Journal of Applied Mathematics and Physics, 2020, 8, 1291-1303
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Continuous Dependence for the Linear Differential Equations of Thermo-Diffusion

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How to cite this paper: Shi, J.C. (2020) Continuous Dependence for the Linear Differential Equations of Thermo-Diffusion. *Journal of Applied Mathematics and Physics*, **8**, 1291-1303.
https://doi.org/10.4236/jamp.2020.87099

Received: June 17, 2020
Accepted: July 12, 2020
Published: July 15, 2020

Abstract

In this paper, we establish the structural stability for the linear differential equations of thermo-diffusion in a semi-infinite pipe flow. Using the technology of a second-order differential inequality, we prove the continuous dependence on the density ρ and the coefficient of thermal conductivity K. These results show that small changes for these coefficients can't cause tremendous changes for the solutions.

Keywords

Differential Equations of Thermo-Diffusion, Structural Stability, Continuous Dependence

1. Introduction

The question of continuous dependence of solutions of problems in partial differential equations on coefficients in the equations has been extensively studied in recent years for a variety of problems. This is sometimes referred to as the question of structural stability and numerous references may be found, for instance, in the book of Ames and Straughan [1] and the monograph of Straughan [2]. For more papers one can see [3]-[8]. In structural stability the emphasis is on continuous dependence (convergence result) on changes in the model itself rather than on the initial data. This means changes in coefficients in the partial differential equations and changes in the equations and may be reflected physically by changes in constitutive parameters. What's more, the inevitable error that arises in both numerical computation and the physical measurement of data can exist. It is relevant to know the magnitude of the effect of such errors on the solution.

In the 1970s, W. Nowacki in his papers [9] [10] gave the differential equations of thermodiffusion in one dimensional space and many papers in the literatures have studied this system. For example, [10] [11] [12] investigated the initial-boundary value problem for the linear system of thermodiffusion using different arguments. [13] proved the existence, uniqueness and regularity of the solution to the initial-boundary value problems for the linear system of thermodiffusion in a solid body. L^p-L^q time decay estimates for the solution of the associated linear Cauchy problem were obtained by [14]. However, in this paper, we considered the differential equations of thermodiffusion in three dimensions and we study not only the continuous dependence on the coefficients of the equations, but also the spatial decay estimates for the solution of the system. In fact, much has been written on the subject of spatial decay bounds for various systems of differential equations, e.g., for a review of such works on Saint-Venant's principle, one can refer to [15]-[23] and the papers cited therein. Recently, there are some new results about structural stability, one could see [24]-[28].

We shall assume that a transient flow occupies the interior of a semi-infinite cylindrical pipe $\boldsymbol{R}$ with boundary $\partial\boldsymbol{R}$. The pipe has arbitrary cross section denoted by $\boldsymbol{D}$ and the boundary $\partial\boldsymbol{D}$ and the generators of the pipe are parallel to the x_3 axis. We introduce the notations:

$$\boldsymbol{R}_z=\left\{(x_1,x_2,x_3)\,|\,(x_1,x_2)\in\boldsymbol{D},x_3>z\geq 0\right\},$$

$$\boldsymbol{D}_z=\left\{(x_1,x_2,x_3)\,|\,(x_1,x_2)\in\boldsymbol{D},x_3=z\geq 0\right\},$$

where z is a running variable along the x_3 axis. Clearly, $\boldsymbol{R}_0=\boldsymbol{R}$ and $\boldsymbol{D}_0=\boldsymbol{D}$. Let u_i, $\boldsymbol{T}$, and $\boldsymbol{C}$ denote the displacement, temperature, and chemical potential as independent fields, respectively. These fields depend on the space variable (x_1,x_2,x_3) and the time variable t and satisfy the following system of equations:

$$\rho\ddot{u}_i-v\Delta u_i-(\lambda+v)u_{j,ji}+\gamma_1\boldsymbol{T}_{,i}+\gamma_2\boldsymbol{C}_{,i}=0,\ \text{in}\ \boldsymbol{R}\times\{t\geq 0\},\tag{1.1}$$

$$c\dot{\boldsymbol{T}}-K\Delta\boldsymbol{T}+\gamma_1\dot{u}_{i,i}+d\dot{\boldsymbol{C}}=0,\ \text{in}\ \boldsymbol{R}\times\{t\geq 0\},\tag{1.2}$$

$$n\dot{\boldsymbol{C}}-M\Delta\boldsymbol{C}+\gamma_2\dot{u}_{i,i}+d\dot{\boldsymbol{T}}=0,\ \text{in}\ \boldsymbol{R}\times\{t\geq 0\},\tag{1.3}$$

with the initial-boundary conditions

$$u_i=0,\quad \boldsymbol{T}=\boldsymbol{C}=0\quad \text{on}\ \partial\boldsymbol{D}\times\{t\geq 0\},\tag{1.4}$$

$$u_i=\dot{u}_i=0,\quad \boldsymbol{T}=\boldsymbol{C}=0\quad \text{in}\ \boldsymbol{R}\times\{t=0\}.\tag{1.5}$$

$$u_i=f_i(x_1,x_2,t),\boldsymbol{T}=\boldsymbol{F}(x_1,x_2,t),\boldsymbol{C}=\boldsymbol{G}(x_1,x_2,t)\ \text{on}\ \boldsymbol{D}_0\times\{t\geq 0\},\tag{1.6}$$

$$u_i,u_{i,j},\boldsymbol{T},\boldsymbol{T}_{,i},\boldsymbol{C},\boldsymbol{C}_{,i},p=o\left(x_3^{-1}\right)\ \text{uniformly in}\ x_1,x_2,t\ \text{as}\ x_3\to\infty.\tag{1.7}$$

In Equations (1.1)-(1.3), Δ is the Laplacian operator; ρ represents the density; γ_1 and γ_2 are the coefficients of thermal and diffusion dilatation; λ and v are the material coefficients; K is the coefficient of thermal conductivity; M is the coefficient of diffusion. n,c,d are the coefficients of thermodiffusion. All the above constants are positive and satisfy

$$cn-d^2>0, \tag{1.8}$$

which implies that (1.1)-(1.3) is a hyperbolic-parabolic system of partial differential equations. In the following several sections, we may use the below inequality. Let $\boldsymbol{D}$ be a plane domain $\boldsymbol{D}$ with the boundary $\partial\boldsymbol{D}$. If $w=0$ on $\partial\boldsymbol{D}$, then

$$\int_{\boldsymbol{D}} w_{,\alpha}w_{,\alpha}\mathrm{d}A \ge \lambda_1\int_{\boldsymbol{D}} w^2\mathrm{d}x, \tag{1.9}$$

where λ_1 is the smallest eigenvalue of the problem

$$\Delta\phi+\lambda\phi=0 \quad \text{in } \boldsymbol{D},$$

$$\phi=0 \quad \text{on } \partial\boldsymbol{D}.$$

This inequality has been well studied (see [29] [30]). Throughout this paper, the usual summation convention is employed with repeated Latin subscripts summed from 1 to 3 and repeat Greek subscript summed from 1 to 2. The comma is used to indicate partial differentiation, *i.e.* $u_{i,j}=\dfrac{\partial u_i}{\partial x_j}$, $\varphi_{\alpha,\alpha}=\sum_{\alpha=1}^{2}\dfrac{\partial\varphi_\alpha}{\partial x_\alpha}$ and $\dot{u}_i$ denotes $\dfrac{\partial u_i}{\partial t}$.

The paper is structured as follows: In Section 2, we derive the continuous dependence on ρ. Section 3 is devoted to seeking the continuous dependence on *K*.

2. Continuous Dependence on the Parameter ρ

Lemma 1. The energy $\boldsymbol{E}(z,t)$ defined in [31]

$$\begin{aligned}\boldsymbol{E}(z,t)=&\frac{1}{2}\rho\int_{\boldsymbol{R}_z}\dot{u}_i\dot{u}_i\mathrm{d}x+\frac{\nu}{2}\int_{\boldsymbol{R}_z}u_{i,j}u_{i,j}\mathrm{d}x+\frac{\lambda+\nu}{2}\int_{\boldsymbol{R}_z}u_{i,i}^2\mathrm{d}x\\&+K\int_0^t\int_{\boldsymbol{R}_z}\boldsymbol{T}_{,j}\boldsymbol{T}_{,j}\mathrm{d}x\mathrm{d}\eta+M\int_0^t\int_{\boldsymbol{R}_z}\boldsymbol{C}_{,j}\boldsymbol{C}_{,j}\mathrm{d}x\mathrm{d}\eta\\&+\int_{\boldsymbol{R}_z}\left[\frac{c}{2}\boldsymbol{T}^2+d\boldsymbol{C}\boldsymbol{T}+\frac{n}{2}C^2\right]\mathrm{d}x,\end{aligned} \tag{2.1}$$

satisfies the following estimates

$$\boldsymbol{E}(z,t)\le\boldsymbol{E}(0,t)\mathrm{e}^{-\frac{1}{m_1(t)}z}. \tag{2.2}$$

where

$$m_1(t)=\frac{\sqrt{\nu t}}{\sqrt{\rho}}+\frac{\sqrt{\lambda+\nu t}}{\sqrt{\rho}}+\frac{\gamma_1\sqrt{t}}{2\sqrt{\rho\lambda_1 K}}+\frac{\gamma_2\sqrt{t}}{2\sqrt{\rho\lambda_1 M}}+\frac{1}{2\sqrt{\lambda_1}}. \tag{2.3}$$

Proof. These results are the main results of paper [31].

Theorem 1. The energy expression $\tilde{\boldsymbol{\varphi}}(z,t)$ satisfies the following estimates:

$$\tilde{\boldsymbol{\varphi}}(z,t)\le\frac{2\pi^2}{\rho\rho^*}\tilde{\boldsymbol{E}}(0,t)\mathrm{e}^{-\frac{1}{2m_1(t)}z}+\frac{\pi^2t^2}{\rho^*\rho}\tilde{\boldsymbol{E}}(0,t)\left(\mathrm{e}^{-\frac{1}{2m_1(t)}z}-\mathrm{e}^{-\frac{3}{2m_1(t)}z}\right). \tag{2.4}$$

Proof. To investigate continuous dependence on ρ, we have to seek a bound for $\frac{\rho}{2}\int_{\boldsymbol{R}_z}\ddot{u}_i\ddot{u}_i\mathrm{d}x$. To do this, we first differentiate (1.1), and then multiply with

$\ddot{u}_i$, integrate over the region $\boldsymbol{R}_z \times [0,t]$ to obtain

$$0=\int_0^t\int_{\boldsymbol{R}_z}\left[\rho\ddot{u}_i-\nu\Delta\dot{u}_i-(\lambda+\nu)\dot{u}_{j,ji}+\gamma_1\dot{\boldsymbol{T}}_{,i}+\gamma_2\dot{\boldsymbol{C}}_{,i}\right]\ddot{u}_i\mathrm{d}x\mathrm{d}\eta, \tag{2.5}$$

which follows that

$$\begin{aligned}0&=\frac{\rho}{2}\int_{\boldsymbol{R}_z}\ddot{u}_i\ddot{u}_i\mathrm{d}x+\nu\int_0^t\int_{\boldsymbol{D}_z}\dot{u}_{i,3}\ddot{u}_i\mathrm{d}A\mathrm{d}\eta+\frac{\nu}{2}\int_{\boldsymbol{R}_z}\dot{u}_{i,j}\dot{u}_{i,j}\mathrm{d}x\\&+(\lambda+\nu)\int_0^t\int_{\boldsymbol{D}_z}\dot{u}_{j,j}\ddot{u}_3\mathrm{d}A\mathrm{d}\eta+\frac{\lambda+\nu}{2}\int_{\boldsymbol{R}_z}\dot{u}_{j,j}^2\mathrm{d}x\\&+\gamma_1\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{T}}_{,i}\ddot{u}_i\mathrm{d}x\mathrm{d}\eta+\gamma_2\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{C}}_{,i}\ddot{u}_i\mathrm{d}x\mathrm{d}\eta,\end{aligned} \tag{2.6}$$

where we have supposed that $\ddot{u}_i$ vanish at $t=0$. Similarly, we have

$$\begin{aligned}0&=c\int_0^t\int_{\boldsymbol{R}_z}\ddot{\boldsymbol{T}}\dot{\boldsymbol{T}}\mathrm{d}x\mathrm{d}\eta-K\int_0^t\int_{\boldsymbol{R}_z}\Delta\dot{\boldsymbol{T}}\dot{\boldsymbol{T}}\mathrm{d}x\mathrm{d}\eta+\gamma_1\int_0^t\int_{\boldsymbol{R}_z}\ddot{u}_{i,i}\dot{\boldsymbol{T}}\mathrm{d}x\mathrm{d}\eta+d\int_0^t\int_{\boldsymbol{R}_z}\ddot{\boldsymbol{C}}\dot{\boldsymbol{T}}\mathrm{d}x\mathrm{d}\eta\\&=\frac{c}{2}\int_{\boldsymbol{R}_z}\dot{\boldsymbol{T}}^2\mathrm{d}x+K\int_0^t\int_{\boldsymbol{D}_z}\dot{\boldsymbol{T}}_{,3}\dot{\boldsymbol{T}}\mathrm{d}A\mathrm{d}\eta+K\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{T}}_{,i}\dot{\boldsymbol{T}}_{,i}\mathrm{d}x\mathrm{d}\eta\\&-\gamma_1\int_0^t\int_{\boldsymbol{D}_z}\ddot{u}_3\dot{\boldsymbol{T}}\mathrm{d}A\mathrm{d}\eta-\gamma_1\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{T}}_{,i}\ddot{u}_i\mathrm{d}x\mathrm{d}\eta+d\int_0^t\int_{\boldsymbol{R}_z}\ddot{\boldsymbol{C}}\dot{\boldsymbol{T}}\mathrm{d}x\mathrm{d}\eta,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}0&=n\int_0^t\int_{\boldsymbol{R}_z}\ddot{\boldsymbol{C}}\dot{\boldsymbol{C}}\mathrm{d}x\mathrm{d}\eta-M\int_0^t\int_{\boldsymbol{R}_z}\Delta\dot{\boldsymbol{C}}\dot{\boldsymbol{C}}\mathrm{d}x\mathrm{d}\eta+\gamma_2\int_0^t\int_{\boldsymbol{R}_z}\ddot{u}_{i,i}\dot{\boldsymbol{C}}\mathrm{d}x\mathrm{d}\eta+d\int_0^t\int_{\boldsymbol{R}_z}\ddot{\boldsymbol{T}}\dot{\boldsymbol{C}}\mathrm{d}x\mathrm{d}\eta\\&=\frac{n}{2}\int_{\boldsymbol{R}_z}\dot{\boldsymbol{C}}^2\mathrm{d}x+M\int_0^t\int_{\boldsymbol{D}_z}\dot{\boldsymbol{C}}_{,3}\dot{\boldsymbol{C}}\mathrm{d}A\mathrm{d}\eta+M\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{C}}_{,i}\dot{\boldsymbol{C}}_{,i}\mathrm{d}x\mathrm{d}\eta\\&-\gamma_2\int_0^t\int_{\boldsymbol{D}_z}\ddot{u}_3\dot{\boldsymbol{C}}dAd\eta-\gamma_2\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{C}}_{,i}\ddot{u}_i\mathrm{d}x\mathrm{d}\eta+d\int_0^t\int_{\boldsymbol{R}_z}\ddot{\boldsymbol{T}}\dot{\boldsymbol{C}}\mathrm{d}x\mathrm{d}\eta,\end{aligned} \tag{2.8}$$

where we also have assumed that $\dot{\boldsymbol{T}}=\dot{\boldsymbol{C}}=0$ at $t=0$. Combining (6)-(8), we set

$$\begin{aligned}\tilde{\boldsymbol{E}}(z,t)&=\frac{\rho}{2}\int_{\boldsymbol{R}_z}\ddot{u}_i\ddot{u}_i\mathrm{d}x+\frac{\nu}{2}\int_{\boldsymbol{R}_z}\dot{u}_{i,j}\dot{u}_{i,j}\mathrm{d}x+\frac{\lambda+\nu}{2}\int_{\boldsymbol{R}_z}\dot{u}_{j,j}^2\mathrm{d}x+K\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{T}}_{,i}\dot{\boldsymbol{T}}_{,i}\mathrm{d}x\mathrm{d}\eta\\&+M\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{C}}_{,i}\dot{\boldsymbol{C}}_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{c}{2}\int_{\boldsymbol{R}_z}\dot{\boldsymbol{T}}^2\mathrm{d}x+\frac{1}{2}\int_{\boldsymbol{R}_z}\left[c\dot{\boldsymbol{T}}^2+2d\dot{\boldsymbol{T}}\dot{\boldsymbol{C}}+n\dot{\boldsymbol{C}}^2\right]\mathrm{d}x\\&=-\nu\int_0^t\int_{\boldsymbol{D}_z}\dot{u}_{i,3}\ddot{u}_i\mathrm{d}A\mathrm{d}\eta-(\lambda+\nu)\int_0^t\int_{\boldsymbol{D}_z}\dot{u}_{j,j}\ddot{u}_3\mathrm{d}A\mathrm{d}\eta-K\int_0^t\int_{\boldsymbol{D}_z}\dot{\boldsymbol{T}}_{,3}\dot{\boldsymbol{T}}\mathrm{d}A\mathrm{d}\eta\\&+\gamma_1\int_0^t\int_{\boldsymbol{D}_z}\ddot{u}_3\dot{\boldsymbol{T}}\mathrm{d}A\mathrm{d}\eta-M\int_0^t\int_{\boldsymbol{D}_z}\dot{\boldsymbol{C}}_{,3}\dot{\boldsymbol{C}}\mathrm{d}A\mathrm{d}\eta+\gamma_2\int_0^t\int_{\boldsymbol{D}_z}\ddot{u}_3\dot{\boldsymbol{C}}\mathrm{d}A\mathrm{d}\eta.\end{aligned} \tag{2.9}$$

Following the method used in [31], we can get

$$\begin{aligned}\tilde{\boldsymbol{E}}(z,t)&=\frac{\rho}{2}\int_{\boldsymbol{R}_z}\ddot{u}_i\ddot{u}_i\mathrm{d}x+\frac{\nu}{2}\int_{\boldsymbol{R}_z}\dot{u}_{i,j}\dot{u}_{i,j}\mathrm{d}x+\frac{\lambda+\nu}{2}\int_{\boldsymbol{R}_z}\dot{u}_{j,j}^2\mathrm{d}x+K\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{T}}_{,i}\dot{\boldsymbol{T}}_{,i}\mathrm{d}x\mathrm{d}\eta\\&+M\int_0^t\int_{\boldsymbol{R}_z}\dot{\boldsymbol{C}}_{,i}\dot{\boldsymbol{C}}_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{c}{2}\int_{\boldsymbol{R}_z}\dot{\boldsymbol{T}}^2\mathrm{d}x+\frac{1}{2}\int_{\boldsymbol{R}_z}\left[c\dot{\boldsymbol{T}}^2+2d\dot{\boldsymbol{T}}\dot{\boldsymbol{C}}+n\dot{\boldsymbol{C}}^2\right]\mathrm{d}x\\&\le\tilde{\boldsymbol{E}}(0,t)\mathrm{e}^{-\frac{1}{m_1(t)}z}.\end{aligned}$$

Also, we employ the argument used in [31] to get that $\tilde{\boldsymbol{E}}(0,t)$ may be bounded by known data. Since $cn>d^2$, we note again $\int_{\boldsymbol{R}_z}\left[c\dot{\boldsymbol{T}}^2+2d\dot{\boldsymbol{T}}\dot{\boldsymbol{C}}+n\dot{\boldsymbol{C}}^2\right]\mathrm{d}x>0$. So, we have

$$\int_{\boldsymbol{R}_z}\ddot{u}_i\ddot{u}_i\mathrm{d}x\le\frac{2}{\rho}\tilde{\boldsymbol{E}}(0,t)\mathrm{e}^{-\frac{1}{m_1(t)}z}. \tag{2.10}$$

Now, we study the continuous dependence on the parameter ρ. Let $(u_i,\boldsymbol{T},\boldsymbol{C})$

and $\left(u_i^*, \boldsymbol{T}^*, \boldsymbol{C}^*\right)$ be the solutions to (1.1)-(1.3) with same initial-boundary conditions, but for different parameters ρ and ρ^*, respectively. Define the difference variables as

$$w_i = u_i - u_i^*, \quad \theta = \boldsymbol{T} - \boldsymbol{T}^*, \quad \Sigma = \boldsymbol{C} - \boldsymbol{C}^*, \quad \pi = \rho - \rho^*. \tag{2.11}$$

Then, (w_i, θ, Σ) satisfy

$$\pi \ddot{u}_i + \rho^* \ddot{w}_i - \nu \Delta w_i - (\lambda + \nu) w_{j,ji} + \gamma_1 \theta_{,i} + \gamma_2 \Sigma_{,i} = 0, \text{ in } \boldsymbol{R} \times \{t \geq 0\}, \tag{2.12}$$

$$c\dot{\theta} - K\Delta\theta + \gamma_1 \dot{w}_{i,i} + d\dot{\Sigma} = 0, \text{ in } \boldsymbol{R} \times \{t \geq 0\}, \tag{2.13}$$

$$n\dot{\Sigma} - M\Delta\Sigma + \gamma_2 \dot{w}_{i,i} + d\dot{\theta} = 0, \text{ in } \boldsymbol{R} \times \{t \geq 0\}, \tag{2.14}$$

with the initial-boundary conditions

$$w_i = 0, \quad \theta = \Sigma = 0 \quad \text{on } \partial\boldsymbol{D} \times \{t \geq 0\}, \tag{2.15}$$

$$w_i = \dot{w}_i = 0, \quad \theta = \Sigma = 0 \quad \text{in } \boldsymbol{R} \times \{t = 0\}. \tag{2.16}$$

$$w_i = \theta = \Sigma = 0 \quad \text{on } \boldsymbol{D}_0 \times \{t \geq 0\}, \tag{2.17}$$

Multiplying (2.12) with $\dot{w}_i$ and integrating by parts, we have

$$\begin{aligned}
0 &= \int_0^t \int_{\boldsymbol{R}_z} \left[\pi \ddot{u}_i + \rho^* \ddot{w}_i - \nu \Delta w_i - (\lambda+\nu) w_{j,ji} + \gamma_1 \theta_{,i} + \gamma_2 \Sigma_{,i}\right] \dot{w}_i \mathrm{d}x\mathrm{d}\eta \\
&= \frac{1}{2}\int_{\boldsymbol{R}_z} \rho^* \dot{w}_i \dot{w}_i dx + \frac{\nu}{2}\int_{\boldsymbol{R}_z} w_{i,j} w_{i,j} \mathrm{d}x + \frac{\lambda+\nu}{2}\int_{\boldsymbol{R}_z} w_{i,i}^2 \mathrm{d}x \\
&+ \pi \int_0^t \int_{\boldsymbol{R}_z} \ddot{u}_i \dot{w}_i \mathrm{d}x\mathrm{d}\eta + \gamma_1 \int_0^t \int_{\boldsymbol{R}_z} \theta_{,i} \dot{w}_i \mathrm{d}x\mathrm{d}\eta + \gamma_2 \int_0^t \int_{\boldsymbol{R}_z} \Sigma_{,i} \dot{w}_i \mathrm{d}x\mathrm{d}\eta \\
&+ \nu \int_0^t \int_{\boldsymbol{D}_z} w_{i,3} \dot{w}_i \mathrm{d}A\mathrm{d}\eta + (\lambda+\nu) \int_0^t \int_{\boldsymbol{D}_z} w_{j,j} \dot{w}_3 \mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{2.18}$$

Similarly, we have

$$\begin{aligned}
0 &= \int_0^t \int_{\boldsymbol{R}_z} \left[c\dot{\theta} - K\Delta\theta + \gamma_1 w_{i,i} + d\dot{\Sigma}\right] \theta \mathrm{d}x\mathrm{d}\eta \\
&= \frac{c}{2}\int_{\boldsymbol{R}_z} \theta^2 \mathrm{d}x\mathrm{d}\eta + K\int_0^t \int_{\boldsymbol{R}_z} \theta_{,i}\theta_{,i} \mathrm{d}x\mathrm{d}\eta - \gamma_1 \int_0^t \int_{\boldsymbol{R}_z} \theta_{,i} \dot{w}_i \mathrm{d}x\mathrm{d}\eta \\
&+ d\int_0^t \int_{\boldsymbol{R}_z} \dot{\Sigma}\theta \mathrm{d}x\mathrm{d}\eta + K\int_0^t \int_{\boldsymbol{D}_z} \theta_{,3}\theta \mathrm{d}A\mathrm{d}\eta - \gamma_1 \int_0^t \int_{\boldsymbol{D}_z} \dot{w}_3 \theta \mathrm{d}A\mathrm{d}\eta,
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
0 &= \int_0^t \int_{\boldsymbol{R}_z} \left[n\dot{\Sigma} - M\Delta\Sigma + \gamma_2 w_{i,i} + d\dot{\theta}\right] \Sigma \mathrm{d}x \\
&= \frac{n}{2}\int_{\boldsymbol{R}_z} \Sigma^2 \mathrm{d}x\mathrm{d}\eta + M\int_0^t \int_{\boldsymbol{R}_z} \Sigma_{,i}\Sigma_{,i} \mathrm{d}x\mathrm{d}\eta - \gamma_2 \int_0^t \int_{\boldsymbol{R}_z} \Sigma_{,i} \dot{w}_i \mathrm{d}x\mathrm{d}\eta \\
&+ d\int_0^t \int_{\boldsymbol{R}_z} \dot{\theta}\Sigma \mathrm{d}x\mathrm{d}\eta + M\int_0^t \int_{\boldsymbol{D}_z} \Sigma_{,3}\Sigma \mathrm{d}A\mathrm{d}\eta - \gamma_2 \int_0^t \int_{\boldsymbol{D}_z} \dot{w}_3 \Sigma \mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{2.20}$$

We define a new function

$$\begin{aligned}
\boldsymbol{\varphi}(z,t) &= \frac{1}{2}\int_{\boldsymbol{R}_z} \rho^* \dot{w}_i \dot{w}_i \mathrm{d}x + \frac{\nu}{2}\int_{\boldsymbol{R}_z} w_{i,j} w_{i,j} \mathrm{d}x + \frac{\lambda+\nu}{2}\int_{\boldsymbol{R}_z} w_{i,i}^2 \mathrm{d}x + K\int_0^t \int_{\boldsymbol{R}_z} \theta_{,i}\theta_{,i} \mathrm{d}x\mathrm{d}\eta \\
&+ M\int_0^t \int_{\boldsymbol{R}_z} \Sigma_{,i}\Sigma_{,i} \mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\int_{\boldsymbol{R}_z} \left[c\theta^2 + 2d\theta\Sigma + n\Sigma^2\right] \mathrm{d}x\mathrm{d}\eta \\
&= \pi \int_0^t \int_{\boldsymbol{R}_z} \ddot{u}_i \dot{w}_i \mathrm{d}x\mathrm{d}\eta - \nu \int_0^t \int_{\boldsymbol{D}_z} w_{i,3} \dot{w}_i \mathrm{d}A\mathrm{d}\eta - (\lambda+\nu) \int_0^t \int_{\boldsymbol{D}_z} w_{j,j} \dot{w}_3 \mathrm{d}A\mathrm{d}\eta \\
&- K\int_0^t \int_{\boldsymbol{D}_z} \theta_{,3}\theta \mathrm{d}A\mathrm{d}\eta + \gamma_1 \int_0^t \int_{\boldsymbol{D}_z} \dot{w}_3 \theta \mathrm{d}A\mathrm{d}\eta \\
&- M\int_0^t \int_{\boldsymbol{D}_z} \Sigma_{,3}\Sigma \mathrm{d}A\mathrm{d}\eta + \gamma_2 \int_0^t \int_{\boldsymbol{D}_z} \dot{w}_3 \Sigma \mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{2.21}$$

Thus, we have

$$-\frac{\partial \varphi(z,t)}{\partial z}=\frac{1}{2}\int_{D_z}\rho^*\dot{w}_i\dot{w}_i\mathrm{d}A+\frac{\nu}{2}\int_{D_z}w_{i,j}w_{i,j}\mathrm{d}A+\frac{\lambda+\nu}{2}\int_{D_z}w_{i,i}^2\mathrm{d}A$$
$$+K\int_0^t\int_{D_z}\theta_{,i}\theta_{,i}\mathrm{d}A\mathrm{d}\eta+M\int_0^t\int_{D_z}\Sigma_{,i}\Sigma_{,i}\mathrm{d}A\mathrm{d}\eta \tag{2.22}$$
$$+\frac{1}{2}\int_{D_z}\left[c\theta^2+2d\theta\Sigma+n\Sigma^2\right]\mathrm{d}A\mathrm{d}\eta.$$

Similar to [31], we have

$$\varphi(z,t)\le\pi\int_0^t\int_{R_z}\ddot{u}_i\dot{w}_i\mathrm{d}x\mathrm{d}\eta+m_1(t)\left(-\frac{\partial\varphi(z,t)}{\partial z}\right), \tag{2.23}$$

where $m_1(t)$ have been defined in (2.3), but ρ in $m_1(t)$ may be replaced by ρ^* here. By Hölder and the AG mean inequalities, from (2.23) we have

$$\varphi(z,t)\le\pi\left(\int_0^t\int_{R_z}\ddot{u}_i\ddot{u}_i\mathrm{d}x\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{R_z}\dot{w}_i\dot{w}_i\mathrm{d}x\mathrm{d}\eta\right)^{\frac{1}{2}}+m_1(t)\left(-\frac{\partial\varphi(z,t)}{\partial z}\right)$$
$$\le\frac{\pi^2t}{\rho^*}\int_0^t\int_{R_z}\ddot{u}_i\ddot{u}_i\mathrm{d}x\mathrm{d}\eta+\frac{\rho^*}{4}\int_{R_z}\dot{w}_i\dot{w}_i\mathrm{d}x+m_1(t)\left(-\frac{\partial\varphi(z,t)}{\partial z}\right). \tag{2.24}$$

Now, we let

$$\tilde{\varphi}(z,t)=\frac{1}{4}\int_{R_z}\rho^*\dot{w}_i\dot{w}_i\mathrm{d}x+\frac{\nu}{2}\int_{R_z}w_{i,j}w_{i,j}\mathrm{d}x+\frac{\lambda+\nu}{2}\int_{R_z}w_{i,i}^2\mathrm{d}x$$
$$+K\int_0^t\int_{R_z}\theta_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta+M\int_0^t\int_{R_z}\Sigma_{,i}\Sigma_{,i}\mathrm{d}x\mathrm{d}\eta \tag{2.25}$$
$$+\frac{1}{2}\int_{R_z}\left[c\theta^2+2d\theta\Sigma+n\Sigma^2\right]\mathrm{d}x\mathrm{d}\eta.$$

From (2.24) and (2.10), we have

$$\tilde{\varphi}(z,t)\le\frac{\pi^2t}{\rho^*}\int_0^t\int_{R_z}\ddot{u}_i\ddot{u}_i\mathrm{d}x\mathrm{d}\eta+2m_1(t)\left(-\frac{\partial\tilde{\varphi}(z,t)}{\partial z}\right), \tag{2.26}$$

which follows that

$$\tilde{\varphi}(z,t)\le\tilde{\varphi}(0,t)\mathrm{e}^{-\frac{1}{2m_1(t)}z}+\frac{\pi^2t^2}{\rho^*\rho}\tilde{E}(0,t)\left(1-\mathrm{e}^{-\frac{1}{m_1(t)}z}\right)\mathrm{e}^{-\frac{1}{2m_1(t)}z}. \tag{2.27}$$

In order to make inequality (2.26) explicit, we need bound for $\tilde{\varphi}(0,t)$. So, we write (2.21) at $z=0$ and use the initial-boundary conditions to obtain

$$\varphi(0,t)=\frac{1}{2}\int_{R_0}\rho^*\dot{w}_i\dot{w}_i\mathrm{d}x+\frac{\nu}{2}\int_{R_0}w_{i,j}w_{i,j}\mathrm{d}x+\frac{\lambda+\nu}{2}\int_{R_0}w_{i,i}^2\mathrm{d}x$$
$$+K\int_0^t\int_{R_0}\theta_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta+M\int_0^t\int_{R_0}\Sigma_{,i}\Sigma_{,i}\mathrm{d}x\mathrm{d}\eta$$
$$+\frac{1}{2}\int_{R_0}\left[c\theta^2+2d\theta\Sigma+n\Sigma^2\right]\mathrm{d}x\mathrm{d}\eta \tag{2.28}$$
$$=\pi\int_0^t\int_{R_0}\ddot{u}_i\dot{w}_i\mathrm{d}x\mathrm{d}\eta$$
$$\le\frac{\pi^2t}{\rho^*}\int_0^t\int_{R_0}\ddot{u}_i\ddot{u}_i\mathrm{d}x\mathrm{d}\eta+\frac{\rho^*}{4}\int_{R_0}\dot{w}_i\dot{w}_i\mathrm{d}x,$$

which results in

$$\tilde{\varphi}(0,t) \le \frac{\pi^2 t}{\rho^*}\int_0^t\int_{R_0} \ddot{u}_i\ddot{u}_i \mathrm{d}x\mathrm{d}\eta \le \frac{2\pi^2}{\rho\rho^*}\tilde{E}(0,t), \tag{2.29}$$

where we have used (2.10). Combining (2.27) and (2.29), we have

$$\tilde{\varphi}(z,t) \le \frac{2\pi^2}{\rho\rho^*}\tilde{E}(0,t)\mathrm{e}^{-\frac{1}{2m_1(t)}z} + \frac{\pi^2 t^2}{\rho^*\rho}\tilde{E}(0,t)\left(\mathrm{e}^{-\frac{1}{2m_1(t)}z} - \mathrm{e}^{-\frac{3}{2m_1(t)}z}\right). \tag{2.30}$$

Inequality (2.30) shows that the amplitude terms in (2.25) become small as $\rho \to \rho^*$ and the continuous dependence on ρ is obtained.

3. Continuous Dependence on the Parameter *K*

Theorem 2. The energy expression $\tilde{\Phi}(z,t_1)$ satisfies the following estimates:

If $h_0 - k_1 - k_2 = 0$, we have

$$\tilde{\Phi}(z,t_1) \le \beta^2 m_4 \mathrm{e}^{-(h_0-k_1)z} + \beta^2 m_3 \mathrm{e}^{-(h_0-k_1)z}. \tag{3.1}$$

If $h_0 - k_1 - k_2 \ne 0$, we have

$$\tilde{\Phi}(z,t_1) \le \beta^2\left[m_4 - \frac{m_3}{h_0-k_1-k_2}\right]\mathrm{e}^{-(h_0-k_1)z} + \frac{\beta^2 m_3}{h_0-k_1-k_2}\mathrm{e}^{-k_2 z}. \tag{3.2}$$

Proof. In this section we compare the solutions of the following two problems

$$\begin{aligned} &\rho\ddot{u}_i - \nu\Delta u_i - (\lambda+\nu)u_{j,ji} + \gamma_1 \boldsymbol{T}_{,i} + \gamma_2 \boldsymbol{C}_{,i} = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \\ &c\dot{\boldsymbol{T}} - K\Delta\boldsymbol{T} + \gamma_1\dot{u}_{i,i} + d\dot{\boldsymbol{C}} = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \\ &n\dot{\boldsymbol{C}} - M\Delta\boldsymbol{C} + \gamma_2\dot{u}_{i,i} + d\dot{\boldsymbol{T}} = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \end{aligned} \tag{3.3}$$

with the initial-boundary conditions

$$\begin{aligned} &u_i = 0, \quad \boldsymbol{T} = \boldsymbol{C} = 0 && \text{on } \partial\boldsymbol{D}\times\{t\ge 0\}, \\ &u_i = \dot{u}_i = 0, \quad \boldsymbol{T} = \boldsymbol{C} = 0 && \text{in } \boldsymbol{R}\times\{t = 0\}, \\ &u_i = f_i(x_1,x_2,t), \boldsymbol{T} = \boldsymbol{F}(x_1,x_2,t), \boldsymbol{C} = \boldsymbol{G}(x_1,x_2,t) && \text{on } \boldsymbol{D}_0\times\{t\ge 0\}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\rho\ddot{u}_i^* - \nu\Delta u_i^* - (\lambda+\nu)u_{j,ji}^* + \gamma_1 \boldsymbol{T}_{,i}^* + \gamma_2 \boldsymbol{C}_{,i}^* = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \\ &c\dot{\boldsymbol{T}}^* - K^*\Delta\boldsymbol{T}^* + \gamma_1\dot{u}_{i,i}^* + d\dot{\boldsymbol{C}}^* = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \\ &n\dot{\boldsymbol{C}}^* - M\Delta\boldsymbol{C}^* + \gamma_2\dot{u}_{i,i}^* + d\dot{\boldsymbol{T}}^* = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \end{aligned} \tag{3.5}$$

with the same initial-boundary conditions (3.4).

Our goal in this section is to derive the continuous dependence on the parameter *K*. If we define

$$w_i = u_i - u_i^*, \quad \theta = \boldsymbol{T} - \boldsymbol{T}^*, \Sigma = \boldsymbol{C} - \boldsymbol{C}^*, \beta = K - K^*, \tag{3.6}$$

then, (w_i,θ,Σ) satisfy the system

$$\begin{aligned} &\rho\ddot{w}_i - \nu\Delta w_i - (\lambda+\nu)w_{j,ji} + \gamma_1\theta_{,i} + \gamma_2\Sigma_{,i} = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \\ &c\dot{\theta} - \beta\Delta\boldsymbol{T} - K^*\Delta\theta + \gamma_1\dot{w}_{i,i} + d\dot{\Sigma} = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \\ &n\dot{\Sigma} - M\Delta\Sigma + \gamma_2\dot{w}_{i,i} + d\dot{\theta} = 0, && \text{in } \boldsymbol{R}\times\{t\ge 0\}, \end{aligned} \tag{3.7}$$

with the initial-boundary conditions

$$
\begin{aligned}
& w_i = 0, \quad \theta = \Sigma = 0 \qquad \text{on } \partial \boldsymbol{D} \times \{t \geq 0\}, \\
& w_i = \dot{w}_i = 0, \quad \theta = \Sigma = 0 \quad \text{in } \boldsymbol{R} \times \{t = 0\}, \\
& w_i = 0, \quad \theta = \Sigma = 0 \qquad \text{on } \boldsymbol{D}_0 \times \{t \geq 0\}.
\end{aligned} \tag{3.8}
$$

We multiply $(3.7)_1$ with w_i and integrate by parts to have

$$
\begin{aligned}
0 &= \int_0^t \int_{\boldsymbol{R}_z} \left[\rho \ddot{w}_i - \nu \Delta w_i - (\lambda+\nu) w_{j,ji} + \gamma_1 \theta_{,i} + \gamma_2 \Sigma_{,i}\right] \dot{w}_i \mathrm{d}x \mathrm{d}\eta \\
&= \frac{\rho}{2} \int_{\boldsymbol{R}_z} \dot{w}_i \dot{w}_i \mathrm{d}x + \frac{\nu}{2} \int_{\boldsymbol{R}_z} w_{i,j} w_{i,j} \mathrm{d}x + \frac{\lambda+\nu}{2} \int_{\boldsymbol{R}_z} w_{i,i}^2 \mathrm{d}x + \gamma_1 \int_0^t \int_{\boldsymbol{R}_z} \theta_{,i} \dot{w}_i \mathrm{d}x \mathrm{d}\eta \\
&\quad + \gamma_2 \int_0^t \int_{\boldsymbol{R}_z} \Sigma_{,i} \dot{w}_i \mathrm{d}x \mathrm{d}\eta + \nu \int_0^t \int_{\boldsymbol{D}_z} w_{i,3} \dot{w}_i \mathrm{d}A \mathrm{d}\eta + (\lambda+\nu) \int_0^t \int_{\boldsymbol{D}_z} w_{j,j} \dot{w}_3 \mathrm{d}A \mathrm{d}\eta.
\end{aligned} \tag{3.9}
$$

Similarly, we have

$$
\begin{aligned}
0 &= \int_0^t \int_{\boldsymbol{R}_z} \left[c\dot{\theta} - \beta \Delta \boldsymbol{T} - K^* \Delta \theta + \gamma_1 w_{i,i} + d\dot{\Sigma}\right] \theta \mathrm{d}x \mathrm{d}\eta \\
&= \frac{c}{2} \int_{\boldsymbol{R}_z} \theta^2 \mathrm{d}x \mathrm{d}\eta + \beta \int_0^t \int_{\boldsymbol{R}_z} T_{,i} \theta_{,i} \mathrm{d}x \mathrm{d}\eta + K^* \int_0^t \int_{\boldsymbol{R}_z} \theta_{,i} \theta_{,i} \mathrm{d}x \mathrm{d}\eta \\
&\quad - \gamma_1 \int_0^t \int_{\boldsymbol{R}_z} \theta_{,i} \dot{w}_i \mathrm{d}x \mathrm{d}\eta + d \int_0^t \int_{\boldsymbol{R}_z} \dot{\Sigma} \theta \mathrm{d}x \mathrm{d}\eta + \beta \int_0^t \int_{\boldsymbol{D}_z} \boldsymbol{T}_{,3} \theta \mathrm{d}A \mathrm{d}\eta \\
&\quad + K^* \int_0^t \int_{\boldsymbol{D}_z} \theta_{,3} \theta \mathrm{d}A \mathrm{d}\eta - \gamma_1 \int_0^t \int_{\boldsymbol{D}_z} \dot{w}_3 \theta \mathrm{d}A \mathrm{d}\eta.
\end{aligned} \tag{3.10}
$$

Combining (2.20), (3.9) and (3.10), we have

$$
\begin{aligned}
\boldsymbol{\Phi}(z,t) &\doteq \frac{\rho}{2} \int_{\boldsymbol{R}_z} \dot{w}_i \dot{w}_i \mathrm{d}x + \frac{\nu}{2} \int_{\boldsymbol{R}_z} w_{i,j} w_{i,j} \mathrm{d}x + \frac{\lambda+\nu}{2} \int_{\boldsymbol{R}_z} w_{i,i}^2 \mathrm{d}x + K^* \int_0^t \int_{\boldsymbol{R}_z} \theta_{,i} \theta_{,i} \mathrm{d}x \mathrm{d}\eta \\
&\quad + M \int_0^t \int_{\boldsymbol{R}_z} \Sigma_{,i} \Sigma_{,i} \mathrm{d}x \mathrm{d}\eta + \frac{1}{2} \int_{\boldsymbol{R}_z} \left[c\theta^2 + 2d\theta\Sigma + n\Sigma^2\right] \mathrm{d}x \mathrm{d}\eta \\
&= -\nu \int_0^t \int_{\boldsymbol{D}_z} w_{i,3} \dot{w}_i \mathrm{d}A \mathrm{d}\eta - (\lambda+\nu) \int_0^t \int_{\boldsymbol{D}_z} w_{j,j} \dot{w}_3 \mathrm{d}A \mathrm{d}\eta - K^* \int_0^t \int_{\boldsymbol{D}_z} \theta_{,3} \theta \mathrm{d}A \mathrm{d}\eta \\
&\quad + \gamma_1 \int_0^t \int_{\boldsymbol{D}_z} \dot{w}_3 \theta \mathrm{d}A \mathrm{d}\eta - M \int_0^t \int_{\boldsymbol{D}_z} \Sigma_{,3} \Sigma \mathrm{d}A \mathrm{d}\eta + \gamma_2 \int_0^t \int_{\boldsymbol{D}_z} \dot{w}_3 \Sigma \mathrm{d}A \mathrm{d}\eta \\
&\quad - \beta \int_0^t \int_{\boldsymbol{D}_z} \boldsymbol{T}_{,3} \theta \mathrm{d}A \mathrm{d}\eta - \beta \int_0^t \int_{\boldsymbol{R}_z} \boldsymbol{T}_{,i} \theta_{,i} \mathrm{d}x \mathrm{d}\eta.
\end{aligned} \tag{3.11}
$$

Now, we define a new function

$$
\begin{aligned}
\boldsymbol{\Gamma}(z,t) &= \int_z^\infty \boldsymbol{\Phi}(\xi,t) \mathrm{d}\xi \\
&= \frac{\rho}{2} \int_{\boldsymbol{R}_z} (\xi - z) \dot{w}_i \dot{w}_i \mathrm{d}x + \frac{\nu}{2} \int_{\boldsymbol{R}_z} (\xi - z) w_{i,j} w_{i,j} \mathrm{d}x + \frac{\lambda+\nu}{2} \int_{\boldsymbol{R}_z} (\xi - z) w_{i,i}^2 \mathrm{d}x \\
&\quad + K^* \int_0^t \int_{\boldsymbol{R}_z} (\xi - z) \theta_{,i} \theta_{,i} \mathrm{d}x \mathrm{d}\eta + M \int_0^t \int_{\boldsymbol{R}_z} (\xi - z) \Sigma_{,i} \Sigma_{,i} \mathrm{d}x \mathrm{d}\eta \\
&\quad + \frac{1}{2} \int_{\boldsymbol{R}_z} (\xi - z) \left[c\theta^2 + 2d\theta\Sigma + n\Sigma^2\right] \mathrm{d}x.
\end{aligned} \tag{3.12}
$$

From (3.11), we have

$$
\begin{aligned}
\boldsymbol{\Gamma}(z,t) &= -\nu \int_0^t \int_{\boldsymbol{R}_z} w_{i,3} \dot{w}_i \mathrm{d}x \mathrm{d}\eta + (\lambda+\nu) \int_0^t \int_{\boldsymbol{R}_z} w_{j,j} \dot{w}_3 \mathrm{d}x \mathrm{d}\eta - K^* \int_0^t \int_{\boldsymbol{R}_z} \theta_{,3} \theta \mathrm{d}x \mathrm{d}\eta \\
&\quad + \gamma_1 \int_0^t \int_{\boldsymbol{R}_z} \dot{w}_3 \theta \mathrm{d}x \mathrm{d}\eta - M \int_0^t \int_{\boldsymbol{R}_z} \Sigma_{,3} \Sigma \mathrm{d}x \mathrm{d}\eta + \gamma_2 \int_0^t \int_{\boldsymbol{R}_z} \dot{w}_3 \Sigma \mathrm{d}x \mathrm{d}\eta \\
&\quad - \beta \int_0^t \int_{\boldsymbol{R}_z} \boldsymbol{T}_{,3} \theta \mathrm{d}x \mathrm{d}\eta - \beta \int_0^t \int_{\boldsymbol{R}_z} (\xi - z) \boldsymbol{T}_{,i} \theta_{,i} \mathrm{d}x \mathrm{d}\eta.
\end{aligned} \tag{3.13}
$$

Following the same procedures which have been used in Section 3 (see

(3.6)-(3.10)), we have

$$\Gamma(z,t) \le m_1(t)\left(-\frac{\partial \Gamma}{\partial z}(z,t)\right) - \beta\int_0^t\int_{R_z} \boldsymbol{T}_{,3}\theta \mathrm{d}x\mathrm{d}\eta - \beta\int_0^t\int_{R_z}(\xi - z)\boldsymbol{T}_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta. \quad (3.14)$$

By Hölder and the AG mean inequalities, from (3.14) we have

$$\begin{aligned}\Gamma(z,t) \le{}& m_1(t)\left(-\frac{\partial \Gamma}{\partial z}(z,t)\right) + \frac{\beta^2}{2K^* m_1(t)\lambda_1}\int_0^t\int_{R_z}\boldsymbol{T}_{,i}\boldsymbol{T}_{,i}\mathrm{d}x\mathrm{d}\eta \\ &+ \frac{m_1(t)K^*}{2}\int_0^t\int_{R_z}\theta_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta + \frac{\beta^2}{2K^*}\int_0^t\int_{R_z}(\xi - z)\boldsymbol{T}_{,i}\boldsymbol{T}_{,i}\mathrm{d}x\mathrm{d}\eta \\ &+ \frac{K^*}{2}\int_0^t\int_{R_z}(\xi - z)\theta_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta.\end{aligned} \quad (3.15)$$

In view of the definitions of the functions $\Phi(z,t)$ and $\Psi(z,t)$, we introduce the functions

$$\begin{aligned}\tilde{\Phi}(z,t) ={}& \frac{\rho}{2}\int_{R_z}\dot{w}_i\dot{w}_i\mathrm{d}x + \frac{\nu}{2}\int_{R_z} w_{i,j}w_{i,j}\mathrm{d}x + \frac{\lambda+\nu}{2}\int_{R_z} w_{i,i}^2\mathrm{d}x \\ &+ \frac{K^*}{2}\int_0^t\int_{R_z}\theta_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta + M\int_0^t\int_{R_z}\Sigma_{,i}\Sigma_{,i}\mathrm{d}x\mathrm{d}\eta \\ &+ \frac{1}{2}\int_{R_z}\left[c\theta^2 + 2d\theta\Sigma + n\Sigma^2\right]\mathrm{d}x\mathrm{d}\eta,\end{aligned} \quad (3.16)$$

and

$$\begin{aligned}\tilde{\Gamma}(z,t) ={}& \int_z^\infty \tilde{\Phi}(\xi,t)\mathrm{d}\xi \\ ={}& \frac{\rho}{2}\int_{R_z}(\xi - z)\dot{w}_i\dot{w}_i\mathrm{d}x + \frac{\nu}{2}\int_{R_z}(\xi - z)w_{i,j}w_{i,j}\mathrm{d}x \\ &+ \frac{\lambda+\nu}{2}\int_{R_z}(\xi - z)w_{i,i}^2\mathrm{d}x + \frac{K^*}{2}\int_0^t\int_{R_z}(\xi - z)\theta_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta \\ &+ M\int_0^t\int_{R_z}(\xi - z)\Sigma_{,i}\Sigma_{,i}\mathrm{d}x\mathrm{d}\eta \\ &+ \frac{1}{2}\int_{R_z}(\xi - z)\left[c\theta^2 + 2d\theta\Sigma + n\Sigma^2\right]\mathrm{d}x\mathrm{d}\eta.\end{aligned} \quad (3.17)$$

Then, inequality (3.15) may be rewritten as

$$\begin{aligned}\tilde{\Gamma}(z,t) \le{}& 3m_1(t)\left(-\frac{\partial \tilde{\Gamma}}{\partial z}(z,t)\right) + \frac{\beta^2}{2K^* m_1(t)\lambda_1}\int_0^t\int_{R_z} T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta \\ &+ \frac{\beta^2}{2K^*}\int_0^t\int_{R_z}(\xi - z)\boldsymbol{T}_{,i}\boldsymbol{T}_{,i}\mathrm{d}x\mathrm{d}\eta.\end{aligned} \quad (3.18)$$

Combining (2.1) and (2.2), we know

$$\int_0^t\int_{R_z}\boldsymbol{T}_{,i}\boldsymbol{T}_{,i}\mathrm{d}x\mathrm{d}\eta \le \frac{1}{K}\boldsymbol{E}(0,t)\mathrm{e}^{-\frac{1}{m_1(t)}z}. \quad (3.19)$$

If we set

$$\hat{\boldsymbol{E}}(z,t) = \int_z^\infty \boldsymbol{E}(\xi,t)\mathrm{d}\xi, \quad (3.20)$$

from (2.2), we have

$$\hat{\boldsymbol{E}}(z,t) \le m_1(t)\boldsymbol{E}(0,t)\mathrm{e}^{-\frac{1}{m_1(t)}z}. \quad (3.21)$$

From the definition of $\boldsymbol{E}(z,t)$ and (3.20), we have

$$\hat{\boldsymbol{E}}(z,t)=\frac{1}{2}\rho\int_{\boldsymbol{R}_z}(\xi-z)\dot{u}_i\dot{u}_i\mathrm{d}x+\frac{\nu}{2}\int_{\boldsymbol{R}_z}(\xi-z)u_{i,j}u_{i,j}\mathrm{d}x$$
$$+\frac{\lambda+\nu}{2}\int_{\boldsymbol{R}_z}(\xi-z)u_{i,i}^2\mathrm{d}x+K\int_0^t\int_{\boldsymbol{R}_z}(\xi-z)\boldsymbol{T}_{,j}\boldsymbol{T}_{,j}\mathrm{d}A\mathrm{d}\eta \tag{3.22}$$
$$+M\int_0^t\int_{\boldsymbol{R}_z}(\xi-z)\boldsymbol{C}_{,j}\boldsymbol{C}_{,j}\mathrm{d}A\mathrm{d}\eta+\int_{\boldsymbol{R}_z}(\xi-z)\left[\frac{c}{2}\boldsymbol{T}^2+d\boldsymbol{C}\boldsymbol{T}+\frac{n}{2}\boldsymbol{C}^2\right]\mathrm{d}x.$$

So, we have the following inequality

$$\int_0^t\int_{\boldsymbol{R}_z}(\xi-z)\boldsymbol{T}_{,j}\boldsymbol{T}_{,j}\mathrm{d}A\mathrm{d}\eta\le\frac{1}{K}m_1(t)\boldsymbol{E}(0,t)\mathrm{e}^{-\frac{1}{m_1(t)}z}. \tag{3.23}$$

Inserting (3.19) and (3.23) back into (3.18), we have

$$\tilde{\Gamma}(z,t)\le m_1(t)\left(-\frac{\partial\tilde{\Gamma}}{\partial z}(z,t)\right)+\beta^2m_3(t)\mathrm{e}^{-\frac{1}{m_1(t)}z}, \tag{3.24}$$

where

$$m_3(t)=\frac{\boldsymbol{E}(0,t)}{2KK^*m_1(t)\lambda_1}+\frac{\boldsymbol{E}(0,t)}{2KK^*}m_1(t). \tag{3.25}$$

For any fixed $t_1>0$, setting $k_1=3m_1(t)$ and $k_2=\frac{1}{m_1(t)}$, from (3.24) we obtain

$$\frac{\partial\tilde{\boldsymbol{\Phi}}}{\partial z}(z,t_1)+\int_z^\infty\tilde{\boldsymbol{\Phi}}(\xi,t_1)\mathrm{d}\xi\le k_1\tilde{\boldsymbol{\Phi}}(z,t_1)+\beta^2m_3\mathrm{e}^{-k_2z}, \tag{3.26}$$

where we have used the fact $\frac{\partial\tilde{\boldsymbol{\Phi}}}{\partial z}(z,t_1)<0$. To get the result we want, we let

$$\boldsymbol{\Pi}(z,t_1)=\mathrm{e}^{-k_1z}\tilde{\boldsymbol{\Phi}}(z,t_1)+h\int_z^\infty\mathrm{e}^{-k_1\xi}\tilde{\boldsymbol{\Phi}}(\xi,t_1)\mathrm{d}\xi. \tag{3.27}$$

Thus, inequality (3.26) may be rewritten as

$$\frac{\partial\boldsymbol{\Pi}}{\partial z}(z,t_1)+h\boldsymbol{\Pi}(z,t_1)\le\beta^2m_3\mathrm{e}^{-(k_1+k_2)z}, \tag{3.28}$$

provided h satisfies the quadratic equation

$$h^2-k_1h-1=0. \tag{3.29}$$

We make the choice of

$$h=h_0=\frac{k_1+\sqrt{k_1^2+4}}{2}. \tag{3.30}$$

For this choice of h, to integrate (3.28), we have to consider the following two cases:

1) If $h_0-k_1-k_2=0$, an integration of (3.28) leads to

$$\boldsymbol{\Pi}(z,t_1)\le\boldsymbol{\Pi}(0,t_1)\mathrm{e}^{-h_0z}+\beta^2m_3z\mathrm{e}^{-h_0z}. \tag{3.31}$$

In light of (3.27), we have

$$\tilde{\boldsymbol{\Phi}}(z,t_1)\le\boldsymbol{\Pi}(0,t_1)\mathrm{e}^{-(h_0-k_1)z}+\beta^2m_3\mathrm{e}^{-(h_0-k_1)z}. \tag{3.32}$$

2) If $h_0-k_1-k_2\ne0$, an integration of (3.28) leads to

$$\Pi(z,t_1) \le \Pi(0,t_1)\mathrm{e}^{-h_0 z} + \frac{\beta^2 m_3}{h_0-k_1-k_2}\left[\mathrm{e}^{(h_0-k_1-k_2)z}-1\right]\mathrm{e}^{-h_0 z}. \quad (3.33)$$

It is easy to proof that the second term on the right of (3.33) is positive either $h_0-k_1-k_2>0$ or $h_0-k_1-k_2<0$. In view of (3.27), we have

$$\tilde{\Phi}(z,t_1) \le \left[\Pi(0,t_1) - \frac{\beta^2 m_3}{h_0-k_1-k_2}\right]\mathrm{e}^{-(h_0-k_1)z} + \frac{\beta^2 m_3}{h_0-k_1-k_2}\mathrm{e}^{-k_2 z}. \quad (3.34)$$

In order to make inequalities (3.32) and (3.34) explicit, we need bound for $\Pi(0,t_1)$. From the definition of $\Pi(z,t_1)$ in (3.27), we may write

$$\Pi(0,t_1) = \tilde{\Phi}(0,t_1) + h\int_0^\infty \mathrm{e}^{-k_1\xi}\tilde{\Phi}(\xi,t_1)\mathrm{d}\xi \le \tilde{\Phi}(0,t_1) + h\tilde{\Gamma}(0,t_1). \quad (3.35)$$

From (3.35), to bound $\Pi(0,t_1)$ we only need to bound $\tilde{\Phi}(0,t_1)$ and $\tilde{\Psi}(0,t_1)$. From (3.11), we have

$$\Phi(0,t_1) = -\beta\int_0^t\int_R T_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta \le \frac{\beta^2}{2K^*}\int_0^t\int_R T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta + \frac{K^*}{2}\int_0^t\int_R \theta_{,i}\theta_{,i}\mathrm{d}x\mathrm{d}\eta \quad (3.36)$$

Using (3.12) and (3.16), we have

$$\tilde{\Phi}(0,t_1) \le \frac{\beta^2}{2KK^*}E(0,t_1). \quad (3.37)$$

From (3.24) and using (3.37), we can get

$$\tilde{\Psi}(0,t_1) \le \frac{m_1\beta^2}{2KK^*}E(0,t_1). \quad (3.38)$$

Combining (3.35), (3.37) and (3.38), we have

$$\Pi(0,t_1) \le \beta^2 m_4, \quad (3.39)$$

where $m_4 = \frac{E(0,t_1)}{2KK^*} + \frac{hm_1E(0,t_1)}{2KK^*}$. Combining the above discussions, we can conclude:

If $h_0-k_1-k_2=0$, we have

$$\tilde{\Phi}(z,t_1) \le \beta^2 m_4 \mathrm{e}^{-(h_0-k_1)z} + \beta^2 m_3 \mathrm{e}^{-(h_0-k_1)z}. \quad (3.40)$$

If $h_0-k_1-k_2\ne 0$, we have

$$\tilde{\Phi}(z,t_1) \le \beta^2\left[m_4 - \frac{m_3}{h_0-k_1-k_2}\right]\mathrm{e}^{-(h_0-k_1)z} + \frac{\beta^2 m_3}{h_0-k_1-k_2}\mathrm{e}^{-k_2 z}. \quad (3.41)$$

Inequalities (3.40) and (3.41) exhibit not only exponential decay in *z*, but also show that the amplitude terms in (3.40) and (3.41) become small as $K \to K^*$.

4. Conclusion

In view of the Equations (1.2) and (1.3), we may also obtain the continuous dependence on the coefficient *M* by employing the methods which have been used in Section 3. Our method is also valid to study other equations. In the future, we will use the method proposed in this paper to study the structural stability for the fluid flow in porous media. We think we will get some interesting results.

Funding

The work was supported by the science foundation of Huashang College Guangdong University of Finance & Economics (Grant # 2019HSDS28).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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