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# Exactly Solvable Two-Parameter Models of Relativistic $\delta_s'$-Sphere Interactions in Quantum Mechanics

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## Abstract

We make a systematic study of two-parameter models of $\delta_s'$-sphere interaction and $\delta_s'$-sphere plus a Coulomb interaction. Where $\delta_s'$ interaction denotes the $\delta'$-sphere interaction of the second kind. We provide the mathematical definitions of Hamiltonians and obtain new results for both models, in particular the resolvents equations, spectral properties and some scattering quantities.

## Keywords

Boundary Conditions Problem, $\delta'$-Sphere Interactions, Self-Adjoint Operator, Resolvent Equation, Spectral Properties, Scattering Theory

## 1. Introduction

The study of the $\delta$-sphere interactions in quantum mechanics has interested many authors since the twentieth century both from the mathematical point of view and for their applications in modeling of physical phenomena [1]-[6]. In nuclear physics, the delta shell interaction has been applied in a calculation of the energy levels of isotopes of Pb, Sn and Ni, Po 210, and nuclei belonging to the 82-neutron shell [7]. In Ref. [8], the authors show that the effective two-nucleon interaction in 2 s 12-1 d 32-shell nuclei can be well approximated by a delta function, which acts only at the nuclear surface. The application in solid state physics and molecular physics may be found respectively in Ref. [9] and Ref. [10].

These studies have used the Von Neumann formalism of self-adjoint extensions of symmetric linear operators in Hilbert space [11] [12]. In Ref. [13], the authors have built a rigorous mathematical model based on the self-adjoint extensions of symmetric operators in the Hilbert space defining the $\delta$-sphere interactions in non-relativistic quantum mechanics.

In relativistic quantum mechanics, Dittrich *et al.* study Dirac operator with a contact interaction supported by a sphere by restricting their attention to the operators that are rotationally and space-reflection symmetric. They define the self-adjoint extensions of the Dirac radial operator and discuss spectral properties.

In the continuation of this work, other researches were made in this field, thus enriching the knowledge of the basic properties of Dirac operator with a $\delta$ shell potential [14] [15] [16] [17] [18].

To the best of our knowledge, the study of $\delta'$-sphere interactions using the theory of self-adjoint extensions of symmetric operators in the Hilbert space began in the twentieth century [19]. And so far, little work has been done in this area both in relativistic and non-relativistic quantum mechanics [20] [21]. Yet these interactions are exactly solvable models and their systematic study allows us to better understand their properties.

In Ref. [21] and Ref. [22], authors study the one and N parameters models of relativistic $\delta'$-sphere interactions called the first and the second kind. For these models, their work provides basic properties and discusses the stationary scattering theory. Nevertheless, in Ref. [23], using the theory mentioned above, the authors introduced in a different way than in the previous case a rigorous mathematical definition of $\delta'$-sphere interactions. But, in Ref. [23], the study of the basic properties of relativistic $\delta'$-sphere interactions was missing. This is the aim of this paper. In addition, as indicated in Ref. [24], the study of the 2 N parameters models that unify the N parameters models of $\delta'$-sphere interactions of the first and the second kind allows us to better understand the dynamic of the perturbed physical system in term of scattering data. Therefore, we discuss the basic properties of two-parameter models of relativistic $\delta'_s$-sphere and $\delta'_s$-sphere plus Coulomb interactions in three space dimensions using the theory of the self-adjoint extensions of symmetric closed operators in Hilbert space, where $\delta'_s$ interaction denotes $\delta'$ interaction of the second kind.

The paper is organised as follows. In Section 2, we provide a mathematical definition of the Hamiltonian describing the two-parameter models of relativistic $\delta'_s$-sphere interactions and obtain new results on the resolvent equation, the spectral properties and the scattering data (scattering matrix, amplitude, length and the differential scattering cross section). In Section 3, we generalize the results of Section 2 to the case of a two-parameter relativistic $\delta'_s$-sphere interaction plus a Coulomb interaction.

## 2. The $\delta'_s$-Sphere Interaction

### 2.1. The Definition of the Hamiltonian

In this section, we discuss the properties of the $\delta'$-sphere interaction of second kind called "$\delta'_s$-sphere interaction". Using the theory of self-adjoint of symmetric closed operators in Hilbert space, we provide the mathematical definition of quantum Hamiltonian describing the $\delta_s'$-sphere interaction formally given by:

$$H_{\hat{G}} = H_D + \hat{G}\delta_s'\left(|x| - R\right),\ x \in \mathbb{R}^3,\ R \in \mathbb{R} \tag{1}$$

where $H_D$ is the Dirac Hamiltonian and $\hat{G}$ is a real $4 \times 4$ matrix defined by:

$$\hat{G} = \begin{pmatrix} \mathbb{1}A_{jl} & 0 \\ 0 & \mathbb{1}B_{jl} \end{pmatrix},$$

$A_{jl}$ and $B_{jl}$ are real constants and $R$ is the radius of a sphere in $\mathbb{R}^3$ centered at the origin.

## 2.2. The Radial Operators

In the Hilbert space $\mathcal{H} = L^2\left(\mathbb{R}^3\right) \otimes \mathbb{C}^4$, consider the Dirac Hamiltonian $H_D$ defined by:

$$\begin{aligned} H_D &= -\mathrm{i}\hat{\alpha}\nabla + \hat{\beta}\frac{c^2}{2}, \\ \mathcal{D}\left(H_D\right) &= H^{1,2}\left(\mathbb{R}^3\right) \otimes \mathbb{C}^4 \end{aligned} \tag{2}$$

where we have used the following definitions and notations:

1) $c$ is the velocity of the light;
2) $H^{m,n}$ is the Sobolev space of indices $(m, n)$;
3) $\hat{\alpha}$ and $\hat{\beta}$ are $4 \times 4$ Dirac matrix given by:

$$\hat{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix},\ \hat{\beta} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{3}$$

where $\sigma$ are Pauli's spin matrix defined by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Consider in $L^2\left(\mathbb{R}^3\right) \otimes \mathbb{C}^4$, the symmetric closed operator $\hat{H}_D$ defined by:

$$\begin{aligned} \hat{H}_D &= -\mathrm{i}\hat{\alpha}\nabla + \hat{\beta}\frac{c^2}{2}, \\ \mathcal{D}\left(\hat{H}_D\right) &= \left\{\chi_{jl} \in H^{1,2}\left(\mathbb{R}^3\right) \otimes \mathbb{C}^4 / \chi_{jl}\left(S_R\right) = 0\right\}, \end{aligned} \tag{5}$$

where $S_R = \left\{x \in \mathbb{R}^3 : |x| = R_n\right\}$ is the sphere of radius $R$ in $\mathbb{R}^3$ centered at the origin.

The operator $\hat{H}_D$ admits a large number of self-adjoint extensions [25]. In this case, only those of $\hat{H}_D$ corresponding to $H_{\hat{G}}$, which are rotationally and space-reflection symmetric, will be considered.

Under these assumptions, one may decompose the space $\mathcal{H}$ in the following way:

$$\mathcal{H} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} \bigoplus_{\mu=-j}^{j} \mathcal{H}_{jl\mu}, \tag{6}$$

where we have:

1) $\mathcal{H}_{jl\mu}=\left\{\chi_{jl}\in\mathcal{H}\,|\,\chi_{jl}(r,n)=\begin{pmatrix} f(r)\Omega_{jl\mu}\\ g(r)\Omega_{jl'\mu}\end{pmatrix}; f,g\in L^2\left((0,\infty),r^2\mathrm{d}r\right)\right\},$ (7)

2) The spherical spinors $\Omega_{jl\mu}$ are defined by:

$$\begin{aligned}\Omega_{jl\mu}&=\begin{pmatrix}\sqrt{\dfrac{j+\mu}{2l+1}}Y_{l,\mu-\frac{1}{2}}(\theta,\varphi)\\ \sqrt{\dfrac{j-\mu}{2l+1}}Y_{l,\mu+\frac{1}{2}}(\theta,\varphi)\end{pmatrix}\quad\text{for } l=j-\frac{1}{2},\\ \Omega_{jl'\mu}&=\begin{pmatrix}-\sqrt{\dfrac{j-\mu+1}{2l+1}}Y_{l,\mu-\frac{1}{2}}(\theta,\varphi)\\ \sqrt{\dfrac{j+\mu+1}{2l+1}}Y_{l,\mu+\frac{1}{2}}(\theta,\varphi)\end{pmatrix}\quad\text{for } l=j+\frac{1}{2}.\end{aligned}\tag{8}$$

3) $l'=j\mp\frac{1}{2}$ for $l=j\pm\frac{1}{2}$.

The following isomorphism $U_{jl}$ defined by:

$$U_{lj}:L^2\left((0,\infty);r^2\mathrm{d}r\right)\otimes\mathbb{C}^2\to\tilde{\mathcal{H}}=L^2\left((0,\infty);\mathrm{d}r\right)\otimes\mathbb{C}^2\tag{9}$$

$$\left(U_{jl}\right)\chi_{jl}\psi(r)=\begin{pmatrix} rf(r)\\ (-1)^{j-l-\frac{1}{2}}rg(r)\end{pmatrix}\tag{10}$$

is introduced to allow us to represent $\mathcal{H}$ in the form:

$$\mathcal{H}=\bigoplus_{j=\frac{1}{2}}^{\infty}\bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}}\bigoplus_{\mu=-j}^{j}\left[U_{jl}^{-1}\tilde{\mathcal{H}}\right]\otimes\left[\Omega_{jl\mu}(\theta,\varphi)\right],\tag{11}$$

where $\left[\Omega_{jl}(\theta,\varphi)\right]$ stands for the vector space generated by the spherical spinors.

With respect to the decomposition (11), $\hat{H}_D$ reads:

$$\hat{H}_D=\bigoplus_{j=\frac{1}{2}}^{\infty}\bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}}\left[U_{jl}^{-1}h_{jl}U_{jl}\right]\otimes\mathbb{1}.\tag{12}$$

The operator $h_{jl}$ in $L^2\left((0,\infty)\right)\otimes\mathbb{C}^2$ is given by:

$$\begin{aligned}h_{jl}&=\begin{pmatrix}\dfrac{c^2}{2} & -c\dfrac{\mathrm{d}}{\mathrm{d}r}+c\dfrac{\kappa_{jl}}{r}\\ c\dfrac{\mathrm{d}}{\mathrm{d}r}+c\dfrac{\kappa_{jl}}{r} & -\dfrac{c^2}{2}\end{pmatrix}\equiv\tau,\ \kappa_{jl}=(-1)^{j-l+\frac{1}{2}}\left(j+\frac{1}{2}\right),\\ \mathcal{D}\left(h_{jl}\right)&=\left\{\chi_{jl}\in L^2\left((0,\infty)\right)\otimes\mathbb{C}^2\,|\,\chi_{jl}\in AC_{loc}\left((0,\infty)\right);\ \chi_{jl}(R\pm)=0;\right.\\ &\qquad\left.\tau\chi_{jl}\in L^2\left((0,\infty)\right)\otimes\mathbb{C}^2\right\},\end{aligned}\tag{13}$$

where $AC_{loc}(\Omega)$ denotes the set of locally absolutely continuous functions on $\Omega$ and

$$\chi_{jl}(x\pm)=\lim_{\varepsilon\to0^+}\chi_{jl}(x\pm\varepsilon).\tag{14}$$

The adjoint $h_{jl}^*$ of $h_{jl}$ is given by:

$$\begin{aligned} &h_{jl}^* = \tau, \\ &\mathcal{D}\left(h_{jl}^*\right) = \left\{ f \in L^2\left((0,\infty)\right) \otimes \mathbb{C}^2 \mid f \in AC_{loc}\left((0,\infty)\setminus\{R\}\right),\ \tau f \in L^2\left((0,\infty)\right) \otimes \mathbb{C}^2 \right\}. \end{aligned} \tag{15}$$

### 2.3. The Self-Adjoint Extensions of $h_{jl}$

Given the following equation:

$$\left(h_{jl}^* - z\right)\chi_{jl} = 0,\ \chi_{jl} \in \mathbb{C} - \left\{\left(-\infty, -\frac{c^2}{2}\right] \cup \left[\frac{c^2}{2}, \infty\right)\right\} \tag{16}$$

One can write Equation (16) in the following form:

$$\begin{aligned} &\chi_{jl,1}'' + \left[k^2(z) - \frac{\kappa_{jl}\left(\kappa_{jl}+1\right)}{r^2}\right]\chi_{jl,1} = 0 \\ &\chi_{jl,2}'' + \left[k^2(z) - \frac{\kappa_{jl}\left(\kappa_{jl}-1\right)}{r^2}\right]\chi_{jl,2} = 0 \end{aligned} \tag{17}$$

where

$$k(z) = \frac{1}{c}\sqrt{z^2 - \frac{c^4}{4}}. \tag{18}$$

Equation (17) has two linearly independent solutions:

$$\chi_{jl}^{(1)} = \begin{cases} \begin{pmatrix} F_{jl}(z,r) \\ \tilde{F}_{jl}(z,r) \end{pmatrix}, r < R, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}; r > R, \end{cases} \tag{19}$$

$$\chi_{jl}^{(2)} = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}; r < R, \\ \begin{pmatrix} G_{jl}(z,r) \\ \tilde{G}_{jl}(z,r) \end{pmatrix}, r > R, \end{cases} \tag{20}$$

where:

$$\begin{aligned} &F_{jl}(z,r) = \left(\frac{k}{2}\right)^{-\kappa_{jl}-\frac{1}{2}} \Gamma\left(\kappa_{jl}+\frac{3}{2}\right)\left(k^2 - \frac{\kappa_{jl}^2}{R^2}\right)^{-1} r^{\frac{1}{2}} J_{\kappa_{jl}+\frac{1}{2}}(kr), \\ &\tilde{F}_{jl}(z,r) = \frac{1}{c}\left(\frac{1}{2}\right)^{-\kappa_{jl}-\frac{1}{2}} k^{-\kappa_{jl}+\frac{1}{2}} \Gamma\left(\kappa_{jl}+\frac{3}{2}\right)\left(k^2 - \frac{\kappa_{jl}^2}{R^2}\right)^{-1} r^{\frac{1}{2}} J_{\kappa_{jl}-\frac{1}{2}}(kr), \\ &G_{jl}(z,r) = \mathrm{i}\frac{\pi}{2}\left(\frac{k}{2}\right)^{\kappa_{jl}+\frac{1}{2}} \Gamma\left(\kappa_{jl}+\frac{3}{2}\right)^{-1} r^{\frac{1}{2}} H^{(1)}_{\kappa_{jl}+\frac{1}{2}}(kr), \\ &\tilde{G}_{jl}(z,r) = \mathrm{i}\frac{\pi}{2c}\left(\frac{1}{2}\right)^{\kappa_{jl}+\frac{1}{2}} k^{\kappa_{jl}+\frac{3}{2}} \Gamma\left(\kappa_{jl}+\frac{3}{2}\right)^{-1} r^{\frac{1}{2}} H^{(1)}_{\kappa_{jl}-\frac{1}{2}}(kr). \end{aligned} \tag{21}$$

$J_p(.)$ is the Bessel function and $H_p^{(1)}(.)$ is the Hankel function of the first kind of order *p*.

The solutions Equation (19) and Equation (20) have been normalized in such a way that:

$$\det\begin{bmatrix} G'_{jl}(z,r) & F'_{jl}(z,r) \\ \tilde{G}'_{jl}(z,r) & \tilde{F}'_{jl}(z,r) \end{bmatrix} = \frac{1}{c}. \quad (22)$$

Therefore, $h_{jl}$ has indices (2, 2) and consequently, all self-adjoint(sa) extensions of $h_{jl}$ are given by a four-parameter family of self-adjoint operators [26]. Since the matrix $\hat{G}$ in Equation (28) depends on two parameters, it follows that the self-adjoint extension $h_{jl,\hat{G}}$ of $h_{jl}$ corresponding to the interaction $V(r) = \hat{G}\delta(r-R)$ is a special two-parameter family.

The relation $h_{jl,\hat{G}} \subset h^*_{jl}$ implies that the domain $\mathcal{D}(h_{jl,\hat{G}})$ contains those functions $\chi_{jl} \in \mathcal{D}(h^*_{jl})$ which satisfy suitable boundary conditions at $r = R$.

**Theorem 2.1:** Any self-adjoint extension $\hat{h}_{jl}$ of $h_{jl}$ reads [25]:

$$\hat{h}_{jl} = \begin{pmatrix} \frac{c^2}{2} & -c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} \\ c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix},$$
$$\mathcal{D}(\hat{h}_{jl}) = \left\{\chi_{jl} \in \mathcal{D}(h^*_{jl}) \mid \chi_{jl} \text{ satifies } \mathrm{cond}_1 \text{ or } \mathrm{cond}_2\right\}, \quad (23)$$

where $\mathrm{cond}_1$ and $\mathrm{cond}_2$ are given by [25]:

$$\mathrm{cond}_1 : \chi_{jl}(R-) = \mathrm{e}^{\mathrm{i}\varphi}\bar{M}\chi_{jl}(R+), \varphi \in [0,\pi), \quad (24)$$

and $\bar{M}$ is 2 × 2 matrix with det $\bar{M} = 1$.

$$\mathrm{cond}_2 : \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}\chi_{jl}(R-) + \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}\chi_{jl}(R+) = 0 \quad (25)$$

where $a_1, a_2, b_1$ and $b_2$ are real and both matrices are nonzero. Conversely, any operator of this form is self-adjoint extensions of $h_{jl}$.

**Theorem 2.2:** [25] The general form of boundary conditions is given by:

$$1 - \tau_0\frac{\hat{G}_{jl}}{2c}\chi_{jl}(R+) - 1 + \tau_0\frac{\hat{G}_{jl}}{2c}\chi_{jl}(R-) = 0 \quad (26)$$

where $\tau_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\hat{G}_{jl} = \begin{pmatrix} A_{jl} & 0 \\ 0 & B_{jl} \end{pmatrix}$.

Let us now construct the self-adjoint extension corresponding to the radial Dirac operator with the potential:

$$V(r) = \hat{G}_{jl}\delta'_s(r-R), \hat{G}_{jl} = \begin{pmatrix} A_{jl} & 0 \\ 0 & B_{jl} \end{pmatrix}, A_{jl}, B_{jl} \in \mathbb{R}. \quad (27)$$

Suppose that $\chi_{jl}$ satisfies the following equation given by:

$$\left[\tau + \hat{G}_{jl}\delta(r-R)\right]\chi_{jl} = E\chi_{jl}, \quad (28)$$

$$\tau = \begin{pmatrix} \frac{c^2}{2} & -c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} \\ c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix}, \tag{29}$$

where the limits $\chi(R \pm \varepsilon)$ exist when $\varepsilon \to 0$. A simple computation shows that integrating Equation (28) over $(R-\varepsilon, R+\varepsilon)$, and taking the limit $\varepsilon \to 0$ we get the following conditions:

$$\begin{cases} \chi_{jl,1}(R+) - \chi_{jl,1}(R-) = -\frac{1}{2C} B_{jl} \left[ \chi_{jl,2}(R+) + \chi_{jl2}(R-) \right] \\ \chi_{jl,2}(R+) - \chi_{jl,2}(R-) = \frac{1}{2C} A_{jl} \left[ \chi_{jl,1}(R+) + \chi_{jl,1}(R-) \right] \end{cases} \tag{30}$$

By replacing the function $\chi_{jl}$ in Equation (30) by its derivative, we get the following conditions:

$$\begin{cases} \chi'_{jl,1}(R+) - \chi'_{jl,1}(R-) + \frac{1}{2C} B_{jl} \left[ \chi'_{jl,2}(R+) + \chi'_{jl,2}(R-) \right] = 0 \\ \chi'_{jl,2}(R+) - \chi'_{jl,2}(R-) - \frac{1}{2C} A_{jl} \left[ \chi'_{jl,1}(R+) + \chi'_{jl,1}(R-) \right] = 0 \end{cases} \tag{31}$$

The boundary conditions Equation (31) can be written in the general form:

$$1 - \tau_0 \frac{\hat{G}_{jl}}{2c} \chi'_{jl}(R+) - 1 + \tau_0 \frac{\hat{G}_{jl}}{2c} \chi'_{jl}(R-) = 0. \tag{32}$$

A straightforward computation shows that these boundary conditions are symmetric and linearly independent and can be written in the form of Equation (26).

One can construct the self-adjoint extension of $h_{jl}$ considering its adjoint $h^*_{jl}$ where:

$$\mathcal{D}\left(\hat{h}_{jl}\right) = \left\{ \chi_{jl} \in \mathcal{D}\left(h^*_{jl}\right) \mid \chi_{jl} \text{ satifies Eq.}(32) \right\}.$$

The boundary conditions of Equation (31) characterize the potential of Equation (27) and introduce a new exactly solvable model of relativistic $\delta'_s$-sphere interactions in quantum mechanics.

Let us consider in $L^2((0,\infty)) \otimes \mathbb{C}^2$ the operator $h_{jl,\hat{G}_{jl}}$ defined by:

$$h_{jl,\hat{G}_{jl}} = \begin{pmatrix} \frac{c^2}{2} & -c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} \\ c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix} \equiv \tau$$

$$\mathcal{D}\left(h_{jl,\hat{G}_{jl}}\right) = \left\{ \chi_{jl} \in \mathcal{D}\left(h^*_{jl}\right) \mid \begin{matrix} \chi'_{jl,1}(R+) - \chi'_{jl,1}(R-) + \frac{1}{2C} B_{jl} \left[ \chi'_{jl,2}(R+) + \chi'_{jl,2}(R-) \right] = 0 \\ \chi'_{jl,2}(R+) - \chi'_{jl,2}(R-) - \frac{1}{2C} A_{jl} \left[ \chi'_{jl,1}(R+) + \chi'_{jl,1}(R-) \right] = 0 \end{matrix} \right\}, \tag{33}$$

$$l \in \left[ j - \frac{1}{2}, j + \frac{1}{2} \right],\ j \in \left[ \frac{1}{2}, \infty \right).$$

The operator $h_{jl,\hat{G}_{jl}}$ is the self-adjoint extension of the symetric operator $h_{jl}$.

The operator $h_{jl,\hat{G}_{jl}}$ gives the mathematical definition of the formal expression:

$$h_{jl,\hat{G}_{jl}} = h_D + \hat{G}_{jl}\delta'_s(r-R), \tag{34}$$

where $\delta'_s$ is the $\delta'$-sphere interaction of the second kind characterized by the boundary conditions of Equation (31) and $h_D$ is the radial Dirac Hamiltonian defined by:

$$h_D = \begin{pmatrix} \frac{c^2}{2} & -c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} \\ c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix},$$
$$\mathcal{D}(h_D) = \left\{ \chi_{jl} \in \mathcal{D}(h^*_{jl}) \mid \begin{matrix} \chi'_{jl}(R+) = \chi'_{jl}(R-) \\ \chi_{jl}(R+) = \chi_{jl}(R-) \end{matrix} \right\}, l \in \left[ j-\frac{1}{2}, j+\frac{1}{2} \right], j \in \left[ \frac{1}{2}, \infty \right). \tag{35}$$

The particular case $\hat{G}_{jl} = 0$ in Equation (33) yields the radial Dirac Hamiltonian $h_{jl,0} \equiv h_D$. The case $A_{jl} \neq 0, B_{jl} = 0$ in Equation (33) gives the one parameter $\delta'_s$-sphere interaction defined by:

$$h_{jl,A_{jl}} = \tau$$
$$\mathcal{D}(h_{jl,A_{jl}}) = \left\{ \chi_{jl} = \begin{pmatrix} \chi_{jl,1} \\ \chi_{jl,2} \end{pmatrix} \in \mathcal{D}(h^*_{jl}) \mid \begin{matrix} \chi'_{jl,1}(R+) = \chi'_{jl,1}(R-) \equiv \chi'_{jl,1}(R), \\ \chi'_{jl,2}(R+) - \chi'_{jl,2}(R-) = \frac{A_{jl}}{c}\chi'_{jl,1}(R) \end{matrix} \right\}.$$

Even, the case $A_{jl} = 0, B_{jl} \neq 0$ in Equation (33) provides the one parameter $\delta'_s$-sphere interaction defined by:

$$h_{jl,B_{jl}} = \tau$$
$$\mathcal{D}(h_{jl,B_{jl}}) = \left\{ \chi_{jl} = \begin{pmatrix} \chi_{jl,1} \\ \chi_{jl,2} \end{pmatrix} \in \mathcal{D}(h^*_{jl}) \mid \begin{matrix} \chi'_{jl,2}(R+) = \chi'_{jl,2}(R-) \equiv \chi'_{jl,2}(R), \\ \chi'_{jl,1}(R+) - \chi'_{jl,1}(R-) = -\frac{B_{jl,2}}{c}\chi'_{jl,2}(R) \end{matrix} \right\}. \tag{36}$$

Let $\hat{G} = \{\hat{G}_{jl}\}, j-\frac{1}{2} \leq l \leq j+\frac{1}{2}, \frac{1}{2} \leq j < \infty$. The decomposition of Equation (12) implies that the operator $H_{\hat{G}}$ in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ defined by:

$$H_{\hat{G}} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} \left[ U^{-1}_{jl} h_{jl,\hat{G}_{jl}} U_{jl} \right] \otimes \mathbb{1}. \tag{37}$$

provides the mathematical definition of the formal expression Equation (2). The operator $H_{\hat{G}}$ in Equation (35) defines the $\delta'_s$ interaction in the tree-dimensional space.

The case $\hat{G} = 0$, *i.e.* $\hat{G}_{jl} = 0$ for all *j* and *l* yields the Dirac Hamiltonian $H_D$ defined by Equation (2).

## 2.4. The Resolvent Equation of $h_{jl,\hat{G}_{jl}}$

**Theorem 2.3:** The resolvent of $h_{jl,\hat{G}_{jl}}$ leads:

$$\begin{aligned}&\left(h_{jl,\hat{G}_{jl}}-z\right)^{-1}(.)\\&=\left(h_D-z\right)^{-1}(.)+\Omega_{jl}\left(z,R\right)\left\{\left(\overline{Q_{jl}^{2}(.)},.\right)\left[A_{jl}Q_{jl}^{(2)}(.)+\frac{A_{jl}B_{jl}}{2c}\tilde{Q}_{jl}^{(1)}(.)\right]\right.\\&\quad\left.+\left(\overline{\tilde{Q}_{jl}^{(2)}(.)},.\right)\left[B_{jl}\tilde{Q}_{jl}^{(2)}(.)-\frac{A_{jl}B_{jl}}{2c}Q_{jl}^{(1)}(.)\right]\right\},\\&\quad z\in\rho\left(h_{jl,\hat{G}_{jl}}\right),\ \operatorname{Im}k>0,\ l\in\left[j-\frac{1}{2},j+\frac{1}{2}\right],\ j\in\left[\frac{1}{2},\infty\right),\end{aligned} \tag{38}$$

where $\rho(.)$ is the resolvent set and where $\left(h_D-z\right)^{-1}$, $\operatorname{Im}k(z)>0$ is the radial Dirac resolvent with kernel:

$$G^{jl}\left(z,r,R\right)=\begin{pmatrix}G_{11}^{jl}\left(z,r,R\right) & G_{12}^{jl}\left(z,r,R\right)\\ G_{21}^{jl}\left(z,r,R\right) & G_{22}^{jl}\left(z,r,R\right)\end{pmatrix}, \tag{39}$$

where

$$\begin{aligned}G_{11}^{jl}\left(z,r,R\right)&=\begin{cases}G_{jl}\left(z,R\right)F_{jl}\left(z,r\right), r<R\\ F_{jl}\left(z,R\right)G_{jl}\left(z,r\right), r>R,\end{cases}\\ G_{12}^{jl}\left(z,r,R\right)&=\begin{cases}\tilde{G}_{jl}\left(z,R\right)F_{jl}\left(z,r\right), r<R\\ \tilde{F}_{jl}\left(z,R\right)G_{jl}\left(z,r\right), r>R,\end{cases}\\ G_{21}^{jl}\left(z,r,R\right)&=\begin{cases}G_{jl}\left(z,R\right)\tilde{F}_{jl}\left(z,r\right), r<R\\ F_{jl}\left(z,R\right)\tilde{G}_{jl}\left(z,r\right), r>R,\end{cases}\\ G_{22}^{jl}\left(z,r,R\right)&=\begin{cases}\tilde{G}_{jl}\left(z,R\right)\tilde{F}_{jl}\left(z,r\right), r<R\\ \tilde{F}_{jl}\left(z,R\right)\tilde{G}_{jl}\left(z,r\right), r>R.\end{cases}\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\Omega_{jl}\left(z,A_{jl},B_{jl},R\right)\\&=-\left[1-\frac{B_{jl}A_{jl}}{4c^2}+B_{jl}\tilde{F}'_{jl}\left(z,R\right)\tilde{G}'_{jl}\left(z,R\right)+A_{jl}F'_{jl}\left(z,R\right)G'_{jl}\left(z,R\right)\right]^{-1},\end{aligned} \tag{41}$$

$$Q_{jl}^{(2)}=\begin{cases}\begin{pmatrix}G'_{jl}\left(z,R\right)F_{jl}\left(z,r\right)\\ G'_{jl}\left(z,R\right)\tilde{F}_{jl}\left(z,r\right)\end{pmatrix}; r<R,\\ \begin{pmatrix}F'_{jl}\left(z,R\right)G_{jl}\left(z,r\right)\\ F'_{jl}\left(z,R\right)\tilde{G}_{jl}\left(z,r\right)\end{pmatrix}; r>R,\end{cases} \tag{42}$$

$$\tilde{Q}_{jl}^{(2)}=\begin{cases}\begin{pmatrix}\tilde{G}'_{jl}\left(z,R\right)F_{jl}\left(z,r\right)\\ \tilde{G}'_{jl}\left(z,R\right)\tilde{F}_{jl}\left(z,r\right)\end{pmatrix}; r<R,\\ \begin{pmatrix}\tilde{F}'_{jl}\left(z,R\right)G_{jl}\left(z,r\right)\\ \tilde{F}'_{jl}\left(z,R\right)\tilde{G}_{jl}\left(z,r\right)\end{pmatrix}; r>R,\end{cases} \tag{43}$$

$$\tilde{Q}_{jl}^{(1)}=\begin{cases}\begin{pmatrix}\tilde{G}'_{jl}\left(z,R\right)F_{jl}\left(z,r\right)\\ \tilde{G}'_{jl}\left(z,R\right)\tilde{F}_{jl}\left(z,r\right)\end{pmatrix}; r<R,\\ -\begin{pmatrix}\tilde{F}'_{jl}\left(z,R\right)G_{jl}\left(z,r\right)\\ \tilde{F}'_{jl}\left(z,R\right)\tilde{G}_{jl}\left(z,r\right)\end{pmatrix}; r>R,\end{cases} \tag{44}$$

$$Q_{jl}^{(1)} = \begin{cases} \begin{pmatrix} G'_{jl}(z,R) F_{jl}(z,r) \\ G'_{jl}(z,R) \tilde{F}_{jl}(z,r) \end{pmatrix}; r < R, \\ -\begin{pmatrix} F'_{jl}(z,R) G_{jl}(z,r) \\ F'_{jl}(z,R) \tilde{G}_{jl}(z,r) \end{pmatrix}; r > R, \end{cases} \tag{45}$$

with $F_{jl}(z,r), \tilde{F}_{jl}(z,r), G_{jl}(z,r), \tilde{G}_{jl}(z,r)$ and $k(z)$ defined by Equation (21) and Equation (18).

**Proof**

One can use the Krein resolvent formula which yields the following relation for the resolvent of $h_{jl,\hat{G}_{jl}}$:

$$\left(h_{jl,\hat{G}_{jl}} - z\right)^{-1} = \left(h_D - z\right)^{-1} + \sum_{p,q=1}^{2} \gamma_{qp} \left(\overline{\chi_{jl}^{(p)}(.)},.\right) \chi_{jl}^{(q)}(.), \ z \in \rho\left(h_{jl,\hat{G}_{jl}}\right), \ \operatorname{Im} k(z) > 0, \tag{46}$$

where $\chi_{jl}^{(q)}(r), q = 1,2$ are given by Equations (19) and (20) respectively.

Taking the function $\chi_{jl} = \begin{pmatrix} \chi_{jl,1} \\ \chi_{jl,2} \end{pmatrix} \in L((0,\infty)) \otimes \mathbb{C}^2$, we can define the function $\eta_{jl}$ by:

$$\eta_{jl}(z,r) = \left(\left(h_{jl,\hat{G}_{jl}} - z\right)^{-1} \chi_{jl}\right)(r). \tag{47}$$

As the function $\eta_{jl} \in \mathcal{D}\left(h_{\hat{G}_{jl}}\right)$, it follows that $\eta_{jl}$ satisfies the boundary conditions in Equation (33). The implementation of these boundary conditions provides the constants $\gamma_{qp}(z)$. When we insert $\gamma_{qp}(z)$ into Equation (46), we obtain Equation (38).

In particular case $A_{jl} \neq 0, B_{jl} = 0$ and $A_{jl} = 0, B_{jl} \neq 0$, the resolvent Equation (40) provides respectively:

$$\begin{gathered} \left(h_{jl,A_{jl}} - z\right)^{-1}(.) = \left(h_D - z\right)^{-1}(.) + A_{jl}\Omega_{jl}\left(z, A_{jl}, 0, R\right)\left(\overline{Q_{jl}^{(2)}(.)},.\right) Q_{jl}^{(2)}(.), \\ z \in \rho\left(h_{jl,A_{jl}}\right), \operatorname{Im} k > 0, l \in \left[j - \frac{1}{2}, j + \frac{1}{2}\right], \ j \in \left[\frac{1}{2}, \infty\right), \end{gathered} \tag{48}$$

and

$$\begin{gathered} \left(h_{jl,B_{jl}} - z\right)^{-1}(.) = \left(h_D - z\right)^{-1}(.) + B_{jl}\Omega_{jl}\left(z, 0, B_{jl}, R\right)\left(\overline{\tilde{Q}_{jl}^{(2)}(.)},.\right) \tilde{Q}_{jl}^{(2)}(.), \\ z \in \rho\left(h_{jl,B_{jl}}\right), \operatorname{Im} k > 0, l \in \left[j - \frac{1}{2}, j + \frac{1}{2}\right], \ j \in \left[\frac{1}{2}, \infty\right), \end{gathered} \tag{49}$$

## 2.5. The Spectral Properties of $h_{jl,\hat{G}_{jl}}$

**Theorem 2.4:** For $A_{jl}, B_{jl} \in (\infty, \infty)$, $l \in \left[j - \frac{1}{2}, j + \frac{1}{2}\right]$, $j \in \left[\frac{1}{2}, \infty\right)$, the essential spectrum of $h_{jl,\hat{G}_{jl}}$ is purely absolutely continuous and coincides with $\left(-\infty, -\frac{c^2}{2}\right] \cup \left[\frac{c^2}{2}, \infty\right)$. Its singularly continuous and residual spectra are empty.

**Proof**

Proposition 6.1 and Theorem 6.2.in Ref. [25] provide detailed proof.

The eigenvalues $E$ of $h_{jl,\hat{G}_{jl}}$ located in $\left[-\frac{c^2}{2},\frac{c^2}{2}\right]$ are given by the pole of the resolvent equation in physical sheet $\operatorname{Im} k > 0$, *i.e.* the solution of:

$$1-\frac{B_{jl}A_{jl}}{4c^2}+B_{jl}\tilde{F}'_{jl}(k,R)\tilde{G}'_{jl}(k,R)+A_{jl}F'_{jl}(k,R)G'_{jl}(k,R)\Big|_{k(z)=\mathrm{i}\sqrt{-P}}=0, P<0 \quad (50)$$

where $k=\mathrm{i}\sqrt{-P}=\frac{1}{c}\sqrt{E^2-\frac{c^4}{4}}$.

A straightforward computation shows that Equation (50) reads:

$$\frac{B_{jl}A_{jl}}{4c^2}-1=\left(P-\frac{\kappa_{jl}^2}{R^2}\right)^{-1}\left(\frac{B_{jl}}{c^2}P\left[r^{\frac{1}{2}}K_{\kappa_{jl}-\frac{1}{2}}\left(\sqrt{-P}r\right)\right]'_{r=R}\left[r^{\frac{1}{2}}I_{\kappa_{jl}-\frac{1}{2}}\left(\sqrt{-P}r\right)\right]'_{r=R}\right.$$
$$\left.+A_{jl}\left[r^{\frac{1}{2}}K_{\kappa_{jl}+\frac{1}{2}}\left(\sqrt{-P}r\right)\right]'_{r=R}\left[r^{\frac{1}{2}}I_{\kappa_{jl}+\frac{1}{2}}\left(\sqrt{-P}r\right)\right]'_{r=R}\right) \quad (51)$$

In **Figure 1**, we consider the following normalization of the energy $\hbar=2m=c^2=1$. Consider also $K_{\kappa_{jl}}=-2$ with the following condition for the coupling constants: $B_{jl}A_{jl}<0$. $Y_2=\frac{B_{jl}A_{jl}}{4c^2}-1$ and $Y_1$ represents the second member of Equation (51). Using the graphical resolution method, *i.e.* by analyzing the intersection of the curves $Y_2$ and $Y_1$, one can show easily that Equation (51) has two solutions which correspond to the two eigenvalues $\{E_1,E_2\}$ of $h_{jl,\hat{G}_{jl}}$ in $\left[-\frac{c^2}{2},\frac{c^2}{2}\right]$.

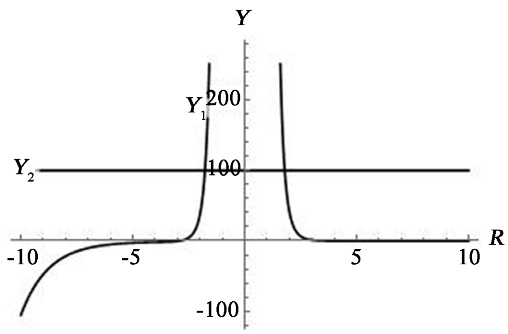


**Figure 1.** Existence of two eigenvalues in $\left[-\frac{c^2}{2},\frac{c^2}{2}\right]$.

The resonances of $h_{jl,\hat{G}_{jl}}$ are defined as poles of the resolvent Equation (38) in the unphysical sheet $\operatorname{Im} k<0$.

## 2.6. The Nonrelativistic Limit

Following the strategy of Gesztesy *et al.* [27], in the case of point interactions, one can discuss the nonrelativistic limit of $h_{jl,\hat{G}_{jl}}$ as $c \to \infty$.

**Theorem 2.5:** For spin-$\frac{1}{2}$ particules, the operator $h_{jl,\hat{G}_{jl}} - \frac{1}{2}$ converges in norm resolvent sense to the Shrödinger operator $h_{l,\hat{\alpha}_l}$ times the projector onto $\mathcal{H}_+ = L^2((0,\infty))$:

$$n \cdot \lim_{c\to\infty}\left(h_{jl,\hat{G}_{jl}} - \frac{c^2}{2} - z\right)^{-1} = \left(h_{l,\hat{\alpha}_l} - z\right)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ z \in \mathbb{C} \setminus \mathbb{R} \tag{52}$$

where $h_{l,\hat{\alpha}_l}$ is defined by:

$$\begin{aligned} h_{l,\hat{\alpha}_l} &= -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2}, \\ \mathcal{D}\left(h_{l,\hat{\alpha}_l}\right) &= \Big\{ f \in L^2((0,\infty)) \mid f, f' \in AC_{loc}((0,\infty)\setminus\{R\}),\ f(0+) = 0, \text{ if } l = 0, \\ &\quad \left(1+\frac{\alpha_l}{2}\right) f'(R+) + \left(1-\frac{\alpha_l}{2}\right) f'(R-) = 0, \\ &\quad \left(1+\frac{\beta_l}{2}\right) f(R+) + \left(1-\frac{\beta_l}{2}\right) f(R-) = 0, \\ &\quad f'' + l(l+1) r^{-2} f \in L^2((0,\infty)) \Big\}, \\ &\quad \hat{\alpha}_l = \{\alpha_l, \beta_l\}, -\infty < \alpha_l, \beta_l < \infty, l \in \mathbb{N}_0 \end{aligned} \tag{53}$$

The boundary conditions in (53) define a self-adjoint extension of the radial schrödinger operator $\dot{h}_l$ defined by:

$$\begin{aligned} \dot{h}_l &= -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2}, \\ \mathcal{D}\left(\dot{h}_l\right) &= \Big\{ f \in L^2((0,\infty)) \mid f, f' \in AC_{loc}((0,\infty)),\ f(0+) = 0, \text{ if } l = 0, \\ &\quad f(R\pm) = f'(R\pm) = 0, -f'' + l(l+1) r^{-2} f \in L^2((0,\infty)) \Big\}, l \in \mathbb{N}_0 \end{aligned} \tag{54}$$

**Proof**

One may follow step by step Ref. [27], where a similar result was obtained in the case of the point interactions. The Hamiltonian $h_{l,\hat{\alpha}_l}$ defines a two-parameter model of nonrelativistic $\delta'$-sphere interaction in quantum mechanics.

## 2.7. The Scattering Theory for the Pair ($h_{jl,\hat{G}_{jl}}; h_D$)

Let us define for $k > 0$ the following function:

$$\begin{aligned} \begin{pmatrix} \eta_{jl,1}(z,r) \\ \eta_{jl,2}(z,r) \end{pmatrix} &= \begin{pmatrix} F_{jl}(z,r) \\ \tilde{F}_{jl}(z,r) \end{pmatrix} + \Omega_{jl}(z, A_{jl}, B_{jl}, R) \Big\{ F'_{jl}(z,R)\left[ A_{jl} Q^{(2)}_{jl}(r) + \frac{A_{jl}B_{jl}}{2c} \tilde{Q}^{(1)}_{jl}(r) \right] \\ &\quad + \tilde{F}'_{jl}(z,R)\left[ B_{jl} \tilde{Q}^{(2)}_{jl}(r) - \frac{A_{jl}B_{jl}}{2c} Q^{(1)}_{jl}(r) \right] \Big\} \end{aligned} \tag{55}$$

where the functions $F_{jl}(z,r), \tilde{F}_{jl}(z,r), \Omega_{jl}(z, A_{jl}, B_{jl}, R), Q^{(2)}_{jl}, \tilde{Q}^{(2)}_{jl}, \tilde{Q}^{(1)}_{jl}$ and $Q^{(1)}_{jl}$ are defined by Equation (21), Equations (41)-(45) respectively. A straightforward computation shows that $\begin{pmatrix} \eta_{jl,1}(z,r) \\ \eta_{jl,2}(z,r) \end{pmatrix}$ are scattering wave functions of

$h_{jl,\hat{G}_{jl}}$.

For the cases $A_{jl} \neq 0$ and $B_{jl} = 0$ and $A_{jl} = 0, B_{jl} \neq 0$, (55) yields, respectively to:

$$\begin{pmatrix} \eta_{jl,1}(z,r) \\ \eta_{jl,2}(z,r) \end{pmatrix} = F_{jl}(z,r)\tilde{F}_{jl}(z,r) + A_{jl}\Omega_{jl}(z, A_{jl}, 0, R) F'_{jl}(z,r) Q^{(2)}_{jl}(.) \quad (56)$$

and

$$\begin{pmatrix} \eta_{jl,1}(z,r) \\ \eta_{jl,2}(z,r) \end{pmatrix} = F_{jl}(z,r)\tilde{F}_{jl}(z,r) + B_{jl}\Omega_{jl}(z, 0, B_{jl}, R) \tilde{F}'_{jl}(z,r) \tilde{Q}^{(2)}_{jl}(.) \quad (57)$$

Equation (56) and Equation (57) define the scattering wave functions corresponding to the Hamiltonian $h_{jl,A_{jl}}$ and $h_{jl,B_{jl}}$ describing two one parameter relativistic $\delta'_s$-sphere interactions.

Let us determine the phase shift and the elements of the on-shell scattering matrix corresponding to $h_{jl,\hat{G}_{jl}}$ using the asymptotic behavior of $\eta_{jl}(z,r)$.

The asymptotic behavior of $\begin{pmatrix} \eta_{jl,1}(z,r) \\ \eta_{jl,2}(z,r) \end{pmatrix}$ as $r \to \infty$ yields to:

$$\begin{aligned} &\begin{pmatrix} \eta_{jl,1}(z,r) \\ \eta_{jl,2}(z,r) \end{pmatrix} \xrightarrow[r\to\infty]{k>0} \\ &\begin{pmatrix} C_{jl}(z)\sin\left[kr - \kappa_{jl}\dfrac{\pi}{2}\right] \\ \tilde{C}_{jl}(z)\sin\left[kr - (\kappa_{jl}-1)\dfrac{\pi}{2}\right] \end{pmatrix} + \Omega_{jl}(z, A_{jl}, B_{jl}, R)\Big[A_{jl}F'^{2}_{jl}(z,R) \\ &+ B_{jl}\tilde{F}'^{2}_{jl}(z,R)\Big] \begin{pmatrix} D_{jl}(z)\exp -\mathrm{i}\left[kr - \kappa_{jl}\dfrac{\pi}{2}\right] \\ \tilde{D}_{jl}(z)\exp -\mathrm{i}\left[kr - (\kappa_{jl}-1)\dfrac{\pi}{2}\right] \end{pmatrix} \\ &= \begin{pmatrix} \left[D_1^2(z) + D_2^2(z)\right]^{\frac{1}{2}} \sin\left[kr - \kappa_{jl}\dfrac{\pi}{2} + \varphi_{G_{jl}}(z)\right] \\ \left[D_3^2(z) + D_4^2(z)\right]^{\frac{1}{2}} \sin\left[kr - (\kappa_{jl}-1)\dfrac{\pi}{2} + \tilde{\varphi}_{G_{jl}}(z)\right] \end{pmatrix} \end{aligned} \quad (58)$$

where

$$\begin{aligned} C_{jl}(z) &= 2^{-\kappa_{jl}} k^{-\kappa_{jl}-1}\left(k^2 - \frac{\kappa_{jl}^2}{R^2}\right)^{-1}\Gamma(2\kappa_{jl}+2)\Gamma(\kappa_{jl}+1)^{-1}, \\ \tilde{C}_{jl}(z) &= \frac{1}{c}2^{-\kappa_{jl}} k^{-\kappa_{jl}}\left(k^2 - \frac{\kappa_{jl}^2}{R^2}\right)^{-1}\Gamma(2\kappa_{jl}+2)\Gamma(\kappa_{jl}+1)^{-1}, \\ D_{jl}(z) &= 2^{\kappa_{jl}} k^{\kappa_{jl}}\Gamma(2\kappa_{jl}+2)^{-1}\Gamma(\kappa_{jl}+1), \\ \tilde{D}_{jl}(z) &= \frac{1}{c}2^{\kappa_{jl}} k^{\kappa_{jl}+1}\Gamma(2\kappa_{jl}+2)^{-1}\Gamma(\kappa_{jl}+1). \end{aligned} \quad (59)$$

In this case, the phase shifts corresponding to $h_{jl,\hat{G}_{jl}}$ are defined as:

$$\begin{aligned}\varphi_{\hat{G}_{jl}} &= -\arctan\frac{D_2}{D_1} = -\arctan\frac{D_4}{D_3} \\ &= -\arctan\frac{D_{jl}(z)\bar{\Omega}_{jl}(z)}{C_{jl}(z)-\mathrm{i}D_{jl}(z)\bar{\Omega}_{jl}(z)} \\ &= -\arctan\frac{\tilde{D}_{jl}(z)\bar{\Omega}_{jl}(z)}{\tilde{C}_{jl}(z)-\mathrm{i}\tilde{D}_{jl}(z)\bar{\Omega}_{jl}(z)},\end{aligned} \tag{60}$$

where

$$\bar{\Omega}_{jl}(z) = \Omega_{jl}(z, A_{jl}, B_{jl}, R)\left[A_{jl}F'_{jl}(z,R)F'_{jl}(z,R) + B_{jl}\tilde{F}'_{jl}(z,R)\tilde{F}'_{jl}(z,R)\right] \tag{61}$$

The elements of the on-shell scattering matrix are given by:

$$S_{\hat{G}_{jl}}(z) = \exp\left[2\mathrm{i}\varphi_{\hat{G}_{jl}}(z)\right]. \tag{62}$$

The partial wave scattering amplitude is given by:

$$f_{\hat{G}_{jl}}(z) = \frac{\exp\left[2\mathrm{i}\varphi_{\hat{G}_{jl}}(z)\right]-1}{2\mathrm{i}k}. \tag{63}$$

## 3. The $\delta'_s$-Sphere plus Coulomb Interaction

### 3.1. The Definition of the Hamiltonian

Let us consider the formal expression given by:

$$H_{\alpha,\hat{G}} = H_D + \frac{\alpha}{|x|} + \hat{G}\delta'_s(|x|-R),\ \alpha \in \mathbb{R},\ x \in \mathbb{R}^3,\ R > 0, \tag{64}$$

where $\hat{G}$ is a real matrix of the form:

$$\hat{G} = \begin{pmatrix} A_{jl} & 0 \\ 0 & B_{jl} \end{pmatrix} \tag{65}$$

Let use the decomposition Equation (12) and introduce in $L^2((0,\infty))\otimes\mathbb{C}^2$ the operator:

$$\hat{H}_{\alpha,\hat{G}} = \bigoplus_{j=\frac{1}{2}}^{\infty}\bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}}\left[U_{jl}^{-1}h_{jl,\alpha}U_{jl}\right]\otimes\mathbb{1}, \tag{66}$$

where $h_{jl,\alpha}$ is given by:

$$h_{jl,\alpha} = \begin{pmatrix} \frac{c^2}{2}+\frac{\alpha}{r} & -c\frac{\mathrm{d}}{\mathrm{d}r}+c\frac{\kappa_{jl}}{r} \\ c\frac{\mathrm{d}}{\mathrm{d}r}+c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2}+\frac{\alpha}{r} \end{pmatrix} \equiv \tau_\alpha,$$

$$\begin{aligned}\mathcal{D}(h_{jl,\alpha}) = \{&\chi_{jl} \in L^2((0,\infty))\otimes\mathbb{C}^2 \mid \chi_{jl} \in AC_{loc}((0,\infty));\ \chi_{jl}(R\pm)=0; \\ &\tau_\alpha\chi_{jl} \in L^2((0,\infty))\otimes\mathbb{C}^2\}.\end{aligned} \tag{67}$$

$\kappa_{jl}$ is defined by Equation (13).

The adjoint $\hat{H}^*_{\alpha,\hat{G}}$ of $\hat{H}_{\alpha,\hat{G}}$ reads:

$$
\begin{aligned}
&\hat{H}^{*}_{\alpha,\hat{G}} = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} \left[ U_{jl}^{-1} h^{*}_{jl,\alpha} U_{jl} \right] \otimes \mathbb{1}, \\
&h^{*}_{jl,\alpha} = \begin{pmatrix} \frac{c^2}{2} + \frac{\alpha}{r} & -c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} \\ c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} + \frac{\alpha}{r} \end{pmatrix} \equiv \tau_{\alpha}, \\
&\mathcal{D}\left(h^{*}_{jl,\alpha}\right) = \left\{ \chi_{jl} \in L^2\left((0,\infty)\right) \otimes \mathbb{C}^2 \mid \chi_{jl} \in AC_{loc}\left((0,\infty) \setminus \{R\}\right), \right. \\
&\qquad \left. \tau_{\alpha}\chi_{jl} \in L^2\left((0,\infty)\right) \otimes \mathbb{C}^2 \right\}.
\end{aligned}
\tag{68}
$$

### 3.2. The Self-Adjoint Extension of $h_{jl,\alpha}$

Let us consider the following equation:

$$
\left(h^{*}_{jl,\alpha} - z\right)\chi_{jl} = 0,\ \chi_{jl} = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}\left(h^{*}_{jl,\alpha}\right),\ z \in \mathbb{C} - \left\{ \left(-\infty, -\frac{c^2}{2}\right] \cup \left[\frac{c^2}{2}, \infty\right) \right\}, \tag{69}
$$

and introduce the following notations:

$$
\begin{aligned}
k &= \frac{1}{c}\sqrt{z^2 - \frac{c^4}{4}}, \\
\zeta &= \sqrt{\kappa_{jl}^2 c^2 - \alpha^2}, \\
\tilde{\zeta} &= \frac{1}{c}\zeta, \\
\tilde{\alpha} &= \frac{2z\alpha}{c^2}.
\end{aligned}
\tag{70}
$$

Equation (69) has two linearly independent solutions:

$$
\chi^{(1)}_{jl,\alpha,z} = \begin{cases} \begin{pmatrix} f_{jl,\alpha,1}(z,r) \\ f_{jl,\alpha,2}(z,r) \end{pmatrix}, & r < R, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}; & r > R, \end{cases} \tag{71}
$$

$$
\chi^{(2)}_{jl,\alpha,z} = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}; & r < R, \\ \begin{pmatrix} g_{jl,\alpha,1}(z,r) \\ g_{jl,\alpha,2}(z,r) \end{pmatrix}, & r > R, \end{cases} \tag{72}
$$

where

$$
f_{jl,\alpha,1}(z,r) = \left(1 - \frac{\alpha^2}{\left(\kappa_{jl}c + \zeta\right)^2}\right)^{-\frac{1}{2}} \left[\cos\left(\arctan\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]^{-\frac{1}{2}} \left[F_{jl,\alpha}(z,r) - \frac{\alpha}{\kappa_{jl}c + \zeta}\tilde{F}_{jl,\alpha}(z,r)\right],
$$

$$
f_{jl,\alpha,2}(z,r) = \left(1 - \frac{\alpha^2}{\left(\kappa_{jl}c + \zeta\right)^2}\right)^{-\frac{1}{2}} \left[\cos\left(\arctan\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]^{-\frac{1}{2}} \left[\tilde{F}_{jl,\alpha}(z,r) - \frac{\alpha}{\kappa_{jl}c + \zeta}F_{jl,\alpha}(z,r)\right],
$$

$$g_{jl,\alpha,1}(z,r)=\gamma(\kappa_{jl},R)\left(1-\frac{\alpha^2}{(\kappa_{jl}c+\zeta)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]^{-\frac{1}{2}}\left[G_{jl,\alpha}(z,r)-\frac{\alpha}{\kappa_{jl}c+\zeta}\tilde{G}_{jl,\alpha}(z,r)\right],$$
$$g_{jl,\alpha,2}(z,r)=\gamma(\kappa_{jl},R)\left(1-\frac{\alpha^2}{(\kappa_{jl}c+\zeta)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]^{-\frac{1}{2}}\left[\tilde{G}_{jl,\alpha}(z,r)-\frac{\alpha}{\kappa_{jl}c+\zeta}G_{jl,\alpha}(z,r)\right]. \quad (73)$$

and

$$F_{jl,\alpha}(z,r)=r^{\tilde{\zeta}+1}e^{-ikr}\,{}_1F_1\left(\tilde{\zeta}+1-i\frac{\tilde{\alpha}}{2k},2\tilde{\zeta}+2,2ikr\right),$$
$$G_{jl,\alpha}(z,r)=\Gamma(2\tilde{\zeta}+2)^{-1}\Gamma\left(\tilde{\zeta}+1-\frac{i\tilde{\alpha}}{2k}\right)(2ik)^{2\tilde{\zeta}+1}r^{\tilde{\zeta}+1}e^{-ikr}U\left(\tilde{\zeta}+1-i\frac{\tilde{\alpha}}{2k},2\tilde{\zeta}+2,2ikr\right),$$
$$\tilde{F}_{jl,\alpha}(z,r)=\frac{\tilde{\zeta}}{c}(2\tilde{\zeta}+1)\left|\Gamma\left(\tilde{\zeta}+\frac{i\tilde{\alpha}}{2k}\right)\right|\left|\Gamma\left(\tilde{\zeta}+1+\frac{i\tilde{\alpha}}{2k}\right)\right|^{-1}r^{\tilde{\zeta}}e^{-ikr}\times{}_1F_1\left(\tilde{\zeta}-i\frac{\tilde{\alpha}}{2k},2\tilde{\zeta},2ikr\right), \quad (74)$$
$$\tilde{G}_{jl,\alpha}(z,r)=\frac{\tilde{\zeta}^{-1}}{c}(2\tilde{\zeta}+1)^{-1}\Gamma(2\tilde{\zeta})^{-1}\Gamma\left(\tilde{\zeta}+\frac{i\tilde{\alpha}}{2k}\right)\left|\Gamma\left(\tilde{\zeta}+\frac{i\tilde{\alpha}}{2k}\right)\right|^{-1}\left|\Gamma\left(\tilde{\zeta}+1+\frac{i\tilde{\alpha}}{2k}\right)\right|$$
$$\times k^2(2ik)^{2\tilde{\zeta}-1}r^{\tilde{\zeta}}e^{-ikr}U\left(\tilde{\zeta}-i\frac{\tilde{\alpha}}{2k},2\tilde{\zeta},2ikr\right),$$

${}_1F_1(a,b,r)(U(a,b,r))$ denotes the regular (respectively, irregular) confluent hypergeometric functions.

$\gamma(\kappa_{jl},R)$ is the normalization constants chosen in such way:

$$\det\begin{bmatrix} g'_{jl,\alpha,1}(z,r) & f'_{jl,\alpha,1}(z,r) \\ g'_{jl,\alpha,2}(z,r) & f'_{jl,\alpha,2}(z,r)\end{bmatrix}=\frac{1}{c} \quad (75)$$

For the particular case $\alpha\to 0+$, we obtain:

$$\begin{pmatrix} f'_{jl,\alpha,1}(z,r) \\ f'_{jl,\alpha,2}(z,r)\end{pmatrix}\xrightarrow[\alpha\to 0+]{}\begin{pmatrix} F'_{jl}(z,r) \\ \tilde{F}'_{jl}(z,r)\end{pmatrix} \quad (76)$$

$$\begin{pmatrix} g'_{jl,\alpha,1}(z,r) \\ g'_{jl,\alpha,2}(z,r)\end{pmatrix}\xrightarrow[\alpha\to 0+]{}\begin{pmatrix} G'_{jl}(z,r) \\ \tilde{G}'_{jl}(z,r)\end{pmatrix} \quad (77)$$

where $F'_{jl}(z,r),\tilde{F}'_{jl}(z,r),G'_{jl}(z,r)$ and $\tilde{G}'_{jl}(z,r)$ are respectively the derivatives of the functions $F_{jl}(z,r),\tilde{F}_{jl}(z,r),G_{jl}(z,r)$ and $\tilde{G}_{jl}(z,r)$ defined by Equation (21).

The operator $h_{jl,\alpha}$ has deficiency indices (2, 2), and consequently, all its self-adjoint extensions may be parametrized by a four-parameter family of self-adjoint operators.

Consider the following two-parameters family of self-adjoint extensions of $h_{jl,\alpha}$:

$$h_{jl,\alpha,G_{jl}}=\begin{pmatrix} \frac{c^2}{2}+\frac{\alpha}{r} & -c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} \\ -c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2}+\frac{\alpha}{r}\end{pmatrix},$$

$$\mathcal{D}\left(h_{jl,\alpha,G_{jl}}\right)=\left\{\chi_{jl}\in\mathcal{D}\left(h_{jl}^{*}\right)\mid\left\{\begin{array}{l}\chi'_{jl,1}(R+)-\chi'_{jl,1}(R-)+\frac{1}{2C}B_{jl}\left[\chi'_{jl,2}(R+)+\chi'_{jl,2}(R-)\right]=0\\ \chi'_{jl,2}(R+)-\chi'_{jl,2}(R-)-\frac{1}{2C}A_{jl}\left[\chi'_{jl,1}(R+)+\chi'_{jl,1}(R-)\right]=0\end{array}\right\}\right\} \quad (78)$$

$$A_{jl},B_{jl}\in\mathbb{R},\ l\in\left[j-\frac{1}{2},j+\frac{1}{2}\right];\ j\in\left[\frac{1}{2},\infty\right).$$

The operator $h_{jl,\alpha,G_{jl}}$ gives the mathematical definition of the formal expression:

$$h_{jl,\alpha,G_{jl}}=h_D+\frac{\alpha}{r}+G_{jl}\delta'_s(r-R),\ R>0. \quad (79)$$

The case $G_{jl}=0$ in Equation (78) gives the radial Dirac-Coulomb Hamiltonian $h_{jl,\alpha,0}\equiv h_{\alpha,D}$,

$$\begin{aligned}&h_{\alpha,D}=h_D+\frac{\alpha}{r}\equiv\tau_\alpha,\\&\mathcal{D}\left(h_{\alpha,D}\right)=\left\{\chi_{jl}\in\mathcal{D}\left(h_{jl,\alpha}^{*}\right)\mid\begin{array}{l}\chi_{jl}(R+)=\chi_{jl}(R-)\\ \chi'_{jl}(R+)=\chi'_{jl}(R-)\end{array}\right\};\left[j-\frac{1}{2},j+\frac{1}{2}\right],\left[\frac{1}{2},\infty\right).\end{aligned} \quad (80)$$

In particular case when $A_{jl}\neq 0$, $B_{jl}=0$ and $A_{jl}=0$, $B_{jl}\neq 0$ in Equation (78) simplifies respectively to one parameter $\delta'_s$-sphere plus Coulomb interaction. Therefore, the model in Equation (64) is defined in $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$ by:

$$\hat{H}_{\alpha,\hat{G}}=\bigoplus_{j=\frac{1}{2}}^{\infty}\bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}}\left[U_{jl}^{-1}h_{jl,\alpha,\hat{G}_{jl}}U_{jl}\right]\otimes\mathbb{1}. \quad (81)$$

The particular case $\hat{G}_{jl}=0$ for all *j* and *l* provides the Dirac-Coulomb Hamiltonian $H_{\alpha,D}$,

$$H_{\alpha,D}=H_D+\frac{\alpha}{|x|},\ \mathcal{D}\left(H_{\alpha,D}\right)=H^{1,2}(\mathbb{R}^3)\otimes\mathbb{C}^4. \quad (82)$$

### 3.3. The Resolvent Equation of $h_{jl,\alpha,\hat{G}_{jl}}$

**Theorem 3.1:** The resolvent of $h_{jl,\alpha,\hat{G}_{jl}}$ reads:

$$\begin{aligned}&\left(h_{jl,\alpha,G_{jl}}-z\right)^{-1}(.)=\left(h_{\alpha,D}-z\right)^{-1}(.)+\Omega_{jl,\alpha}(z,R)\left\{\left(\overline{Q_{jl,\alpha}^{(2)}(.)},.\right)\right.\\&\times\left[A_{jl}Q_{jl,\alpha}^{(2)}(.)+\frac{A_{jl}B_{jl}}{2c}\tilde{Q}_{jl,\alpha}^{(1)}(.)\right]+\left(\overline{\tilde{Q}_{jl,\alpha}^{(2)}(.)},.\right)\left.\left[B_{jl}\tilde{Q}_{jl,\alpha}^{(2)}(.)-\frac{A_{jl}B_{jl}}{2c}Q_{jl,\alpha}^{(1)}(.)\right]\right\},\\&z\in\rho\left(h_{jl,\alpha,\hat{G}_{jl}}\right),\ \operatorname{Im}k>0,\ l\in\left[j-\frac{1}{2},j+\frac{1}{2}\right],\ j\in\left[\frac{1}{2},\infty\right),\end{aligned} \quad (83)$$

where $\rho(.)$ is the resolvent set and where $\left(h_{\alpha,D}-z\right)^{-1}$, $\operatorname{Im}k(z)>0$ is the radial Dirac resolvent with kernel:

$$G^{jl,\alpha}(z,r,R)=\begin{pmatrix}G_{11}^{jl,\alpha}(z,r,R)&G_{12}^{jl,\alpha}(z,r,R)\\G_{21}^{jl,\alpha}(z,r,R)&G_{22}^{jl,\alpha}(z,r,R)\end{pmatrix}, \quad (84)$$

and

$$G_{11}^{jl,\alpha}(z,r,R)=\begin{cases} g'_{jl,\alpha,1}(z,R)\,f_{jl,\alpha,1}(z,r), & r<R \\ f'_{jl,\alpha,1}(z,R)\,g_{jl,1}(z,r), & r>R, \end{cases}$$
$$G_{12}^{jl,\alpha}(z,r,R)=\begin{cases} g'_{jl,\alpha,2}(z,R)\,f_{jl,\alpha,1}(z,r), & r<R \\ f'_{jl,\alpha,2}(z,R)\,g_{jl,\alpha,1}(z,r), & r>R, \end{cases}$$
$$G_{21}^{jl,\alpha}(z,r,R)=\begin{cases} g'_{jl,\alpha,1}(z,R)\,f_{jl,\alpha,2}(z,r), & r<R \\ f'_{jl,\alpha,1}(z,R)\,g_{jl,\alpha,2}(z,r), & r>R, \end{cases} \tag{85}$$
$$G_{22}^{jl,\alpha}(z,r,R)=\begin{cases} g'_{jl,\alpha,2}(z,R)\,f_{jl,\alpha,2}(z,r), & r<R \\ f'_{jl,\alpha,2}(z,R)\,g_{jl,\alpha,2}(z,r), & r>R \end{cases}$$

$$\Omega_{jl,\alpha}(z,R) = -\left[1-\frac{B_{jl}A_{jl}}{4c^2}+B_{jl}\,f'_{jl,\alpha,2}(z,R)\,g'_{jl,\alpha,2}(z,R)+A_{jl}\,f'_{jl,\alpha,1}(z,R)\,g'_{jl,\alpha,1}(z,R)\right]^{-1}, \tag{86}$$

$$Q_{jl,\alpha}^{(2)}=\begin{cases} \begin{pmatrix} g'_{jl,\alpha,1}(z,R)\,f_{jl,\alpha,1}(z,r) \\ g'_{jl,\alpha,1}(z,R)\,f_{jl,\alpha,2}(z,r) \end{pmatrix}, & r<R, \\ \begin{pmatrix} f'_{jl,\alpha,1}(z,R)\,g_{jl,\alpha,1}(z,r) \\ f'_{jl,\alpha,1}(z,R)\,g_{jl,\alpha,2}(z,r) \end{pmatrix}, & r>R, \end{cases} \tag{87}$$

$$\tilde{Q}_{jl,\alpha}^{(2)}=\begin{cases} \begin{pmatrix} g'_{jl,\alpha,2}(z,R)\,f_{jl,\alpha,1}(z,r) \\ g'_{jl,\alpha,2}(z,R)\,f_{jl,\alpha,2}(z,r) \end{pmatrix}, & r<R, \\ \begin{pmatrix} f'_{jl,\alpha,2}(z,R)\,g_{jl,\alpha,1}(z,r) \\ f'_{jl,\alpha,2}(z,R)\,g_{jl,\alpha,2}(z,r) \end{pmatrix}, & r>R, \end{cases} \tag{88}$$

$$\tilde{Q}_{jl,\alpha}^{(1)}=\begin{cases} \begin{pmatrix} g'_{jl,\alpha,2}(z,R)\,f_{jl,\alpha,1}(z,r) \\ g'_{jl,\alpha,2}(z,R)\,f_{jl,\alpha,2}(z,r) \end{pmatrix}, & r<R, \\ \begin{pmatrix} -f'_{jl,\alpha,2}(z,R)\,g_{jl,\alpha,1}(z,r) \\ f'_{jl,\alpha,2}(z,R)\,g_{jl,\alpha,2}(z,r) \end{pmatrix}, & r>R, \end{cases} \tag{89}$$

$$Q_{jl,\alpha}^{(1)}=\begin{cases} \begin{pmatrix} g'_{jl,\alpha,1}(z,R)\,f_{jl,\alpha,1}(z,r) \\ g'_{jl,\alpha,1}(z,R)\,f_{jl,\alpha,2}(z,r) \end{pmatrix}, & r<R, \\ \begin{pmatrix} -f'_{jl,\alpha,1}(z,R)\,g_{jl,\alpha,1}(z,r) \\ f'_{jl,\alpha,1}(z,R)\,g_{jl,\alpha,2}(z,r) \end{pmatrix}, & r>R, \end{cases} \tag{90}$$

with the functions $f_{jl,\alpha,1}(z,r), f_{jl,\alpha,2}(z,r), g_{jl,\alpha,1}(z,r), g_{jl,\alpha,2}(z,r)$ defined by Equation (73).

### 3.4. The Spectral Properties of $h_{jl,\alpha,G_{jl}}$

**Theorem 3.2:** For $A_{jl}, B_{jl} \in (-\infty,\infty)$, $l \in \left[j-\frac{1}{2}, j+\frac{1}{2}\right]$, $j \in \left[\frac{1}{2},\infty\right)$ and $\alpha \in \mathbb{R}$, the essential spectrum of $h_{jl,\alpha,G_{jl}}$ is purely absolutely continuous and coincide with $\left(\infty,-\frac{c^2}{2}\right] \cup \left[\frac{c^2}{2},\infty\right)$. Its singular continuous and residual spectra are empty. The bound states of $h_{jl,\alpha,\hat{G}_{jl}}$ in the gap $\left(-\frac{c^2}{2},\frac{c^2}{2}\right)$ coincide with the poles of the resolvent Equation (90) in $\operatorname{Im} k > 0$.

**Proof**

Follow step by step the proof of the Theorem 2.4.

## 3.5. The Scattering Theory for the Pair ( $h_{jl,\alpha,\hat{G}_{jl}}, h_{\alpha,D}$ )

Let us define, for $k > 0$, the following function:

$$\begin{pmatrix}\eta_{jl,\alpha,1}(z,r)\\ \eta_{jl,\alpha,2}(z,r)\end{pmatrix}=\begin{pmatrix}f_{jl,\alpha,1}(z,r)\\ f_{jl,\alpha,2}(z,r)\end{pmatrix}+\Omega_{jl,\alpha}(z,R)\Big\{f'_{jl,\alpha,1}(z,R)\Big[A_{jl}Q^{(2)}_{jl,\alpha}(z,r)+\frac{A_{jl}B_{jl}}{2c}\tilde{Q}^{(1)}_{jl,\alpha}(z,r)\Big]+f'_{jl,\alpha,2}(z,R)\Big[B_{jl}\tilde{Q}^{(2)}_{jl,\alpha}(z,r)-\frac{A_{jl}B_{jl}}{2c}Q^{(1)}_{jl,\alpha}(z,r)\Big]\Big\}, \quad (91)$$

with the functions $f_{jl,\alpha,1}(z,r), f_{jl,\alpha,2}(z,r), \Omega_{jl,\alpha}(z,R), Q^{(2)}_{jl,\alpha}(.), \tilde{Q}^{(2)}_{jl,\alpha}, \tilde{Q}^{(1)}_{jl,\alpha}$ and $Q^{(1)}_{jl,\alpha}(.)$ defined by Equation (73), Equations (86)-(90) respectively.

A straightforward computation shows that $\begin{pmatrix}\eta_{jl,\alpha,1}\\ \eta_{jl,\alpha,2}\end{pmatrix}$ are scattering wave functions of $h_{jl,\alpha,\hat{G}_{jl}}$. The cases $A_{jl}\neq 0$, $B_{jl}=0$ and $A_{jl}=0, B_{jl}\neq 0$ in Equation (91) simplifies, respectively to:

$$\begin{pmatrix}\eta_{jl,\alpha,1}(z,r)\\ \eta_{jl,\alpha,2}(z,r)\end{pmatrix}=\begin{pmatrix}f_{jl,\alpha,1}(z,r)\\ F_{jl,\alpha,2}(z,r)\end{pmatrix}+A_{jl}\Omega_{jl,\alpha}(z,A_{jl},0,R)\,f'_{jl,\alpha,1}(z,R)Q^{(2)}_{jl}(z,r) \quad (92)$$

and

$$\begin{pmatrix}\eta_{jl,\alpha,1}(z,r)\\ \eta_{jl,\alpha,2}(z,r)\end{pmatrix}=\begin{pmatrix}f_{jl,\alpha,1}(z,r)\\ f_{jl,\alpha,2}(z,r)\end{pmatrix}+B_{jl}\Omega_{jl,\alpha}(z,0,B_{jl},R)\,f'_{jl,\alpha,2}(z,R)\tilde{Q}^{(2)}_{jl,\alpha}(z,r) \quad (93)$$

Equation (92) and Equation (93) define the scattering wave functions corresponding to the Hamiltonian $h_{jl,\alpha,A_{jl}}$ and $h_{jl,\alpha,B_{jl}}$ describing two one parameter relativistic $\delta'_s$-sphere interactions.

Let us determine the phase shift and the elements of the on-shell scattering matrix corresponding to $h_{jl,\alpha,\hat{G}_{jl}}$ using the asymptotic behavior of $\eta_{jl,\alpha}(z,r)$.

The asymptotic behavior of $\begin{pmatrix}\eta_{jl,\alpha,1}\\ \eta_{jl,\alpha,2}\end{pmatrix}$ as $r\to\infty$ yields [28]:

$$\begin{pmatrix}\eta_{jl,\alpha,1}(z,r)\\ \eta_{jl,\alpha,2}(z,r)\end{pmatrix}\xrightarrow[r\to\infty]{k>0}\begin{pmatrix}V_1(z)\sin\left[y_1+\delta^0_{\xi}\right]+V_2(z)\cos\left[y_1+\delta^0_{\xi}\right]\\ V_3(z)\sin\left[y_2+\delta^0_{\xi-1}\right]+V_4(z)\cos\left[y_2+\delta^0_{\xi-1}\right]\end{pmatrix}=\begin{pmatrix}\left[V_1^2(z)+V_2^2(z)\right]^{\frac{1}{2}}\sin\left[y_1+\delta^0_{\xi}+\delta^C_{\alpha,G_{jl},1}\right]\\ \left[V_3^2(z)+V_4^2(z)\right]^{\frac{1}{2}}\sin\left[y_2+\delta^0_{\xi-1}+\delta^C_{\alpha,G_{jl},2}\right]\end{pmatrix}, \quad (94)$$

where

$$y_1 = k - \frac{\tilde{\alpha}}{k}\ln(2kr) - \tilde{\zeta}\frac{\pi}{2},$$
$$y_2 = k - \frac{\tilde{\alpha}}{k}\ln(2kr) - (\tilde{\zeta}-1)\frac{\pi}{2}. \tag{95}$$

and

$$\delta_{\zeta}^{0}(z) = \delta_{\zeta-1}^{0}(z) + \arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right),$$
$$\delta_{\zeta}^{0}(z) = \arg\Gamma\left(\tilde{\zeta} + 1 + \mathrm{i}\frac{\tilde{\alpha}}{2k}\right). \tag{96}$$

The Coulomb modified phase shift $\varphi^{C}_{\alpha,\hat{G}_{jl}}$ corresponding to $h_{jl,\alpha,\hat{G}_{jl}}$ is given by:

$$\begin{pmatrix} \varphi^{C}_{\alpha,\hat{G}_{jl},1} \\ \varphi^{C}_{\alpha,\hat{G}_{jl},2} \end{pmatrix} = \begin{pmatrix} -\arctan\dfrac{V_2(z)}{V_1(z)} \\ -\arctan\dfrac{V_4(z)}{V_3(z)} \end{pmatrix}. \tag{97}$$

The constants $V_i\,(i=1,2,3,4)$ are defined by:

$$V_1(z) = d_1(z) + \left[d_2(z) - \mathrm{i}\bar{\Omega}_{jl,\alpha}(z)d_4(z)\right]\sin\left[\arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]$$
$$-\bar{\Omega}_{jl,\alpha}(z)\left(\mathrm{i}d_3(z) + d_4(z)\right)\cos\left[\arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right],$$
$$V_2(z) = \left[d_2(z) - \mathrm{i}\bar{\Omega}_{jl,\alpha}(z)d_4(z)\right]\cos\left[\arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]$$
$$+\bar{\Omega}_{jl,\alpha}(z)\left(d_3(z) + d_4(z)\right)\sin\left[\arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right],$$
$$V_3(z) = \frac{\alpha}{\kappa_{jl}c+\zeta}\left[-d_1(z) + \mathrm{i}\bar{\Omega}_{jl,\alpha}(z)d_3(z)\right]\sin\left[\arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]$$
$$-\left(\frac{\alpha}{\kappa_{jl}c+\zeta}\right)^{-1}d_2(z) + \bar{\Omega}_{jl,\alpha}(z)\left[-\frac{\alpha}{\kappa_{jl}c+\zeta}d_3(z)\cos\left[\arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]\right.$$
$$\left.+\mathrm{i}\left(\frac{\alpha}{\kappa_{jl}c+\zeta}\right)^{-1}d_4(z)\right], \tag{98}$$
$$V_4(z) = \frac{\alpha}{\kappa_{jl}c+\zeta}\left[-d_1(z) + \mathrm{i}\bar{\Omega}_{jl,\alpha}(z)d_3(z)\right]\cos\left[\arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]$$
$$+\bar{\Omega}_{jl,\alpha}(z)\left[-\frac{\alpha}{\kappa_{jl}c+\zeta}d_3(z)\sin\left[\arctan\left(\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right] - \left(\frac{\alpha}{\kappa_{jl}c+\zeta}\right)^{-1}d_4(z)\right],$$

with $\bar{\Omega}_{jl,\alpha}(z)$ defined by:

$$\bar{\Omega}_{jl,\alpha}(z) = \Omega_{jl,\alpha}(z,R)\left[A_{jl}\,f'_{jl,\alpha,1}(z,R)\,f'_{jl,\alpha,1}(z,R) + B_{jl}\,f'_{jl,\alpha,2}(z,R)\,f'_{jl,\alpha,2}(z,R)\right]. \tag{99}$$

The constants $d_i,\ i=1,2,3,4$ are given by:

$$d_1(z)=\left(1-\frac{\alpha^2}{(\kappa_{jl}c+\zeta)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]^{-\frac{1}{2}}2^{-\tilde{\zeta}}k^{-\tilde{\zeta}-1}\Gamma\left(2\tilde{\zeta}+2\right)$$
$$\times\left|\Gamma\left(\tilde{\zeta}+1+\mathrm{i}\frac{\tilde{\alpha}}{2k}\right)\right|^{-1}\mathrm{e}^{\pi\frac{\tilde{\alpha}}{2k}},$$
$$d_2(z)=-\frac{\alpha}{c(\kappa_{jl}c+\zeta)}\left(1-\frac{\alpha^2}{(\kappa_{jl}c+\zeta)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]^{-\frac{1}{2}}2^{-\tilde{\zeta}}k^{-\tilde{\zeta}}$$
$$\times\Gamma\left(2\tilde{\zeta}+2\right)\left|\Gamma\left(\tilde{\zeta}+1+\mathrm{i}\frac{\tilde{\alpha}}{2k}\right)\right|^{-1}\mathrm{e}^{\pi\frac{\tilde{\alpha}}{2k}},$$
$$d_3(z)=\left(1-\frac{\alpha^2}{(\kappa_{jl}c+\zeta)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]^{-\frac{1}{2}}2^{\tilde{\zeta}}k^{\tilde{\zeta}}\Gamma\left(2\tilde{\zeta}+2\right)^{-1}$$
$$\times\left|\Gamma\left(\tilde{\zeta}+1+\mathrm{i}\frac{\tilde{\alpha}}{2k}\right)\right|\mathrm{e}^{-\pi\frac{\tilde{\alpha}}{2k}},\qquad(100)$$
$$d_4(z)=-\frac{\alpha}{c(\kappa_{jl}c+\zeta)}\left(1-\frac{\alpha^2}{(\kappa_{jl}c+\zeta)^2}\right)^{-\frac{1}{2}}\left[\cos\left(\arctan\frac{\tilde{\alpha}}{2k\tilde{\zeta}}\right)\right]^{-\frac{1}{2}}2^{\tilde{\zeta}}k^{\tilde{\zeta}+1}$$
$$\times\Gamma\left(2\tilde{\zeta}+2\right)^{-1}\left|\Gamma\left(\tilde{\zeta}+1+\mathrm{i}\frac{\tilde{\alpha}}{2k}\right)\right|\mathrm{e}^{-\pi\frac{\tilde{\alpha}}{2k}}.$$

The limit $\alpha\to 0+$, in (97) yields:

$$\lim_{\alpha\to 0+}\delta^C_{\alpha,\hat{G}_{jl},1}=\lim_{\alpha\to 0+}\varphi^C_{\alpha,\hat{G}_{jl},2}=\varphi_{\hat{G}_{jl}}(z)\qquad(101)$$

where $\varphi_{\hat{G}_{jl}}(z)$ is defined by (60).

The Coulomb-modified on-shell scattering matrix is given by:

$$S_{\alpha,\hat{G}_{jl},n}=\exp\left[2\mathrm{i}\phi^C_{\alpha,\hat{G}_{jl},n}(z)\right],\ n=1,2.\qquad(102)$$

## 4. Conclusion

In this paper, using the self-adjoint theory of symmetric operator in Hilbert space, we studied the basic properties of two-parameter models of relativistic $\delta'_s$-sphere and $\delta'_s$-sphere plus Coulomb interaction (where a charged particle is perturbed by a $\delta'_s$-sphere interaction). For both interactions, we obtain interesting results on resolvent equations, spectral properties and scattering data (the phase shift, scattering matrix, scattering amplitude, and scattering cross section). As a perspective, one can use simulations or real-world data to validate the theoretical models proposed in this paper. Also, in our future paper in preparation, we intend to study the case where the relativistic $\delta'_s$-sphere interaction is centered on finitely many concentric spheres.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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