Journal of Applied Mathematics and Physics, 2020, 8, 727-736
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Spectrum of a Class of Difference Operators with Indefinite Weights

Congmin Yang, Yunlan Gao*, Kang Sun

College of Sciences, Inner Mongolia University of Technology, Hohhot, China
Email: *gaogaoyyyyy@sina.com

How to cite this paper: Yang, C.M., Gao, Y.L. and Sun, K. (2020) Spectrum of a Class of Difference Operators with Indefinite Weights. *Journal of Applied Mathematics and Physics*, **8**, 727-736.
https://doi.org/10.4236/jamp.2020.84056

Received: March 7, 2020
Accepted: April 21, 2020
Published: April 24, 2020

Abstract

In this study, we use analytical methods and Sylvester inertia theorem to research a class of second order difference operators with indefinite weights and coupled boundary conditions. The eigenvalue problem with sign-changing weight has lasted a long time. The number of eigenvalues and the number of sign changes of the corresponding eigenfunctions of discrete equations under different boundary conditions are mainly studied. For the discrete Sturm-Liouville problems, similar conclusions about the properties of eigenvalues and the number of sign changes of the corresponding eigenfunctions are obtained under different boundary conditions, such as periodic boundary conditions, antiperiodic boundary conditions and separated boundary conditions etc. The purpose of this paper is to extend the similar conclusion to the coupled boundary conditions, which is of great significance to the perfection of the theory of the discrete Sturm-Liouville problems. We came to the following conclusions: first, the eigenvalues of the problem are real and single, the number of the positive eigenvalues is equal to the number of positive elements in the weight function, and the number of negative eigenvalues is equal to the number of negative elements in the weight function. Second, under some conditions, we obtain the sign change of the eigenfunction corresponding to the j-th positive/negative eigenvalue.

Keywords

Spectrum, Difference Operator, Coupled Boundary Conditions, Indefinite Weight

1. Introduction

Spectral theory for Sturm-Liouville boundary value problems has important physical meaning and practical significance. The study of discrete Sturm-Liouville boundary value problems has a very important role in problems when it can be described by discrete and continuous mathematical models. In this paper, we deal with eigenvalue problems for the discrete Sturm-Liouville equations.

In this paper, the following equations and boundary conditions are studied.

$$\Delta\left[p(t-1)\Delta u(t-1)\right]-q(t)u(t)+\lambda m(t)u(t)=0,\quad t\in\Gamma \tag{1}$$

$$\begin{pmatrix} p(0)\Delta u(0) \\ -p(T)\Delta u(T) \end{pmatrix} = \boldsymbol{K}\begin{pmatrix} u(0) \\ u(T) \end{pmatrix}. \tag{2}$$

λ is the spectrum parameter, $\Gamma=\{1,2,\cdots,T\}$ ($T>1$ is an integer), the real matrix $\boldsymbol{K}=\begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$ is positive definite and $k_{12}\neq 0$. The real function $p(t)$, $q(t)$ and $m(t)$ are all defined on Γ, and

$$p(1)=0,\quad p(t)>0,\quad t\in\{0,2,3,\cdots,T\},$$

$$q(t)\geq 0,\quad \forall t\in\Gamma.$$

The weight function $m(t)\neq 0$ on Γ and changes sign. It's easy to know

$$T^{+}+T^{-}=T.$$

In the introduction part, we also mention some other boundary conditions listed below,

$$u(0)-\alpha u(1)=u(T+1)-\beta u(T)=0, \tag{3}$$

$$u(0)=u(T),\quad u(1)=u(T+1) \tag{4}$$

$$u(0)+u(T)=0,\quad u(1)+u(T+1)=0 \tag{5}$$

$$\Delta u(0)=\Delta u(T)=0 \tag{6}$$

$$\alpha u(0)-\beta\Delta u(0)=0,\quad \gamma u(T+1)+\delta\Delta u(T)=0, \tag{7}$$

The study of the eigenvalue problem with sign-changing weight has lasted a long time. In 1914, Bôcher [1] studied such problems in the continuous case. Subsequently, many scholars have studied the differential operator problem with indefinite weight function and obtained a series of important results (for example [2] [3]).

However, there are few results on the spectra of discrete second-order linear eigenvalue problems when the weight function $m(t)$ changes its sign on *T*.

In 2007, Ji and Yang [4] [5] studied the structure of the eigenvalues of problem (1) (3) when the weight function $m(t)$ changes its sign, and they obtained that the number of positive eigenvalues is equal to the number of positive elements in the weight function, and the number of negative eigenvalues is equal to the number of negative elements in the weight function.

In 2008, Ji and Yang [6] discussed the eigenvalues of (1) (4) and (1) (5); by using the matrix theory, they got a very interesting result: the numbers of positive eigenvalues are equal to the numbers of positive elements in the weight function, and the numbers of negative eigenvalues are equal to the numbers of negative elements in the weight function.

In 2013, Ma, Gao and Lu [7] discussed the spectra of the discrete second-order Neumann Eigenvalue problem (1) (6), By the analytical methods, he not only gives the properties of the eigenvalues, but also gives the number of sign changes of the eigenfunction corresponding to the *j*-th positive/negative eigenvalue.

In 2015, C. Gao, R. Ma [8] studied the problem (1) (4), they find that the problems have T real eigenvalues (including the multiplicity). Furthermore, the numbers of positive eigenvalues are equal to the numbers of positive elements in the weight function, and the numbers of negative eigenvalues are equal to the numbers of negative elements in the weight function.

In 2018, R. Ma, C. Gao, Y. Lu [9] studied the problem (1) (7), by the Sylvester inertia theorem, they obtained the following conclusion:

a) (1) (7) has real and simple eigenvalues, which can be ordered as follows

$$\lambda_{T,T^-}^- < \lambda_{T,T^--1}^- < \cdots < \lambda_{T,1}^- < 0 < \lambda_{T,1}^+ < \lambda_{T,2}^+ < \cdots < \lambda_{T,T^+}^+$$

b) every eigenfunction $\Psi_{T,k}^{v}$ corresponding to the eigenvalue $\lambda_{T,k}^{v}$ changes its sign exactly $k-1$ times.

It is the purpose of this paper to establish the discrete analogue of the above conclusion.

In this paper, we will study the problems (1) (2) under coupled boundary conditions, using analytical methods and Sylvester inertia theorem, two conclusions are given. First, the eigenvalues of the questions are real and single, the number of positive eigenvalues is equal to the number of positive elements in the weight function, and the number of negative eigenvalues is equal to the number of negative elements in the weight function. Second, under some conditions, we obtain sign changes of the eigenfunction corresponding to the *j*-th positive/negative eigenvalue. We discuss different situations and get a series of important conclusions.

2. Theory

2.1. Main Theorem

Theorem 1. The question (1) (2) has T real and simple eigenvalues. And the number of positive eigenvalues is equal to the number of positive elements in the weight function, the number of negative eigenvalues is equal to the number of negative elements in the weight function. And the eigenvalues can be sorted as follows:

$$\lambda_{T,T^-}^- < \lambda_{T,T^--1}^- < \cdots < \lambda_{T,1}^- < 0 < \lambda_{T,1}^+ < \lambda_{T,2}^+ < \cdots < \lambda_{T,T^+}^+ .$$

Theorem 2. In this paper, let $\lambda_{T,i}^{v}$ be the eigenvalue of (1) (2), and $u\left(t,\lambda_{T,i}^{v}\right)$ be the eigenfunction corresponding to the eigenvalue $\lambda_{T,i}^{v}$, where $v\in\{+,-\}$. When $k_{22}-p(T)<0$, we will give the sign-change times in different cases:

Case 1: If $m(1)>0$ and $\left\|D^{-1}J\right\|<\dfrac{c(1)}{m(1)}$ hold, or $m(1)<0$ hold, where $\left\|D^{-1}J\right\|$ is the matrix norm of $D^{-1}J$. Then

1) When $k_{12} > 0$

If i is even, the number of sign changes of $u\left(t,\lambda_{T,i}^{+}\right)$ is $i-1$; if i is odd, the number of sign changes of $u\left(t,\lambda_{T,i}^{+}\right)$ is i.

2) When $k_{12} < 0$

If i is even, the number of sign changes of $u\left(t,\lambda_{T,i}^{+}\right)$ is i; if i is odd, the number of sign changes of $u\left(t,\lambda_{T,i}^{+}\right)$ is $i-1$.

Case 2: If $m(1) < 0$ and $\left\|D^{-1}J\right\| < -\dfrac{c(1)}{m(1)}$ hold, or $m(1) > 0$ hold. Then

1) When $k_{12} > 0$

If i is even, the number of sign changes of $u\left(t,\lambda_{T,i}^{-}\right)$ is $i-1$; if i is odd, the number of sign changes of $u\left(t,\lambda_{T,i}^{-}\right)$ is i.

2) When $k_{12} < 0$

If i is even, the number of sign changes of $u\left(t,\lambda_{T,i}^{-}\right)$ is i; if i is odd, the number of sign changes of $u\left(t,\lambda_{T,i}^{-}\right)$ is $i-1$.

2.2. Lemma and Proof of the Theorem

Let $c(1) = q(1) + \dfrac{k_{11}p(0)}{k_{11}+p(0)}$, $c(t) = p(t) + p(t-1) + q(t)$, $t = 2,3,\cdots,T-1$, $c(T) = p(T-1) + q(T) - \dfrac{k_{12}^2}{k_{11}+p(0)} + k_{22}$, then problem (1) (2) can be transformed into a matrix equation

$$\boldsymbol{Ju} = \lambda \boldsymbol{Du},$$

where

$$\boldsymbol{J} = \begin{pmatrix} c(1) & 0 & 0 & 0 & \cdots & 0 & 0 & \dfrac{k_{12}p(0)}{k_{11}+p(0)} \\ 0 & c(2) & -p(2) & 0 & \cdots & 0 & 0 & 0 \\ 0 & -p(2) & c(3) & -p(3) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -p(T-2) & c(T-1) & -p(T-1) \\ \dfrac{k_{12}p(0)}{k_{11}+p(0)} & 0 & 0 & 0 & \cdots & 0 & -p(T-1) & c(T) \end{pmatrix},$$

$$\boldsymbol{D} = diag\left(m(1), m(2), \cdots, m(T)\right).$$

We can know $\boldsymbol{J}$ and $\boldsymbol{J}_j$ ($j = 1,2,\cdots,T-1$) all are positive definite matrixes. In fact, for every $\boldsymbol{X} = (x_1, x_2, \cdots, x_T)$, where $x_i \in R$, we have

$$\boldsymbol{XJX}^{\mathrm{T}} = \sum_{i=2}^{T-1} p(i)(x_{i+1}-x_i)^2 + \sum_{i=1}^{T} q(i)x_i^2 + \frac{|K|}{k_{11}+p(0)}x_T^2 + \frac{p(0)\left[(k_{11}x_1 + k_{12}x_T)^2 + |K|x_T^2\right]}{(k_{11}+p(0))k_{11}} \ge 0$$

If $\boldsymbol{XJX}^{\mathrm{T}} = 0$, then $\boldsymbol{X} = 0$, so $\boldsymbol{J}$ is a positive definite matrix. Similarly, all matrixes $\boldsymbol{J}_j$ ($j = 1,2,\cdots,T-1$) are positive definite matrixes.

Let $Q_j(\lambda)$ denote the *j*-th principal subdeterminant of $\boldsymbol{J}-\lambda\boldsymbol{D}$ deleting the first row and the first column, suppose that $Q_0(\lambda)=1$, $Q_T(\lambda)=\det(\boldsymbol{J}-\lambda\boldsymbol{D})$, and

$$Q_0(\lambda)=1; \tag{8}$$

$$Q_1(\lambda)=c(2)-\lambda m(2); \tag{9}$$

$$Q_j(\lambda)=\left(c(j+1)-\lambda m(j+1)\right)Q_{j-1}(\lambda)-p(j)^2 Q_{j-2}(\lambda);\quad j=2,3,\cdots,T-1, \tag{10}$$

$$Q_T(\lambda)=\left(c(1)-\lambda m(1)\right)Q_{T-1}(\lambda)-\left(\frac{k_{12}p(0)}{k_{11}+p(0)}\right)^2 Q_{T-2}(\lambda). \tag{11}$$

As we know, finding the eigenvalues of (1) (2) is equivalent to finding the zeros of $Q_T(\lambda)$.

For $j\in\{1,2,\cdots,T-1\}$, let j^+ be the number of the elements in $\{m(i)\mid m(i)>0\}$ for some $i\in\{2,\cdots,j+1\}$, and j^- be the number of the elements in $\{m(i)\mid m(i)<0\}$ for some $i\in\{2,\cdots,j+1\}$.

Lemma 1. For $j\in\{1,2,\cdots,T\}$, we have

$$\lim_{\lambda\to-\infty}(-1)^{j^-}Q_j(\lambda)=+\infty,\quad \lim_{\lambda\to+\infty}(-1)^{j^+}Q_j(\lambda)=+\infty.$$

Proof. For $j\in\{1,2,\cdots,T-1\}$, we have

$$Q_j(\lambda)=m(2)\cdots m(j+1)(-\lambda)^j+O\left(\lambda^{j-1}\right),$$

then

$$\begin{aligned}\lim_{\lambda\to-\infty}(-1)^{j^-}Q_j(\lambda)&=\lim_{\lambda\to-\infty}(-1)^{j^-}\left[m(2)\cdots m(j+1)(-\lambda)^j+O\left(\lambda^{j-1}\right)\right]\\&=\lim_{\lambda\to-\infty}\left[(-1)^{j^-}m(2)\cdots m(j+1)(-\lambda)^j+O\left(\lambda^{j-1}\right)\right]\\&=+\infty\end{aligned}$$

when $j=T$, we have

$$Q_T(\lambda)=m(1)m(2)\cdots m(T)(-\lambda)^T+O\left(\lambda^{T-1}\right),$$

$$\begin{aligned}\lim_{\lambda\to-\infty}(-1)^{T^-}Q_T(\lambda)&=\lim_{\lambda\to-\infty}(-1)^{T^-}\left[m(1)m(2)\cdots m(T)(-\lambda)^T+O\left(\lambda^{T-1}\right)\right]\\&=\lim_{\lambda\to-\infty}\left[(-1)^{T^-}m(1)m(2)\cdots m(T)(-\lambda)^T+O\left(\lambda^{T-1}\right)\right]\\&=+\infty\end{aligned}$$

Similarly, we can get that $\lim_{\lambda\to+\infty}(-1)^{j^+}Q_j(\lambda)=+\infty$ ($1\le j\le T$).

Lemma 2. For $j\in\{1,2,\cdots,T\}$, the roots of $Q_j(\lambda)=0$ are real. Moreover, $Q_j(\lambda)=0$ has j^+ positive roots and j^- negative roots.

Lemma 3. Two continuous polynomials $Q_{j-1}(\lambda)$ and $Q_j(\lambda)$ have no common zeros, for $j\in\{1,2,\cdots,T-1\}$

The proof of lemma 2 and Lemma 3 are similar to the proof of lemma 2 and Lemma 3 of [9] respectively.

Lemma 4. Suppose that $\lambda=\lambda_0$ is a root of $Q_j(\lambda)=0$, then we have $Q_{j-1}(\lambda_0)Q_{j+1}(\lambda_0)<0$, for $j\in\{1,2,\cdots,T-1\}$.

Proof. For $j=1,2,\cdots,T-2$, since $Q_j(\lambda_0)=0$, by lemma 3, we have $Q_{j-1}(\lambda_0)\neq 0$, combining this with (8)-(10), we infer that

$$Q_{j-1}(\lambda_0)Q_{j+1}(\lambda_0)=-p(j+1)^2 Q_{j-1}(\lambda_0)^2<0.$$

when $j=T-1$, since $Q_{T-1}(\lambda_0)=0$, by lemma 3, we have $Q_{T-2}(\lambda_0)\neq 0$, combining this with (11), we infer that

$$Q_{T-2}(\lambda_0)Q_T(\lambda_0)=-\left(\frac{k_{12}p(0)}{k_{11}+p(0)}\right)^2 Q_{T-2}(\lambda_0)^2<0.$$

The proof is completed.

Lemma 5. For $j=1,2,\cdots,T$, the roots of $Q_j(\lambda)=0$ is simple. The positive roots of $Q_j(\lambda)=0$ and $Q_{j+1}(\lambda)=0$ separate one another and the negative roots of $Q_j(\lambda)=0$ and $Q_{j+1}(\lambda)=0$ separate one another.

Proof. 1) When $j=1$,

From (9), we have $Q_1(\lambda)=c(2)-\lambda m(2)$, if $m(2)>0$, then $j=1$, $j^+=1$, $j^-=0$ and $\lambda_{1,1}^+=\dfrac{c(2)}{m(2)}>0$. If $m(2)<0$, then $j=1$, $j^+=0$, $j^-=1$ and $\lambda_{1,1}^-=\dfrac{c(2)}{m(2)}<0$.

From (10), $Q_2(\lambda)=\left(c(3)-\lambda m(3)\right)\left(c(2)-\lambda m(2)\right)-p(2)^2$, therefore $Q_2(\lambda)=0$ has two different roots:

$$\lambda_1=\frac{m(2)c(3)+m(3)c(2)-\sqrt{\left(m(2)c(3)-m(3)c(2)\right)^2+4p(2)^2 m(3)m(2)}}{2m(2)m(3)},$$

$$\lambda_2=\frac{m(2)c(3)+m(3)c(2)+\sqrt{\left(m(2)c(3)-m(3)c(2)\right)^2+4p(2)^2 m(3)m(2)}}{2m(2)m(3)}.$$

If $m(2)>0,m(3)>0$, then $j=2$, $j^+=2$, $j^-=0$, by calculation, we know that $0<\lambda_1<\lambda_2$, let $\lambda_{2,1}^+=\lambda_1$ and $\lambda_{2,2}^+=\lambda_2$, we have $0<\lambda_{2,1}^+<\lambda_{1,1}^+<\lambda_{2,2}^+$.

If $m(2)>0,m(3)<0$, then $j=2$, $j^+=1$, $j^-=1$, we know that $\lambda_2<0<\lambda_1$, let $\lambda_{2,1}^+=\lambda_1$ and $\lambda_{2,1}^-=\lambda_2$, we have $\lambda_{2,1}^-<0<\lambda_{2,1}^+<\lambda_{1,1}^+$.

If $m(2)<0,m(3)>0$, then $j=2$, $j^+=1$, $j^-=1$, we know that $\lambda_2<0<\lambda_1$, let $\lambda_{2,1}^+=\lambda_1$ and $\lambda_{2,1}^-=\lambda_2$, we have $\lambda_{1,1}^-<\lambda_{2,1}^-<0<\lambda_{2,1}^+$.

If $m(2)<0,m(3)<0$, then $j=2$, $j^+=0$, $j^-=2$, we know that $\lambda_1<\lambda_2<0$, let $\lambda_{2,2}^-=\lambda_1$ and $\lambda_{2,1}^-=\lambda_2$, we have $\lambda_{2,2}^-<\lambda_{1,1}^-<\lambda_{2,1}^-<0$.

Thus $j=1$, the result holds.

2) When $j=2,\cdots,T-1$, the proof is similar to Lemma 5 of [9].

From the above five lemmas, we can get Theorem 1.

Lemma 6. Let $\omega(\lambda)$ be the sign-change times of $\left\{Q_0(\lambda),Q_1(\lambda),\cdots,Q_{T-1}(\lambda)\right\}$, then for $i\in\left\{2,\cdots,(T-1)^+\right\}$, we have that

$$\lim_{\lambda\to\lambda_{T-1,i-1}^+-0}\omega(\lambda)=i-1,\quad \lim_{\lambda\to\lambda_{T-1,i}^+ +0}\omega(\lambda)=i-1,$$

where $\lambda\to C-0$ means that $\lambda\to C$ from left hand side of *C*, and $\lambda\to C+0$

means that $\lambda \to C$ from right hand side of *C*.

Proof. This proof is similar to the lemma 6 of [9].

Note. From the proof process, we can see that $\omega(\lambda)$ is a nondecreasing function.

Lemma 7. Let $u(.,\lambda)$ be the eigenfunction of (1) (2). When $u(2,\lambda)=1$, we have that

$$Q_k(\lambda)=p(2)\cdots p(k+1)u(k+2),\quad k=1,2,\cdots,T-2, \tag{12}$$

$$Q_{T-1}(\lambda)=-p(2)\cdots p(T-1)\frac{k_{12}p(0)}{k_{11}+p(0)}u(1). \tag{13}$$

Proof. This proof is similar to the lemma 7 of [9].

Finally, we prove theorem 2.

Proof. We only prove case 1. The proof of case 2 is similar.

From (12), we know that the sign of $u(k,\lambda_{T,i}^+)$ is equal to the sign of $Q_{k-2}(\lambda_{T,i}^+)$, for $k=2,\cdots,T$, so the sign-change times of $\{u(2,\lambda_{T,i}^+),\cdots,u(T,\lambda_{T,i}^+)\}$ is equal to the number of sign changes of $\{Q_0(\lambda_{T,i}^+),Q_1(\lambda_{T,i}^+),\cdots,Q_{T-2}(\lambda_{T,i}^+)\}$.

From lemma 5, we have $\lambda_{T-1,i-1}^+<\lambda_{T,i}^+<\lambda_{T-1,i}^+$. Combining this with lemma 6 and note, we infer that

$$\omega(\lambda_{T,i}^+)=i-1. \tag{14}$$

i) If $m(1)>0$ and $\left\|D^{-1}J\right\|<\frac{c(1)}{m(1)}$ hold,

Since $\left\|D^{-1}J\right\|<\frac{c(1)}{m(1)}$, by the theory of matrices, we get that $\rho(D^{-1}J)\le\left\|D^{-1}J\right\|$, therefore, $\lambda_{T,i}^+<\frac{c(1)}{m(1)}$, that is $c(1)-\lambda_{T,i}^+m(1)>0$.

From (11), we know

$$\left(c(1)-\lambda_{T,i}^+m(1)\right)Q_{T-1}(\lambda_{T,i}^+)=\left(\frac{k_{12}p(0)}{k_{11}+p(0)}\right)^2 Q_{T-2}(\lambda_{T,i}^+). \tag{15}$$

then by the (15) with $c(1)-\lambda_{T,i}^+m(1)>0$, we have

$$\operatorname{sgn}Q_{T-1}(\lambda_{T,i}^+)=\operatorname{sgn}Q_{T-2}(\lambda_{T,i}^+)=\operatorname{sgn}u(T,\lambda_{T,i}^+), \tag{16}$$

which implies that the sign-change times of $\{Q_0(\lambda_{T,i}^+),Q_1(\lambda_{T,i}^+),\cdots,Q_{T-2}(\lambda_{T,i}^+)\}$ is $i-1$, so the number of sign changes of $\{u(2,\lambda_{T,i}^+),\cdots,u(T,\lambda_{T,i}^+)\}$ is $i-1$.

From lemma 5, we have $\operatorname{sgn}Q_{T-1}(\lambda_{T,i}^+)=(-1)^{i-1}$, for $i=1,2,\cdots,T^+$.

From (2), we have

$$u(0,\lambda_{T,i}^+)=\frac{p(0)u(1,\lambda_{T,i}^+)-k_{12}u(T,\lambda_{T,i}^+)}{k_{11}+p(0)}, \tag{17}$$

$$-p(T)u(T+1,\lambda_{T,i}^+)=k_{12}u(0,\lambda_{T,i}^+)+(k_{22}-p(T))u(T,\lambda_{T,i}^+). \tag{18}$$

1) When $k_{12}>0$

From (13), we have

$$-\operatorname{sgn} u\left(1,\lambda_{T,i}^{+}\right)=\operatorname{sgn} u\left(T,\lambda_{T,i}^{+}\right)=\left(-1\right)^{i-1}, \tag{19}$$

combining (17) with (19), we have

$$\operatorname{sgn} u\left(1,\lambda_{T,i}^{+}\right)=\operatorname{sgn} u\left(0,\lambda_{T,i}^{+}\right)=\left(-1\right)^{i}, \tag{20}$$

due to $k_{22}-p\left(T\right)<0$, by (18) and (20), then

$$\operatorname{sgn} u\left(T+1,\lambda_{T,i}^{+}\right)=\operatorname{sgn} u\left(T,\lambda_{T,i}^{+}\right)=\left(-1\right)^{i-1}.$$

So we can get the sign change.

2) When $k_{12}<0$

From (13), we have

$$\operatorname{sgn} u\left(1,\lambda_{T,i}^{+}\right)=\operatorname{sgn} u\left(T,\lambda_{T,i}^{+}\right)=\left(-1\right)^{i-1}, \tag{21}$$

combining (17) with (21), we have

$$\operatorname{sgn} u\left(1,\lambda_{T,i}^{+}\right)=\operatorname{sgn} u\left(0,\lambda_{T,i}^{+}\right)=\left(-1\right)^{i-1}, \tag{22}$$

due to $k_{22}-p\left(T\right)<0$, by (18) and (22), then

$$\operatorname{sgn} u\left(T+1,\lambda_{T,i}^{+}\right)=\operatorname{sgn} u\left(T,\lambda_{T,i}^{+}\right)=\left(-1\right)^{i-1}.$$

So we can get the sign change.

ii) If $m\left(1\right)<0$ hold, we infer that

$$c\left(1\right)-\lambda_{T,i}^{+}m\left(1\right)>0.$$

combining this with (15), then (16) holds, which implies that the sign-change times of $\left\{Q_0\left(\lambda_{T,i}^{+}\right),Q_1\left(\lambda_{T,i}^{+}\right),\cdots,Q_{T-2}\left(\lambda_{T,i}^{+}\right)\right\}$ is $i-1$, so we have that the number of sign changes of $\left\{u\left(2,\lambda_{T,i}^{+}\right),\cdots,u\left(T,\lambda_{T,i}^{+}\right)\right\}$ is $i-1$.

The rest proof of the section is as same as the corresponding section in part (i).

3. Conclusion

In this paper, we study a class of second order difference operators with indefinite weights and coupled boundary conditions. Firstly, we transform the problem into matrix form. Then, by using the inertia theorem and analytic method, we get some important conclusions about eigenvalues and eigenfunctions (see Theorem 1 and Theorem 2). Our work has great significance to the perfection of the theory of the discrete Sturm-Liouville problems.

Funding

This work was supported by the National Natural Science Foundation of China (11661059) and the Natural Science Foundation of the Inner Mongolia Autonomous Region of China (2017MS(LH)0103).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Bôcher, M. (1914) The Smallest Characteristic Numbers in a Certain Exceptional Case. *Bulletin of the American Mathematical Society*, **21**, 6-9. https://doi.org/10.1090/S0002-9904-1914-02560-1

[2] Anane, A., Chakrone, O. and Moussa, M. (2002) Spectrum of One Dimensional p-Laplacian Operator with Indefinite Weight. *Electronic Journal of Qualitative Theory of Differential Equations*, No. 17, 1-11. https://doi.org/10.14232/ejqtde.2002.1.17

[3] Constantin, A. (1997) A General-Weighted Sturm-Liouville Problem. *Annali della Scuola normale superiore di Pisa*, *Classe di scienze*, **24**, 767-782.

[4] Ji, J. and Yang, B. (2007) Eigenvalue Comparisons for a Class of Boundary Value Problems of Second Order Difference Equations. *Linear Algebra and Its Applications*, **420**, 218-227. https://doi.org/10.1016/j.laa.2006.07.006

[5] Ji, J. and Yang, B. (2007) Eigenvalue Comparisons for Second Order Difference Equations with Neumann Boundary Conditions. *Linear Algebra and Its Applications*, **425**, 171-183. https://doi.org/10.1016/j.laa.2007.03.021

[6] Ji, J. and Yang, B. (2008) Eigenvalue Comparisons for Second Order Difference Equations with Periodic and Antiperiodic Boundary Conditions. *Journal of Applied Mathematics and Computing*, **27**, 307-324. https://doi.org/10.1007/s12190-008-0039-2

[7] Ma, R., Gao, C. and Lu, Y.Q. (2013) Spectrum of Discrete Second-Order Neumann Boundary Value Problems with Sign-Changing Weight. *Abstract and Applied Analysis*, **2013**, Article ID: 280508. https://doi.org/10.1155/2013/280508

[8] Gao, C. and Ma, R. (2015) Eigenvalues of Discrete Linear Second-Order Periodic and Antiperiodic Eigenvalue Problems with Sign-Changing Weight. *Linear Algebra and its Applications*, **467**, 40-56. https://doi.org/10.1016/j.laa.2014.11.002

[9] Ma, R., Gao, C. and Lu, Y. (2018) Spectrum Theory of Second-Order Difference Equations with Indefinite Weight. *Journal of Spectral Theory*, **8**, 971-985. https://doi.org/10.4171/JST/219

Nomenclature

Δ = forward difference operator $\Delta u(t) = u(t+1) - u(t)$

λ = spectrum parameter

$m(t)$ = weight function·

$\Gamma_+ = \{i \in \Gamma \mid m(i) > 0\}$, $\Gamma \setminus \Gamma_+ = \{i \in \Gamma \mid m(i) < 0\}$

T^+ (T^-) = the number of elements in Γ_+ ($\Gamma \setminus \Gamma_+$)

$\boldsymbol{K}, \boldsymbol{J}, \boldsymbol{D}$ = matrix

R = real number field

$\boldsymbol{X} = (x_1, x_2, \cdots, x_T)$ = vector

$Q_T(\lambda)$ = determinant of $\boldsymbol{J} - \lambda \boldsymbol{D}$

$\boldsymbol{J}_j$ = the *j*-th principal submatrix of $\boldsymbol{J}$ deleting the first row and the first column

$\boldsymbol{D}_j$ = the *j*-th principal submatrix of $\boldsymbol{D}$ deleting the first row and the first column

$Q_j(\lambda)$ = *j*-th principal subdeterminant of $\boldsymbol{J} - \lambda \boldsymbol{D}$ deleting the first row and the first column

$\lambda_{i,j}^{v}$ = *j*-th v root of $Q_i(\lambda)$ $v = \{+, -\}$

$\left\| \boldsymbol{D}^{-1} \boldsymbol{J} \right\|$ = matrix norm of $\boldsymbol{D}^{-1} \boldsymbol{J}$

$\rho\left(\boldsymbol{D}^{-1} \boldsymbol{J}\right)$ = spectral radius of $\boldsymbol{D}^{-1} \boldsymbol{J}$

$u\left(t, \lambda_{T,i}^{v}\right)$ = eigenfunction corresponding to the eigenvalue $\lambda_{T,i}^{v}$