Journal of Applied Mathematics and Physics, 2024, 12, 1126-1133
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

On the Equality of Weighted Bajratarević Means to Quasi-Arithmetic Means

Yaxun Yang

College of Mathematics and Physics, Southwest University of Science and Technology, Mianyang, China
Email: 760365643@qq.com

How to cite this paper: Yang, Y.X. (2024) On the Equality of Weighted Bajratarević Means to Quasi-Arithmetic Means. *Journal of Applied Mathematics and Physics*, **12**, 1126-1133.
https://doi.org/10.4236/jamp.2024.124070

Received: March 11, 2024
Accepted: April 19, 2024
Published: April 22, 2024

Abstract

In this paper, we considered the equality problem of weighted Bajraktarević means with weighted quasi-arithmetic means. Using the method of substituting for functions, we first transform the equality problem into solving an equivalent functional equation. We obtain the necessary and sufficient conditions for the equality equation.

Keywords

Bajraktarević Means, Quasi-Arithmetic Means, Equality Problem, Functional Equation

1. Introduction

Let $I \subseteq \mathbb{R}$ be a nonempty open interval. The *weighted quasi-arithmetic mean* $A_\varphi : I^2 \to I$ is defined by:

$$A_{\varphi;\lambda}(x,y) := \varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right), \quad x, y \in I,$$

where $\lambda \in (0,1)$ and $\varphi : I \to \mathbb{R}$ is a continuous strictly monotone function.

Let $t, s \in \mathbb{R}_+$, the *weighted two-variable Bajraktarević mean* $B_{f,g} : I^2 \to I$ is defined by:

$$B_{f,g;t,s}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{tf(x)+sf(y)}{tg(x)+sg(y)}\right), \quad x, y \in I,$$

where $f, g : I \to \mathbb{R}$ are two continuous functions such that g is nowhere zero on I and the ratio function $\frac{f}{g}$ is strictly monotone on I.

The research on the equality of Bajraktarević means has experienced a long history. As early as 1958, Bajraktarević [1] solved the equality of n-variable quasi-arithmetic means with weight function for a fixed $n \geq 3$, and got the necessary and sufficient condition under twice differentiable assumption. Aczél and Daróczy [2] obtained the same result without differentiability conditions when the equality holds for all $n \ge 2$, $n \in \mathbb{N}$. The case of fixed $n = 2$ is much more difficult and allows considerably more solutions. Losonczi [3] got 32 new families of solution under the six-time differentiable supposition. Several new characterizations of the equality of two-variable Bajraktarević means have been obtained by Losonczi, Páles and Zakaria in [4] under the same regularity assumptions. Recently, Páles and Zakaria [5] obtained the same conclusion under only first-order differentiability. Grünwald and Páles [6] considered the equality problem of generalized Bajraktarević means.

We say that two pairs of functions $(f,g): I \to \mathbb{R}^2$ and $(h,k): I \to \mathbb{R}^2$ are *equivalent* if there exist constants a,b,c,d with $ad \ne cd$ such that:

$$h = af + bg, \quad k = cf + dg, \tag{1}$$

and it can be written by $(f,g) \sim (h,k)$.

We will consider the equality problem of weighted Bajraktarević mean to weighted quasi-arithmetic means, that is:

$$B_{G,F;t,s}(x,y) = A_{h,\lambda}(x,y), \quad x,y \in I, \tag{2}$$

Applying G/H to the both sides of (2) and substituting $f := F \circ h^{-1}$, $g := G \circ h^{-1}$ and $\varphi := \frac{g}{f}$, we get an equivalent formulation of (2) as follows:

$$\left(tf(x) + sf(y)\right) \cdot \varphi\left(\lambda x + (1-\lambda) y\right) = tf(x)\varphi(x) + sf(y)\varphi(y), \quad x,y \in I. \tag{3}$$

For the case $\lambda = \frac{1}{2}, t = s$, this equation was considered and solved in [7] under strict monotonicity and continuity of φ and in [8] under continuity of φ, respectively. For the case $\lambda = t = 1 - s$, this equation was solved in [9].

2. Some Necessary Conditions

Lemma 1. *Let* $\varphi: I \to \mathbb{R}$ *be a strictly monotone function,* $f: I \to \mathbb{R}$ *be an arbitrary function, and* $t,s \in \mathbb{R}_+$, $\lambda \in (0,1)$. *Assume that the functional Equation* (3) *holds, then either f is identically zero, or f is nowhere zero, f and* φ *are infinitely many times differentiable and there exists a nonzero constant* $\gamma \in \mathbb{R}$ *such that*:

$$f^{\mu} \varphi' = \gamma, \tag{4}$$

where

$$\mu := \frac{\lambda s + (1-\lambda)t}{\lambda(1-\lambda)(t+s)} \in (0, +\infty). \tag{5}$$

Proof. If *f* is identically zero, then (3) holds. Now, we assume that there exists a point y_0 such that *f* does not vanish at y_0. Then, for $x \in I$ with $x \ne y_0$, the convex combination $\lambda x + (1-\lambda) y_0$ is strictly between the values x and y_0. Therefore, by the strict monotonicity of φ, we have that

$\left(\varphi(\lambda x+(1-\lambda)y_0)-\varphi(x)\right)\cdot\left(\varphi(y_0)-\varphi(\lambda x+(1-\lambda)y_0)\right)>0$. Then, it follows from (3), that

$$f(x)=\frac{s}{t}\cdot f(y_0)\cdot\frac{\varphi(y_0)-\varphi(\lambda x+(1-\lambda)y_0)}{\varphi(\lambda x+(1-\lambda)y_0)-\varphi(x)}. \tag{6}$$

This implies that $f(x)$ is nonzero for all $x\in I$, furthermore, $f(x)$ has the same sign as $f(y_0)$, *i.e.* the sign of f is constant.

In what follows, we prove that, at every point of I, the function f is continuous at every point where φ is continuous. Denote by D_φ the set of discontinuity point of φ. Then, the monotonicity of φ implies that D_φ is countable.

Let $x_0\in I$ be fixed. Then, $\lambda x_0+(1-\lambda)I$ is a subinterval of I, hence, $I\setminus D_\varphi$ intersects $\lambda x_0+(1-\lambda)I$. There, exists an element $y_0\in I$ such that $\lambda x_0+(1-\lambda)y_0\in I\setminus D_\varphi$. Thus, φ is continuous at $\lambda x_0+(1-\lambda)y_0$. Therefore, (6) yields that f is continuous at x_0. Hence, f is continuous almost everywhere. On the other hand, f is bounded an every compact subinterval of I, it follows that f is Riemann integrable on every compact subinterval of I.

Let $0<\alpha<\frac{1}{2}|I|$ and $I_\alpha:=(I-\alpha)\cap(I+\alpha)$. Then, I_α is an nonempty interval and $I_\alpha+[-\alpha,\alpha]\subseteq I$. Let $u\in I_\alpha$, $v\in[-\alpha,\alpha]$ and substituting $x:=u-(1-\lambda)v$ and $y:=u+\lambda v$ into (3), we obtain that:

$$\begin{aligned}&\left(tf(u-(1-\lambda)v)+sf(u+\lambda v)\right)\varphi(u)\\&=tf(u-(1-\lambda)v)\varphi(u-(1-\lambda)v)+sf(u+\lambda v)\varphi(u+\lambda v),\end{aligned}$$

holds for all $u\in I_\alpha$ and for all $v\in[-\alpha,\alpha]$.

Integrating both sides of the above equation on $v\in[-\alpha,\alpha]$, it follows that:

$$\begin{aligned}&\varphi(u)\int_{-\alpha}^{\alpha}\left(tf(u-(1-\lambda)v)+sf(u+\lambda v)\right)\mathrm{d}v\\&=t\int_{-\alpha}^{\alpha}f(u-(1-\lambda)v)\varphi(u-(1-\lambda)v)\mathrm{d}v+s\int_{-\alpha}^{\alpha}f(u+\lambda v)\varphi(u+\lambda v)\mathrm{d}v.\end{aligned}$$

After simple change of the variable transformations, for all $u\in I_\alpha$, we get:

$$\varphi(u)\left(\frac{t}{1-\lambda}\int_{u-(1-\lambda)\alpha}^{u+(1-\lambda)\alpha}f+\frac{s}{\lambda}\int_{u-\lambda\alpha}^{u+\lambda\alpha}f\right)=\frac{t}{1-\lambda}\int_{u-(1-\lambda)\alpha}^{u+(1-\lambda)\alpha}f\cdot\varphi+\frac{s}{\lambda}\int_{u-\lambda\alpha}^{u+\lambda\alpha}f\cdot\varphi \tag{7}$$

Hence, φ is continuously differentiable on I_α. Since $0<\alpha<\frac{1}{2}|I|$ is arbitrary, it follows that φ is continuously differentiable and f is continuous on $\bigcup_{\alpha>0}I_\alpha=I$. By (6), the continuous differentiability of φ implies that f is also continuously differentiable.

Now, we show that φ and f are twice continuously differentiable. Differentiating (3) with respect to x, we have:

$$\begin{aligned}&\lambda\varphi'(\lambda x+(1-\lambda)y)\cdot(tf(x)+sf(y))+t\varphi(\lambda x+(1-\lambda)y)\cdot f'(x)\\&=tf'(x)\varphi'(x),\quad x,y\in I.\end{aligned} \tag{8}$$

Substituting $x:=u-(1-\lambda)v$ and $y:=u+\lambda v$ into the above equation and integrating both sides on $v\in[-\alpha,\alpha]$, we get:

$$\lambda\varphi'(u)\int_{-\alpha}^{\alpha}\left(tf\left(u-(1-\lambda)v\right)+sf\left(u+\lambda v\right)\right)\mathrm{d}v$$
$$=-t\varphi(u)\int_{-\alpha}^{\alpha}f'\left(u-(1-\lambda)v\right)\mathrm{d}v+t\int_{-\alpha}^{\alpha}\left(f\varphi\right)'\left(u-(1-\lambda)v\right)\mathrm{d}v,\quad u\in I_{\alpha}.$$

After similar change of the variable transformations as (7), for all $u \in I_\alpha$, we obtain:

$$\lambda\varphi'(u)\left(\frac{t}{1-\lambda}\int_{u-(1-\lambda)\alpha}^{u+(1-\lambda)\alpha}f+\frac{s}{\lambda}\int_{u-\lambda\alpha}^{u+\lambda\alpha}f\right)$$
$$=-\frac{t}{1-\lambda}\varphi(u)\int_{u-(1-\lambda)\alpha}^{u+(1-\lambda)\alpha}f'+\frac{t}{1-\lambda}\int_{u-(1-\lambda)\alpha}^{u+(1-\lambda)\alpha}\left(f\varphi\right)'$$

Thus, φ is twice continuously differentiable on I_α and hence on *I*. Then, by (6), this result implied that *f* is two times continuously differentiable on *I*.

To prove that φ and *f* are infinitely many times differentiable, differentiate (8) with respect to *y*, we get:

$$\begin{aligned}&(1-\lambda)\lambda\varphi''\left(\lambda x+(1-\lambda)y\right)\cdot\left(tf(x)+sf(y)\right)\\&+\varphi'\left(\lambda x+(1-\lambda)y\right)\left(\lambda sf'(y)+(1-\lambda)tf'(x)\right)=0.\end{aligned}\tag{9}$$

Substituting $y := x$, we get:

$$\lambda(1-\lambda)(t+s)f\varphi''+\left(\lambda s+(1-\lambda)t\right)f'\varphi'=0,\tag{10}$$

which is equivalent with

$$\left(f^{\mu}\varphi'\right)'=0,$$

where $\mu := \dfrac{\lambda s+(1-\lambda)t}{\lambda(1-\lambda)(t+s)} > 0$.

Hence, there exists a real constant γ such that $f^{\mu}\varphi' = \gamma$. If γ were zero, then this equation would imply that φ' is identically zero, which contradicts the strict monotonicity of φ. As a consequence, (4) holds. Finally, using (4) and (6) repeatedly, we get that φ and *f* are infinitely many times differentiable. □

Lemma 2. *Let* $\varphi : I \to \mathbb{R}$ *be a strictly monotone function,* $f : I \to \mathbb{R}$ *be a non-identically-zero function, and* $t, s \in \mathbb{R}_+$, $\lambda \in (0,1)$. *If* (φ, f) *solves* (3), *then we have* $\lambda = \dfrac{t}{t+s}$. *And what's more,* μ *defined by* (5) *equals* 2.

Proof. Differentiating (9) with respect to *x*, we obtain:

$$\begin{aligned}&\lambda\left(\lambda sf'(y)+2(1-\lambda)tf'(x)\right)\cdot\varphi''\left(\lambda x+(1-\lambda)y\right)\\&+(1-\lambda)tf''(x)\cdot\varphi'\left(\lambda x+(1-\lambda)y\right)\\&+\lambda^2(1-\lambda)\left(tf(x)+sf(y)\right)\cdot\varphi'''\left(\lambda x+(1-\lambda)y\right)=0\end{aligned}\tag{11}$$

Inserting $y := x$, it follows that:

$$\lambda\left(2(1-\lambda)t+\lambda s\right)f'\varphi''+\lambda^2(1-\lambda)(t+s)f\varphi'''+(1-\lambda)tf''\varphi'=0.\tag{12}$$

On the other hand, differentiating (9) with respect to *x*, we obtain:

$$\begin{aligned}&\left(\lambda(1-\lambda)(t+s)+\lambda s+(1-\lambda)t\right)f'\varphi''\\&+\lambda(1-\lambda)(t+s)f\varphi'''+\left(\lambda s+(1-\lambda)t\right)f''\varphi'=0.\end{aligned}\tag{13}$$

Combing (12) and (13), we conclude that:

$$\lambda(1-\lambda)\left(\lambda(t+s)-t\right)f'\varphi''+\left(\lambda^2 s-(1-\lambda)^2 t\right)f''\varphi'=0. \tag{14}$$

Firstly, we assume:

$$\lambda(1-\lambda)\left(\lambda(t+s)-t\right)\neq 0,\quad \lambda^2 s-(1-\lambda)^2 t\neq 0, \tag{15}$$

then Equation (14) can be rewritten by:

$$\frac{\varphi''}{\varphi'}=\tau\frac{f''}{f'}, \tag{16}$$

where $\tau:=\dfrac{(1-\lambda)^2 t-\lambda^2 s}{\lambda(1-\lambda)\left(\lambda(t+s)-t\right)}\neq 0$.

Integrating the above equation, we obtain that there exists a constant $c_1\in\mathbb{R}\setminus\{0\}$ such that:

$$\varphi'=c_1 f'^{\tau}. \tag{17}$$

By (17) and (4), letting $k=\left(\dfrac{\gamma}{c_1}\right)^{1/\tau}\neq 0$, we get:

$$f'=kf^{-\frac{\mu}{\tau}}.$$

Solving the above equation, we get there exist constants $k_1,k_3\in\mathbb{R}$ with $k_1\neq 0$ such that:

$$f(x)=\begin{cases}k_1\mathrm{e}^{kx}, & \mu+\tau=0,\\ k_2(x+k_3)^{\frac{\tau}{\mu+\tau}}, & \mu+\tau\neq 0,\end{cases} \tag{18}$$

where $k_2:=\left(k\cdot\dfrac{\mu+\tau}{\tau}\right)^{\frac{\tau}{\mu+\tau}}$.

Using (4) and (18), we obtain that there exist constants $b_1,b_2\in\mathbb{R}$ such that:

$$\varphi(x)=\begin{cases}a_1\mathrm{e}^{-\mu kx}+b_1, & \mu+\tau=0,\\ a_2(x+k_3)^{1-\frac{\mu\tau}{\mu+\tau}}+b_2, & \mu+\tau\neq 0,\end{cases} \tag{19}$$

where $a_1:=-\gamma\big/\left(\mu k k_1^{\mu}\right)$, $a_2:=\gamma\Big/\left(k_2^{\mu}\left(1-\dfrac{\mu\tau}{\mu+\tau}\right)\right)$.

For the case $\mu+\tau=0$, substituting (18) and (19) into (3), we obtain:

$$\mathrm{e}^{-\mu k(\lambda x+(1-\lambda)y)}\left(t\mathrm{e}^{kx}+s\mathrm{e}^{ky}\right)=t\mathrm{e}^{kx(1-\mu)}+s\mathrm{e}^{ky(1-\mu)}. \tag{20}$$

Comparing the coefficients of x after we make Taylor expansion of the above equation, we can get that:

$$t=\lambda(t+s),$$

which leads to contradictions with (15).

Similarly, for the case $\mu+\tau\neq 0$, substituting (18) and (19) into (3) and comparing the coefficients, we can get:

$$\frac{\mu\tau}{\mu+\tau}=1,$$

that is $\varphi(x)=a_2+b_2$, which leads to contradictions with the assumption of φ strictly monotone function.

Secondly, we assume:

$$\lambda(1-\lambda)(\lambda(t+s)-t)\neq 0,\quad \lambda^2 s-(1-\lambda)^2 t=0, \tag{21}$$

then Equation (14) can be rewritten by:

$$f'\varphi''=0. \tag{22}$$

Combing (4) and (25), it leads to $f'=\varphi''=0$. Therefore, there exits $a,b,c\in\mathbb{R}$ with $a\neq 0$ such that:

$$f(x)=c,\quad \varphi(x)=ax+b,\quad x\in I.$$

Substituting the above equations into (3) and comparing the coefficients of *x*, we obtain:

$$t=\lambda(t+s),$$

which leads to contradictions with (21).

Thirdly, we assume:

$$\lambda(1-\lambda)(\lambda(t+s)-t)=0,\quad \lambda^2 s-(1-\lambda)^2 t\neq 0, \tag{23}$$

that is:

$$\lambda=\frac{t}{t+s},\quad t\neq s, \tag{24}$$

then Equation (14) can be rewritten by:

$$f''\varphi'=0. \tag{25}$$

Using (5), (24) leads to $\mu=2$. Due to (4), φ' is nowhere zero. Therefore, $f''=0$ on *I*. Then, there exist $a,b\in\mathbb{R}$ such that:

$$f(x)=a(x+b),\quad x\in I. \tag{26}$$

Using (4) and $\mu=2$, we get there exist $c_1\in\mathbb{R}$ such that:

$$\varphi(x)=a_1(x+b)^{-1}+c_1, \tag{27}$$

where $a_1:=-\gamma a^{-2}$.

When f,φ satisfy (26) and (27), respectively, it is easy to verify that (3) is valid.

Finally, for the case:

$$\lambda(1-\lambda)(\lambda(t+s)-t)=0,\quad \lambda^2 s-(1-\lambda)^2 t=0, \tag{28}$$

it holds that $\lambda=\frac{1}{2}$ and $s=t$ which lead to $\mu=2$ from (5). □

3. Main Results

Using Lemma 2 and Theorem 5 in [9], we get the following.

Theorem 1. *Let* $\varphi:I\to\mathbb{R}$ *be a strictly monotone function,* $f:I\to\mathbb{R}$ *be a* *non-identically-zero function, and* $t,s\in\mathbb{R}_+$, $\lambda\in(0,1)$. *Then,* (3) *holds if and only if* $\lambda=\frac{t}{t+s}$, *f is nowhere zero and there exists* $p\in\mathbb{R}$ *with* $\left(\lambda-\frac{1}{2}\right)p=0$ *such that*:

$$(f, f\cdot\varphi)\sim(S_p, C_p), \tag{29}$$

where

$$S_p(x):=\begin{cases}\sin\left(\sqrt{-p}x\right), & p<0,\\ x, & p=0,\\ \sinh\left(\sqrt{p}x\right), & p>0,\end{cases} \quad \text{and} \quad C_p(x):=\begin{cases}\cos\left(\sqrt{-p}x\right), & p<0,\\ 1, & p=0,\\ \cosh\left(\sqrt{p}x\right), & p>0.\end{cases}$$

Corollary 1. *Let* $t,s\in\mathbb{R}_+$, $\lambda\in(0,1)$, *and let* $G,F:I\to\mathbb{R}$ *be two continuous functions such that g is nowhere zero on I and the ratio function* $\frac{f}{g}$ *is strictly monotone on I,* $h:I\to\mathbb{R}$ *be a continuous strictly monotone function. Then,* (2) *holds if and only if* $\lambda=\frac{t}{t+s}$ *and there exists* $p\in\mathbb{R}$ *with* $\left(\lambda-\frac{1}{2}\right)p=0$ *such that*:

$$(F,G)\sim(S_p\circ h, C_p\circ h). \tag{30}$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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