define the inner production on E_k as
$$\left((\varsigma, \eta), (\overline{\varsigma}, \overline{\eta}) \right) = \left(\nabla^{2m+k}\varsigma, \nabla^{2m+k}\overline{\varsigma} \right) + \left(\nabla^k \eta, \nabla^k \overline{\eta} \right).$$

According to the above conditions, we receive

$$trL_t(\Psi(\tau) \cdot Q_N(\tau)) = \sum_{j=1}^N \left(L_t(\Psi(\tau)) \cdot Q_N(\tau) \omega_j(\tau), \omega_j(\tau) \right)_{E_k}$$
$$= \sum_{j=1}^N \left(L_t(\Psi(\tau)) \omega_j(\tau), \omega_j(\tau) \right)_{E_k}, \tag{4.11}$$

where
$$\left(L_t(\Psi(\tau)) \omega_j(\tau), \omega_j(\tau) \right)_{E_k} = \left(L_t(\Psi(\tau)) \omega_j(\tau), \omega_j(\tau) \right)_{E_k}$$
$$= -\left(\overline{\wedge_\varepsilon} \omega_j, \omega_j \right) - \left(g'(\phi)\omega_j, \omega_j \right).$$

$$\left(\overline{\wedge_\varepsilon} \omega_j, \omega_j \right) = \left(\left(\varepsilon\xi_j - \eta_j, M\left(\left\| \nabla^m u \right\|_p^p \right) - \varepsilon\beta(-\Delta)^{2m}\xi_j + \varepsilon^2 \xi_j + D \cdot \xi_j \right.\right.$$
$$\left.\left. + \beta(-\Delta)^{2m}\eta_j - \varepsilon\eta_j \right), (\xi_j, \eta_j) \right)$$
$$= \varepsilon \left\| \nabla^{2m+k}\xi_j \right\|^2 + \left(M\left(\left\| \nabla^m u \right\|_p^p \right) - \varepsilon\beta - 1 \right)\left(\nabla^{2m+k}\xi_j, \nabla^{2m+k}\eta_j \right)$$
$$+ \varepsilon^2 \left(\nabla^k \xi_j, \nabla^k \eta_j \right) + \beta \left\| \nabla^{2m+k}\eta_j \right\|^2 - \varepsilon \left\| \nabla^k \xi_j \right\|^2 + \left(\nabla^k D \cdot \xi_j, \nabla^k \eta_j \right) \tag{4.12}$$
$$\geq \left(\varepsilon - \frac{\varepsilon^2(\beta-1)}{2} \right)\left\| \nabla^{2m+k}\xi_j \right\|^2 + \frac{(\beta+1)\lambda_1^{2m} - \varepsilon^2 - C_{16}}{2}\left\| \nabla^k \eta_j \right\|^2$$
$$- \frac{\varepsilon^2 + 2\varepsilon + C_{16}}{2}\left\| \nabla^k \xi_j \right\|^2,$$

$$\left(g'(\phi)\omega_j, \omega_j \right) = \left(\left(0, \rho|u|^\rho \eta_j \right), (\xi_j, \eta_j) \right)$$
$$\geq -\rho \left\| u \right\|_\infty^\rho \left\| \nabla^k \eta_j \right\|^2 \geq -C_{17} \left\| \nabla^k \eta_j \right\|^2. \tag{4.13}$$

According to (4.10)-(4.13), let
$$\delta = \min\left\{ \varepsilon - \frac{\varepsilon^2(\beta-1)}{2}, \frac{(\beta+1)\lambda_1^{2m} - \varepsilon^2 - C_{16}}{2} - C_{17} \right\}, \quad \gamma = \frac{\varepsilon^2 + 2\varepsilon + C_{16}}{2}, \text{ we obtain}$$

$$\sum_{j=1}^N \left(L_t(\Psi(\tau)) \omega_j(\tau), \omega_j(\tau) \right)_{E_k} \leq -N\delta + \gamma \sum_{j=1}^N \left\| \nabla^k \xi_j \right\|^2. \tag{4.14}$$

For almost all t, we have

$$\sum_{j=1}^N \left\| \nabla^k \xi_j \right\|^2 \leq \sum_{j=1}^N \lambda_j^{-\frac{2m-1}{2m}},$$

where λ_j is the eigenvalue of $(-\Delta)^{2m}$. Thus

$$tr\left(L_t\left(\Psi\right)\left(\tau\right)\cdot Q_N\left(\tau\right)\right)\leq -N\delta+\gamma\sum_{j=1}^{N}\lambda_j^{-\frac{2m-1}{2m}}.$$

Suppose

$$q_N\left(t\right)=\sup_{\Psi_0\in A}\inf_{\xi_j\in E_k}\left(\frac{1}{t}\int_0^t trL_t\left(S\left(\tau\right)\Psi_0\right)\cdot Q_N\left(\tau\right)\mathrm{d}\tau\right),$$

$$q_N=\lim_{t\to\infty}q_N\left(t\right).$$

Due to (4.14), we have

$$q_N\leq -N\delta+\gamma\sum_{j=1}^{N}\lambda_j^{-\frac{2m-1}{2m}}.$$

Therefore, the Lyapunov exponents $\sigma_1,\sigma_2,\cdots,\sigma_N$ of B_{0k} are uniformly bounded, moreover

$$\sigma_1+\sigma_2+\cdots+\sigma_N\leq -N\delta+\gamma\sum_{j=1}^{n}\lambda_j^{-\frac{2m-1}{2m}},$$

such that

$$\left(q_j\right)_+\leq -N\delta+\gamma\sum_{j=1}^{n}\lambda_j^{\frac{2m-1}{2m}}\leq\gamma\sum_{j=1}^{N}\lambda_j^{\frac{2m-1}{2m}}\leq-\frac{N\delta}{3},$$

$$q_N\leq -N\delta\left(1-\frac{\gamma}{N\delta}\sum_{j=1}^{N}\lambda_j^{\frac{2m-1}{2m}}\right)\leq-\frac{2N\delta}{3}.$$

Further,

$$\max_{1\leq j\leq N}\frac{\left(q_j\right)_+}{|q_N|}\leq\frac{1}{2}.$$

Thus, we obtain $d_H\left(A_k\right)<\dfrac{N}{2}, d_F\left(A_k\right)<\dfrac{3N}{2}.$ ∎

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Kirchhoff, G. (1883) Vorlesungen fiber Mechanik. Teubner, Stuttgart.

[2] Hosoya, M. and Yamada, Y. (1991) On Some Nonlinear Wave Equations II: Global Existence and Energy Decay of Solutions. *Journal of the Faculty of Science, the University of Tokyo. Section* 1*A, Mathematics*, **38**, 239-250.

[3] Chueshov, I. (2011) Long-Time Dynamics of Kirchhoff Wave Models with Strong Nonlinear Damping. *Journal of Differential Equations*, **252**, 1229-1262. https://doi.org/10.1016/j.jde.2011.08.022

[4] Lin, G.G. and Chen, Y.H. (2020) A Family of Global Attractors for the Higher-Order Kirchhoff-Type Equations and Its Dimension Estimation. *American Journal of Applied Mathematics*, **8**, 300-310.

[5] Lin, G.G. and Guan, L.P. (2019) Global Attractor Family and Its Dimension Estimation of Higher-Order Kirchhoff Type Equations with Strong Damping. *Acta Analysis Functionalis Applicata*, **21**, 268-281.

[6] Lin, G.G. and Wang, W. (2018) Exponential Attractors for the Higher-Order Kirchhoff-Type Equation with Nonlinear Damped Term. *Acta Analysis Functionalis Applicata*, **20**, 19-25.

[7] Li, Q.X. and Tan, Z. (2002) Global Existence of a Kirchhoff Type Equation with Damping and Source Terms. *Journal of Xiamen University* (*Natural Science*), No. 4, 418-422.

[8] Lin, G.G. and Li, Z.X. (2019) Attractor Family and Dimension for a Class of High-Order Nonlinear Kiechhoff Equations. *Journal of Shandong University* (*Natural Science*), **54**, 1-11.

[9] Lin, G.G. (2011) Nonlinear Evolution Equations. Yunnan University Press, Kunming.

[10] Wang, Y.H. and Zhong, C.K. (2013) Upper Semicontinuity of Pullback Attractors for Nonautonomous Kirchhoff Wave Models. *Discrete & Continuous Dynamical Systems*, **33**, 3189-3209. https://doi.org/10.3934/dcds.2013.33.3189

[11] Zhang, S.L., Zhang, J.W. and Wang, H.Y. (2020) Global Attractors for Kirchhoff Wave Equation with Nonlinear Damping and Memory. *Boundary Value Problems*, **2020**, Article No. 116. https://doi.org/10.1186/s13661-020-01413-5

[12] Ye, Y.J. and Tao, X.X. (2019) Initial Boundary Value Problem for Higher-Order Nonlinear Kirchhoff-Type Equation. *Acta Mathematica Sinica, Chinese Series*, **62**, 923-938.

[13] Gao, Y.L., Sun, Y.T. and Lin, G.G. (2016) The Global Attractor for the Higher-Order Kirchhoff-Type Equation with Nonlinear Strongly Damped Term. *International Journal of Modern Nonlinear Theory and Application*, **10**, 6608-6621.

[14] Lin, G.G. and Yang, S.M. (2017) The Global Attractors for the Higher-Order Coupled Kirchhoff-Type Equation with Strongly Damped Terms and Source Term. *European Journal of Mathematics and Computer Science*, **4**, 66-78.

[15] Yang, Z.J., Ding, D.Y. and Li, L. (2016) Longtime Dynamics of the Kirchhoff Equations with Fractional Damping and Supercritical Nonlinearity. *Journal of Mathematical Analysis and Applications*, **422**, 485-510. https://doi.org/10.1016/j.jmaa.2016.04.079

[16] Teman (1988) Infinite Dimensional Dynamical Systems in Mechanics and Physics. Springer-Verlag, New York, 15-26. https://doi.org/10.1007/978-1-4684-0313-8