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Efficient Decomposition Shooting Method for Solving Third-Order Boundary Value Problems

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Abstract

The present paper proposes a mathematical method to numerically treat a class of third-order linear Boundary Value Problems (BVPs). This method is based on the combination of the Adomian Decomposition Method (ADM) and, the modified shooting method. A complete derivation of the proposed method has been provided, in addition to its numerical implementation and, validation via the utilization of the Runge-Kutta method and, other existing methods. The method has been applied to diverse test problems and turned out to perform remarkably. Lastly, the simulated numerical results have been graphically illustrated and, also supported by some absolute error comparison tables.

Keywords

Linear Third Order BVPs, Shooting Method, Adomian Decomposition Method, Two-Point Boundary Value Problem

1. Introduction

Many physical problems arising in engineering, science, economics, social science and, business are modeled mostly using the Ordinary Differential Equations (ODEs) in the form of either the Boundary Value problems (BVPs) or Initial Value Problems (IVPs), or their combination. Additionally, not all of these ODEs have exact analytical solutions through the known analytical techniques. Thus, this reason has forced various mathematicians and researchers to direct their inquisitiveness in the search for optimal computational procedures that will serve as an alternative approach to at least get hold of approximate solutions whenever the exact solutions are not analytically realistic.

However, the ODEs of particular concern in this study are the third-order linear BVPs. Many attempts have been made in the past literature to study different forms of third-order BVPs, including for instance the coupled systems of third-order BVPs and, laying a solid foundation for the existence and stability of the associated solutions, see [1]-[6] and, the references therewith. Moreover, as the methods for BVPs are very limited, Javeed *et al.* [7] proposed a numerical approach based on the application of the shooting method to solve different BVPs by coming up with higher-order initial guesses. It is remarkable to state here that the shooting method utilized in [7] is a promising iterative approach that starts off by transforming the governing BVPs to corresponding IVPs. To go further, some coupled systems of third-order BVPs were tackled using different methods like the finite difference method by Noor *et al.* [8] and, via the application of a meticulous computational procedure by Al-Said [9]. See also the recent work of Nasira *et al.* [10] for a new numerical procedure to solve third-order BVPs with two-point and multipoint Robin boundary data. Furthermore, the well-known semi-analytical method called the Adomian Decomposition Method (ADM) has equally been greatly used in both the past and recent literature for solving various linear and nonlinear BVPs, see [11]-[20] and the references therewith for some related studies and various modifications of the ADM.

Furthermore, since we aim in this study to combine the ADM and shooting method to numerically examine the third-order linear BVPs, it is notable to mention the work of Attili and Syam [21] where the combination of these methods was proposed and used to study the two-point BVPs of the form

$$q(t)u''(t)+q'(t)u'(t)=f(t,u,u'), \tag{1}$$

with the following prescribed Dirichlet boundary data

$$u(a)=\alpha,\ u(b)=\beta, \tag{2}$$

where the functions $q(t),q'(t)$ and $f(t,u,u')$ are known continuous functions. Further, the following inverse operator was particularly utilized while studying the problem [21]

$$L^{-1}(t,D)=\int_a^t \frac{\mathrm{d}x}{q(x)}\int_a^x \frac{\mathrm{d}s}{\xi(s)}u(s), \tag{3}$$

where

$$L(t,D)=\xi(t)Dq(t)D, \tag{4}$$

with $D=\frac{d}{d(t)}$, and $\frac{1}{q(t)}$ is locally integrable; while $q(t)$ and $\xi(t)$ are smooth functions.

However, the present paper proposes a numerical method to study a class of third-order linear BVPs by utilizing the combination of the ADM and, the modified shooting method. A complete derivation of this method will be provided, executed and, validated using promising methods in the literature.

2. Adomian Decomposition Method (ADM)

The ADM is an efficient semi-analytical method that is widely used to solve a variety of functional equations. To demonstrate how this method is utilized, let us consider the following generalized third-order linear ODE

$$y''' = p(x)y'' + q(x)y' + r(x)y + s(x),\ \ a \le x \le b, \tag{5}$$

subject to the following initial conditions

$$y(a) = \lambda,\ \ y'(a) = \alpha,\ \ y''(a) = t. \tag{6}$$

Now, expressing the above equation in an operator form, we write

$$Ly = p(x)y'' + q(x)y' + r(x)y + s(x),\ \ a \le x \le b, \tag{7}$$

where $L = \frac{\mathrm{d}^3}{\mathrm{d}x^3}$, and the functions $p(x), q(x), r(x), s(x)$ are known prescribed functions; while λ, α and t are also known constants. The inverse operator is expressed as

$$L^{-1}(.) = \int_a^x\int_a^x\int_a^x (.)\,\mathrm{d}x\mathrm{d}x\mathrm{d}x. \tag{8}$$

Therefore, applying the inverse operators given in Equation (8) to Equation (7), we have

$$y(x) = \phi(x) + L^{-1}\left(p(x)y'' + q(x)y' + r(x)y + s(x)\right), \tag{9}$$

such that

$$L\phi(x) = 0. \tag{10}$$

Additionally, the ADM decomposed the solution $y(x)$ as follows

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

such that upon substituting it into Equation (9) yields

$$\sum_{n=0}^{\infty} y_n = \phi(x) + L^{-1}\left[p(x)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sum_{n=0}^{\infty} y_n\right) + q(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} y_n\right) + r(x)\left(\sum_{n=0}^{\infty} y_n\right) + s(x)\right],$$

through which the ADM recurrently gives the following iterates

$$y_0 = \phi(x) + L^{-1}\left(s(x)\right),$$

$$y_{n+1} = L^{-1}\left(p(x)y_n'' + q(x)y_n' + r(x)y_n\right),\ \ n \ge 0. \tag{11}$$

such that the overall closed-form solution (approximate solution) of the governing equation given in Equation (5) together with the conditions given in Equation (6) is expressed as

$$y_m(x) = \sum_{n=0}^{m} y_n(x).$$

3. Efficient Decomposition Shooting Method (EDSM)

The shooting method is a promising iterative approach that starts off by transforming the governing BVP to a corresponding system of IVPs with specified initial value conditions. These unknown initial conditions are guessed to solve the

IVPs; whereas the accurateness of the guessed missing initial conditions is thereafter ascertained by putting both the given value at the terminal point and, the computed value of the dependent variable side by side. Furthermore, when these two values differ, a new value should be guessed continuously until an agreed value is attained between the two under a specified degree of accurateness.

At the moment, let us consider the following third-order linear two-point BVP

$$y''' = p(x)y'' + q(x)y' + r(x)y + s(x), \quad a \le x \le b, \tag{12}$$

$$y(a) = \lambda, \quad y'(a) = \alpha, \quad y(b) = \beta, \tag{13}$$

Now, based on the shooting method, the third-order BVP given above will turn into two IVPs where the boundary conditions given in Equation (13) will be replaced with specific initial conditions for each IVP as follows

$$u''' = p(x)u'' + q(x)u' + r(x)u + s(x), \quad a \le x \le b, \tag{14}$$

$$u(a) = \lambda, \quad u'(a) = \alpha, \quad u''(a) = 0, \tag{15}$$

and

$$v''' = p(x)v'' + q(x)v' + r(x)v, \quad a \le x \le b, \tag{16}$$

$$v(a) = 0, \quad v'(a) = 0, \quad v''(a) = 1. \tag{17}$$

Additionally, Equation (14) can be written as

$$Lu = p(x)u'' + q(x)u' + r(x)u + s(x), \tag{18}$$

where the proposed differential operator may take the form of L earlier defined; together with inverse L^{-1} given in Equation (8).

By applying L^{-1} on Equation (18), one gets

$$u(x) = \phi_1(x) + L^{-1}(p(x)u'') + L^{-1}(q(x)u') + L^{-1}(r(x)u) + L^{-1}(s(x)), \tag{19}$$

such that $L\phi_1(x) = 0$.

Next, the ADM decomposes the solution $u(x)$ by the following infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{20}$$

where the component $u_n(x)$ are recurrently determined. More, substituting the above series solution into Equation (19) yields

$$\begin{aligned}\sum_{n=0}^{\infty} u_n &= \phi_1(x) + L^{-1}\left(p(x)\sum_{n=0}^{\infty} u_n''(x)\right) + L^{-1}\left(q(x)\sum_{n=0}^{\infty} u_n'(x)\right) \\ &+ L^{-1}\left(r(x)\sum_{n=0}^{\infty} u_n(x)\right) + L^{-1}(s(x)),\end{aligned} \tag{21}$$

of which the components $u_n(x)$ are recurrently found as follows

$$\begin{aligned} u_0 &= \phi_1(x) + L^{-1}(s(x)), \\ u_1 &= L^{-1}\left(p(x)u_0''(x) + q(x)u_0'(x) + r(x)u_0(x)\right), \\ u_2 &= L^{-1}\left(p(x)u_1''(x) + q(x)u_1'(x) + r(x)u_1(x)\right), \\ &\vdots \end{aligned} \tag{22}$$

and so on. Also, for the sake of numerical computation, the m-term approximant are considered as

$$u = \psi_{1,m}(x) = \sum_{k=0}^{m} u_k(x), \tag{23}$$

Similarly, corresponding to the second BVP given in Equation (16), we can compute

$$v = \psi_{2,m}(x) = \sum_{k=0}^{m} v_k(x), \tag{24}$$

where the components $v_k(x)$ can be determined as

$$\begin{aligned} v_0 &= \phi_2(x), \\ v_1 &= L^{-1}\left(p(x)v_0''(x) + q(x)v_0'(x) + r(x)v_0(x)\right), \\ v_2 &= L^{-1}\left(p(x)v_1''(x) + q(x)v_1'(x) + r(x)v_1(x)\right), \\ &\vdots \end{aligned} \tag{25}$$

and so on. Subsequently, we construct z where z can be written as

$$\begin{aligned} z_1(x) &= u_1(x) + \theta v_1(x), \\ z_2(x) &= u_2(x) + \theta v_2(x), \\ &\vdots \end{aligned} \tag{26}$$

and so on, where θ is a constant number.

Hence, let $u(x)$ and $v(x)$ denote the solutions to the third-order linear IVPs given in Equations (14)-(17), respectively, then, we define

$$z(x) = u(x) + \frac{\beta - u(b)}{v(b)} v(x), \quad v(b) \neq 0, \tag{27}$$

where $z(x)$ is the solution to the third-order linear BVP given Equations ((12), (13)).

4. Numerical Examples

This section examines the proposed methodology for the third-order linear BVPs by demonstrating its applicability to some selected test problems. The method is also compared with the mixture of the fourth-order Runge-Kutta approach and the shooting method to further assess the performance of the proposed method. Similarly, we will compare our results with other numerical methods in the literature [22]-[28] according to the value of m used in them. Therefore, each method used in the comparison was indicated by an abbreviated symbol as follows:

- **LSRKM4**: Fourth-Order Linear Shooting Runge-Kutta Method in [22].
- **FDM**: Finite Difference Method in [22].
- **PAM (3, 3)**: Pade Approximation (3, 3) Method in [22].
- **RCAM**: Rational Chebyshev Approximation Method in [22].
- **QBSM**: Quartic B-Spline Method in [22].
- **HPM2** Second-Order Homotopy Perturbation Method [23].

- **LSHPM2**: Second-Order Least Square Homotopy Perturbation Method [23].
- **RKT3s4:** Three-Stage Fourth-Order Explicit Runge-Kutta Type Method [24].
- **GJGOMM**: Generalized Jacobi-Galerkin Operational Matrix Method in [25].
- **OCM**: Operational Collocation Method in [26].
- **ESM:** Exponential Spline Method [27].
- **QNSM:** Quintic Non-polynomial Spline Method [28].

Further, we present certain supportive **Tables 1-8** and **Figures 1-6** reporting the absolute error difference between the exact analytical solution and, on the other hand, the obtained numerical results using the proposed Efficient Decomposition Shooting Method (E_{EDSM}) and, further validated with the fourth-order Runge-Kutta method (E_{SRKM4}). Furthermore, it is worth mentioning here that, in all the graphical comparative illustrations given in **Figures 1-6**, the exact analytical solution is portrayed using a *blue line*, the proposed approximate solution E_{EDSM} is shown using a *dashed dotted red line* and, finally, the approximate benchmark solution E_{SRKM4} is depicted using asterisked curve black.

Example 1. Consider the third-order linear BVP given by [22]

$$y''' = 2x^2 y'' - 3xy' - 5x^2 y + \mathrm{e}^{2x}\left(3x^3 - x^2 - 5x - 4\right),\ 0 \le x \le 1,$$

$$y(0) = 1,\ y'(0) = 1,\ y(1) = 0.$$

The exact analytical solution is given by $y(x) = \mathrm{e}^{2x}(1-x)$.

Now, based on the proposed modified shooting decomposition method, we make consideration to the following two IVPs

$$u''' = 2x^2 u'' - 3xu' - 5x^2 u + \mathrm{e}^{2x}\left(3x^3 - x^2 - 5x - 4\right), \tag{28}$$

$$u(0) = 1,\ u'(0) = 1,\ u''(0) = 0, \tag{29}$$

and

$$v''' = 2x^2 v'' - 3xv' - 5x^2 v, \tag{30}$$

$$v(0) = 0,\ v'(0) = 0,\ v''(0) = 1. \tag{31}$$

The operator versions of Equations (28) and (30) can be written as

$$Lu = 2x^2 u'' - 3xu' - 5x^2 u + \mathrm{e}^{2x}\left(3x^3 - x^2 - 5x - 4\right), \tag{32}$$

$$Lv = 2x^2 v'' - 3xv' - 5x^2 v. \tag{33}$$

Applying L^{-1} to both sides in Equation (32) using the conditions in Equation (29), we get

$$\begin{aligned} u(x) = {} & \frac{15}{4} + \frac{27}{8}x + \frac{17}{16}x^2 + \frac{3}{8}x^3\mathrm{e}^{2x} - \frac{29}{16}x^2\mathrm{e}^{2x} + \frac{25}{8}x\mathrm{e}^{2x} \\ & - \frac{11}{4}\mathrm{e}^{2x} + L^{-1}\left(2x^2u'' - 3xu' - 5x^2u\right). \end{aligned} \tag{34}$$

Similarly, applying L^{-1} to both sides of Equation (33) using the conditions in Equation (31), we get

$$v(x) = \frac{x^2}{2} + L^{-1}\left(2x^2v'' - 3xv' - 5x^2v\right). \tag{35}$$

What's more, decomposing the respective solutions in Equations (34) and (35) based on the ADM, then respective recursive relationships are thus given by

$$\begin{cases} u_0(x) = \frac{15}{4} + \frac{27}{8}x + \frac{17}{16}x^2 + \frac{3}{8}x^3 e^{2x} - \frac{29}{16}x^2 e^{2x} + \frac{25}{8}x e^{2x} - \frac{11}{4}e^{2x}, \\ u_{n+1}(x) = 2L^{-1}\left(x^2 u_n''\right) - 3L^{-1}\left(x u_n'\right) - 5L^{-1}\left(x^2 u_n\right), \ n \ge 0, \end{cases}$$

and

$$\begin{cases} v_0(x) = \frac{x^2}{2}, \\ v_{n+1}(x) = 2L^{-1}\left(x^2 v_n''\right) - 3L^{-1}\left(x v_n'\right) - 5L^{-1}\left(x^2 v_n\right), \ n \ge 0. \end{cases}$$

In addition, expressing some of the terms from the above recursive relationships, we get

$$\begin{cases} u_0(x) = \frac{15}{4} + \frac{27}{8}x + \frac{17}{16}x^2 + \frac{3}{8}x^3 e^{2x} - \frac{29}{16}x^2 e^{2x} + \frac{25}{8}x e^{2x} - \frac{11}{4}e^{2x}, \\ u_1(x) = -\frac{87}{128}x - \frac{17}{672}x^7 - \frac{9}{64}x^6 - \frac{167}{480}x^5 - \frac{27}{64}x^4 + \frac{9}{64}x^5 e^{x^2} - \frac{57}{64}x^4 e^{x^2} \\ \qquad + \frac{73}{32}x^3 e^{x^2} - \frac{397}{128}x^2 e^{x^2} + \frac{333}{128}x e^{x^2} - \frac{123}{128}e^{x^2} - \frac{23}{128}x^2 + \frac{123}{128}, \\ \vdots \end{cases}$$

and

$$\begin{cases} v_0(x) = \frac{x^2}{2}, \\ v_1(x) = -\frac{1}{60}x^5 - \frac{1}{84}x^7, \\ \vdots \end{cases}$$

Then, the solutions of Equations (28) and (30) with $m = 10$ are obtained in a series form. Finally, the approximate solution $z(x)$ with $m = 10$, $h = \frac{1}{10}$, and $x_k = kh$ for $k = 0, 1, \cdots, m$, is given by

$$z(x_k) = u(x_k) + \frac{-u(1)}{v(1)} v(x_k).$$

In Table 1, we report the absolute error difference between the exact analytical solution and, the proposed solution E_{EDSM} and, further validated with E_{SRKM4} and the other methods described in [22]. In E_{SRKM4} and E_{LSRKM4}, the same method was used, which is the shooting method with Runge-Kutta method of order four, but using different hypotheses, so that in E_{SRKM4}, hypotheses (29) and (31) were used, but in E_{LSRKM4} other hypotheses were used, which are as follows

$$u(0) = 1, \ u'(0) = 0, \ u''(0) = 0,$$

and

$$v(0) = 0, \ v'(0) = 1, \ v''(0) = 0,$$

thus, we note that E_{SRKM4} was more accurate than E_{LSRKM4}.

Table 1. The absolute error for EDSM, SRKM4 and the methods described in [22] when $m = 10$.

x	E_{LSRKM4}	E_{FDM}	$E_{PAM(3,3)}$	E_{RCAM}	E_{QBSM}	E_{SRKM4}	E_{EDSM}
0.0	0	0	0	0	0	0	0
0.1	3.3×10^{-6}	2.5×10^{-4}	1.1×10^{-10}	8.2×10^{-8}	1.5×10^{-4}	7.1×10^{-7}	4.9×10^{-15}
0.2	6.4×10^{-6}	3.5×10^{-4}	1.3×10^{-10}	2.1×10^{-7}	5.9×10^{-4}	9.0×10^{-7}	2.0×10^{-14}
0.3	9.1×10^{-6}	3.1×10^{-4}	6.7×10^{-10}	3.8×10^{-7}	1.2×10^{-3}	6.4×10^{-7}	4.4×10^{-14}
0.4	1.1×10^{-5}	1.3×10^{-4}	1.4×10^{-9}	5.8×10^{-7}	2.0×10^{-3}	2.0×10^{-8}	7.8×10^{-14}
0.5	1.3×10^{-5}	1.5×10^{-4}	2.2×10^{-9}	7.7×10^{-7}	2.8×10^{-3}	8.5×10^{-7}	1.2×10^{-13}
0.6	1.3×10^{-5}	6.3×10^{-4}	2.9×10^{-9}	9.2×10^{-7}	3.5×10^{-3}	1.8×10^{-6}	1.8×10^{-13}
0.7	1.3×10^{-5}	9.0×10^{-4}	3.3×10^{-9}	9.9×10^{-7}	3.9×10^{-3}	2.6×10^{-6}	2.4×10^{-13}
0.8	1.1×10^{-5}	1.1×10^{-3}	3.2×10^{-9}	9.2×10^{-7}	3.7×10^{-3}	3.0×10^{-6}	3.1×10^{-13}
0.9	7.2×10^{-6}	9.6×10^{-4}	2.2×10^{-9}	6.3×10^{-7}	2.5×10^{-3}	2.4×10^{-6}	3.7×10^{-13}
1.0	0	0	0	0	0	0	0

Table 2. Comparison between different methods when $m = 10$.

Numerical Methods	Maximum Error
EDSM	3.7×10^{-13}
SRKM4	3.0×10^{-6}
LSRKM4	1.3×10^{-5}
FDM	1.1×10^{-3}
PAM(3,3)	3.3×10^{-9}
RCAM	9.9×10^{-7}
QBSM	3.9×10^{-3}

From **Table 2**, we can see that EDSM is the most efficient method for solving example 1, compare with the results of the five methods in ref. [22] and SRKM4.

Again, we portray the exact analytical and approximate solutions in **Figure 1**; one would notice an ideal agreement between these solutions.

Example 2. Consider the third-order linear BVP given by [23] [24] [25] [26]

$$y''' - xy = \mathrm{e}^x \left(x^3 - 2x^2 - 5x - 3\right), \quad 0 \le x \le 1$$

$$y(0) = 0, \quad y'(0) = 1, \quad y(1) = 0.$$

The exact analytical solution is given by $y(x) = x(1-x)\mathrm{e}^x$.

The approximate solution $z(x)$ with $m = 10$, $h = \frac{1}{10}$, and $x_k = kh$ for $k = 0, 1, \cdots, m$, is given by

$$z(x_k) = u(x_k) + \frac{-u(1)}{v(1)} v(x_k).$$

In **Table 3**, we report the absolute error difference between the exact analytical solution and, the proposed solution E_{EDSM} and, further validated with E_{SRKM4} and the other methods described in [23].

From **Table 4**, we can see that EDSM is the most efficient method for solving example 2, compare with the results of the all methods in [23] [24] [25] [26] and SRKM4.

Again, we portray the exact analytical and approximate solutions in **Figure 2**; one would notice an ideal agreement between these solutions.

Example 3. Consider the third-order linear singularly perturbed BVP given by [24] [25] [27]:

$$-\epsilon y''' + y = 81\epsilon^2 \cos(3x) + 3\epsilon \sin(3x),\ \ 0 \le x \le 1$$

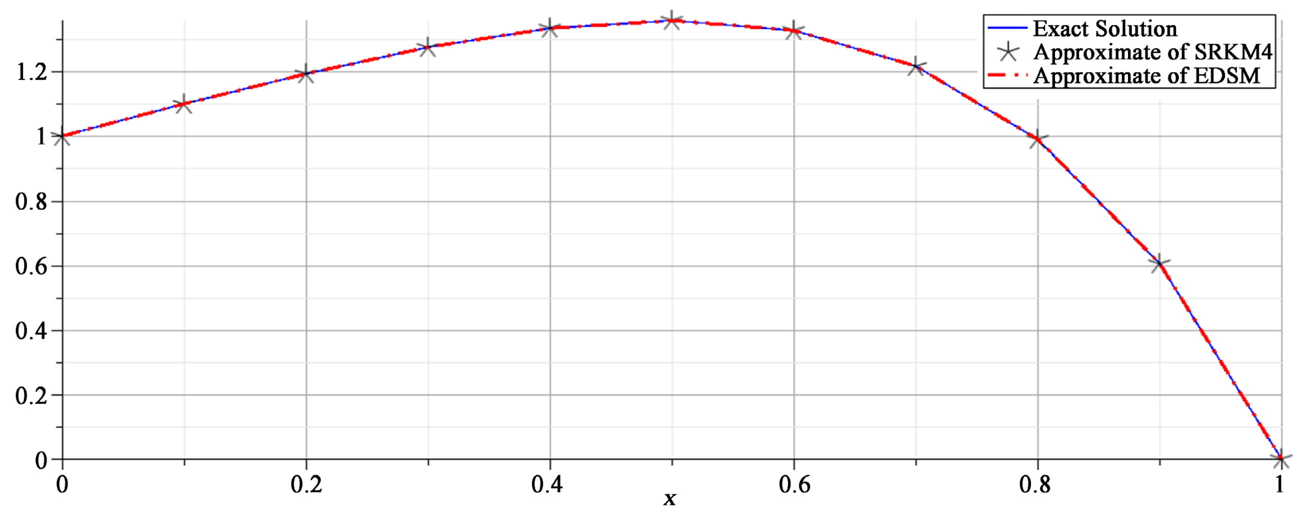

Figure 1. Graphical comparative depictions of the exact and approximate solutions with $m = 10$.

Table 3. The absolute error for EDSM, SRKM4 and the methods described in [23] when $m = 10$.

x	E_{LSHPM2}	E_{HPM2}	E_{SRKM4}	E_{EDSM}
0.0	1.8×10^{-15}	0	0	0
0.1	3.4×10^{-10}	3.1×10^{-7}	3.1×10^{-7}	5.3×10^{-23}
0.2	5.4×10^{-10}	1.2×10^{-6}	6.0×10^{-7}	3.2×10^{-23}
0.3	2.0×10^{-10}	2.7×10^{-6}	8.7×10^{-7}	2.3×10^{-23}
0.4	8.0×10^{-10}	4.7×10^{-6}	1.1×10^{-6}	5.1×10^{-23}
0.5	4.2×10^{-10}	7.0×10^{-6}	1.3×10^{-6}	3.8×10^{-23}
0.6	3.1×10^{-10}	9.3×10^{-6}	1.4×10^{-6}	4.8×10^{-23}
0.7	4.1×10^{-10}	1.1×10^{-5}	1.4×10^{-6}	6.7×10^{-24}
0.8	1.7×10^{-11}	1.0×10^{-5}	1.2×10^{-6}	3.8×10^{-23}
0.9	1.2×10^{-10}	7.2×10^{-6}	7.7×10^{-7}	6.9×10^{-23}
1.0	7.5×10^{-15}	1.6×10^{-13}	0	0

Table 4. Comparison between different methods.

m	Numerical Methods	Maximum Error
4	EDSM	3.2×10^{-16}
	SRKM4	4.7×10^{-5}
	RKT3s4	4.3×10^{-6}
	GJGOMM	1.0×10^{-6}
	OCM	9.9×10^{-7}
6	EDSM	1.3×10^{-24}
	SRKM4	1.0×10^{-5}
	GJGOMM	8.4×10^{-10}
	OCM	8.4×10^{-10}
8	EDSM	9.8×10^{-26}
	SRKM4	3.4×10^{-6}
	RKT3s4	2.3×10^{-7}
	GJGOMM	5.1×10^{-13}
	OCM	5.0×10^{-13}
10	EDSM	6.9×10^{-23}
	SRKM4	1.4×10^{-6}
	LSHPM2	8.0×10^{-10}
	HPM2	1.1×10^{-5}
	GJGOMM	4.9×10^{-16}
	OCM	2.3×10^{-16}

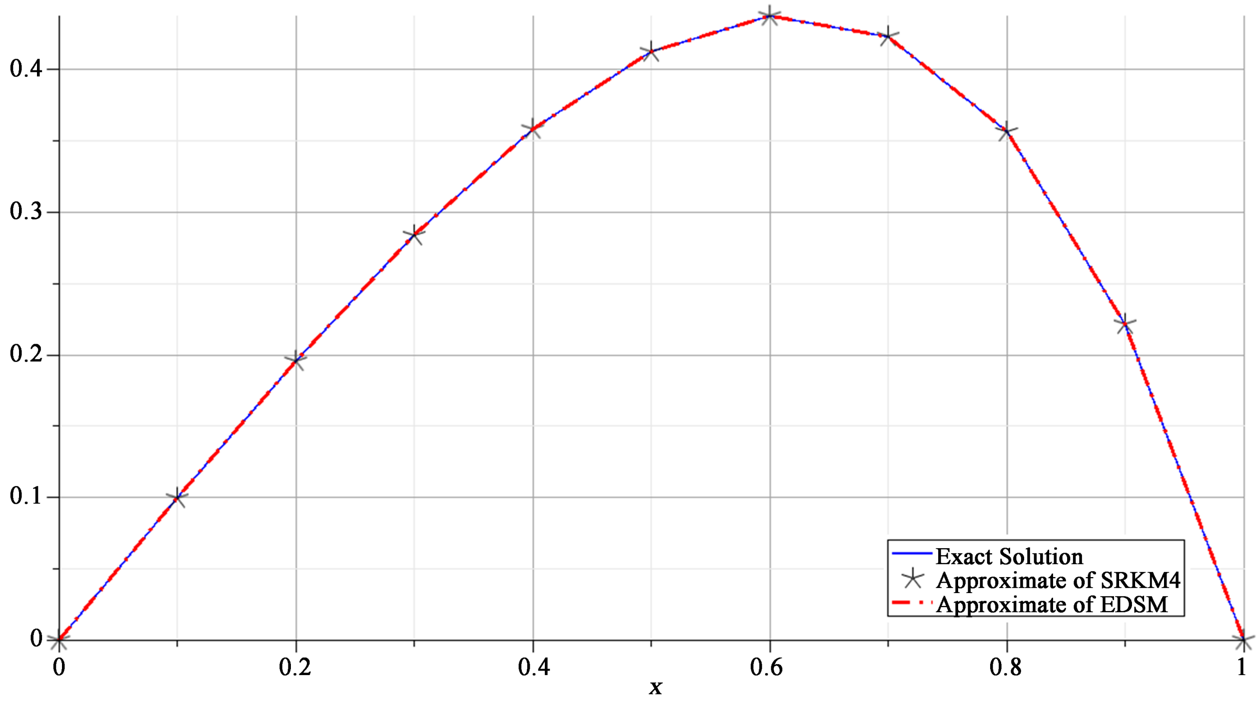

Figure 2. Graphical comparative depictions of the exact and approximate solutions with $m = 10$.

$$y(0)=0,\ y'(0)=9\epsilon,\ y(1)=3\epsilon\sin(3).$$

The exact analytical solution is given by $y(x)=3\epsilon\sin(3x)$.

The approximate solution $z(x)$ with $m=10$, $h=\frac{1}{10}$, and $x_k=kh$ for $k=0,1,\cdots,m$, is given by

$$z(x_k)=u(x_k)+\frac{3\epsilon\sin(3)-u(1)}{v(1)}v(x_k).$$

In **Table 5**, we report the absolute error difference between the exact analytical solution and, the proposed solution E_{EDSM} and, further validated with E_{SRKM4}.

From **Table 6**, we can see that EDSM is the most efficient method for solving example 3, compare with the results of the all methods in [24] [25] [27] and SRKM4.

Again, we portray the exact analytical and approximate solutions in **Figures 3-5**; one would notice an ideal agreement between these solutions.

Example 4. Consider the third-order linear BVP given by [24] [27] [28]:

$$-\epsilon y'''+y=6\epsilon(1-x)^5x^3-6\epsilon^2\Big(6(1-x)^5-90(1-x)^4x$$
$$+180(1-x)^3x^2-60(1-x)^2x^3\Big),\ 0\le x\le 1,$$
$$y(0)=0,\ y'(0)=0,\ y(1)=0.$$

The exact analytical solution is given by $y(x)=6x^3\epsilon(1-x)^5$.

The approximate solution $z(x)$ with $m=10$, $h=10$, and $x_k=kh$ for $k=0,1,\cdots,m$, is given by

Table 5. The absolute error for EDSM and SRKM4 when $m=10$ and different values ϵ.

x	$\epsilon=\frac{1}{16}$		$\epsilon=\frac{1}{32}$		$\epsilon=\frac{1}{64}$	
	E_{SRKM4}	E_{EDSM}	E_{SRKM4}	E_{EDSM}	E_{SRKM4}	E_{EDSM}
0.0	0	0	0	0	0	0
0.1	7.9×10^{-7}	2.6×10^{-28}	3.1×10^{-7}	2.2×10^{-25}	5.8×10^{-8}	1.5×10^{-22}
0.2	2.3×10^{-6}	1.0×10^{-27}	1.4×10^{-6}	8.6×10^{-25}	8.4×10^{-7}	6.2×10^{-22}
0.3	4.4×10^{-6}	2.4×10^{-27}	3.0×10^{-6}	2.0×10^{-24}	2.2×10^{-6}	1.4×10^{-21}
0.4	6.7×10^{-6}	4.2×10^{-27}	5.0×10^{-6}	3.6×10^{-24}	3.8×10^{-6}	2.6×10^{-21}
0.5	8.8×10^{-6}	6.7×10^{-27}	6.8×10^{-6}	5.7×10^{-24}	5.3×10^{-6}	4.3×10^{-21}
0.6	1.0×10^{-5}	9.9×10^{-27}	8.1×10^{-6}	8.6×10^{-24}	6.3×10^{-6}	6.8×10^{-21}
0.7	1.0×10^{-5}	1.4×10^{-26}	8.4×10^{-6}	1.3×10^{-23}	6.6×10^{-6}	1.0×10^{-20}
0.8	9.1×10^{-6}	1.9×10^{-26}	7.4×10^{-6}	1.8×10^{-23}	5.8×10^{-6}	1.6×10^{-20}
0.9	5.7×10^{-6}	2.4×10^{-26}	4.7×10^{-6}	2.4×10^{-23}	3.6×10^{-6}	2.2×10^{-20}
1.0	0	0	0	0	0	0

Table 6. Comparison between different methods when different values m and ϵ.

Numerical Methods	m	Maximum Error		
		$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$
EDSM	10	2.4×10^{-26}	2.4×10^{-23}	2.2×10^{-20}
	20	1.0×10^{-40}	4.5×10^{-39}	7.0×10^{-33}
	40	2.0×10^{-40}	1.3×10^{-37}	2.1×10^{-25}
SRKM4	10	1.0×10^{-5}	8.4×10^{-6}	6.6×10^{-6}
	20	7.0×10^{-7}	5.5×10^{-7}	4.1×10^{-7}
	40	4.5×10^{-8}	3.5×10^{-8}	2.6×10^{-8}
RKT3s4	10	1.9×10^{-5}	3.7×10^{-5}	7.0×10^{-5}
	20	2.6×10^{-6}	5.0×10^{-6}	9.9×10^{-6}
	40	3.4×10^{-7}	6.6×10^{-7}	1.3×10^{-6}
GJGOMM	10	9.3×10^{-13}	4.2×10^{-13}	4.2×10^{-13}
ESM	10	4.4×10^{-8}	1.9×10^{-8}	7.9×10^{-9}
	20	2.1×10^{-10}	8.9×10^{-11}	3.6×10^{-11}
	40	1.1×10^{-12}	4.5×10^{-13}	1.8×10^{-13}
QNSM	10	3.1×10^{-7}	1.3×10^{-7}	5.7×10^{-8}
	20	4.9×10^{-9}	2.1×10^{-9}	8.5×10^{-10}
	40	7.5×10^{-11}	3.2×10^{-11}	1.3×10^{-11}

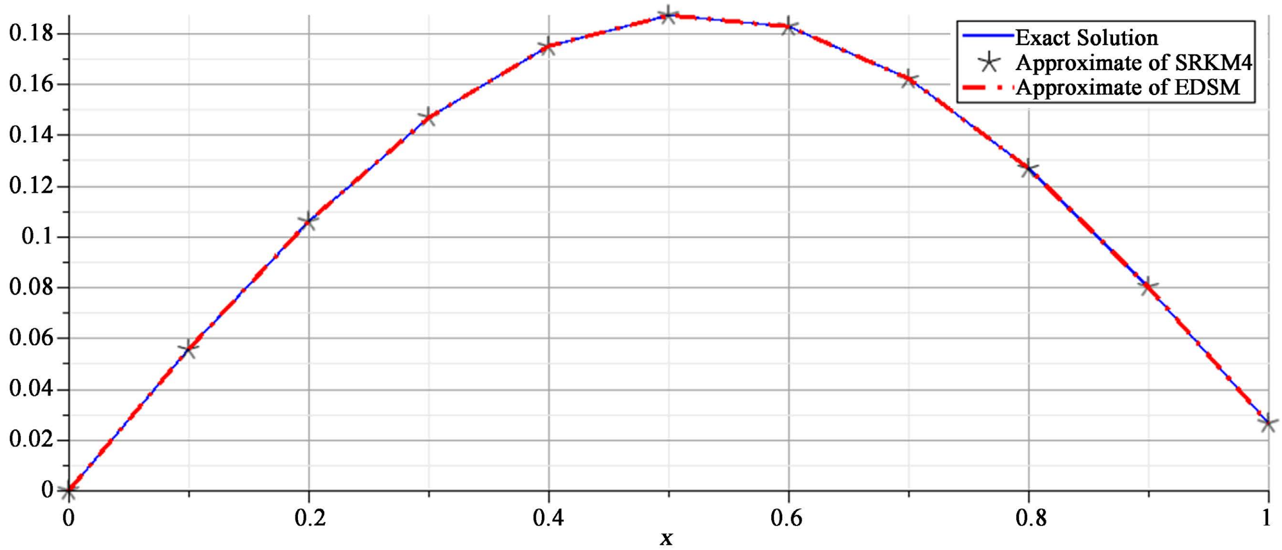

Figure 3. Graphical comparative depictions of the exact and approximate solutions with $m = 10$ and $\epsilon = \frac{1}{16}$.

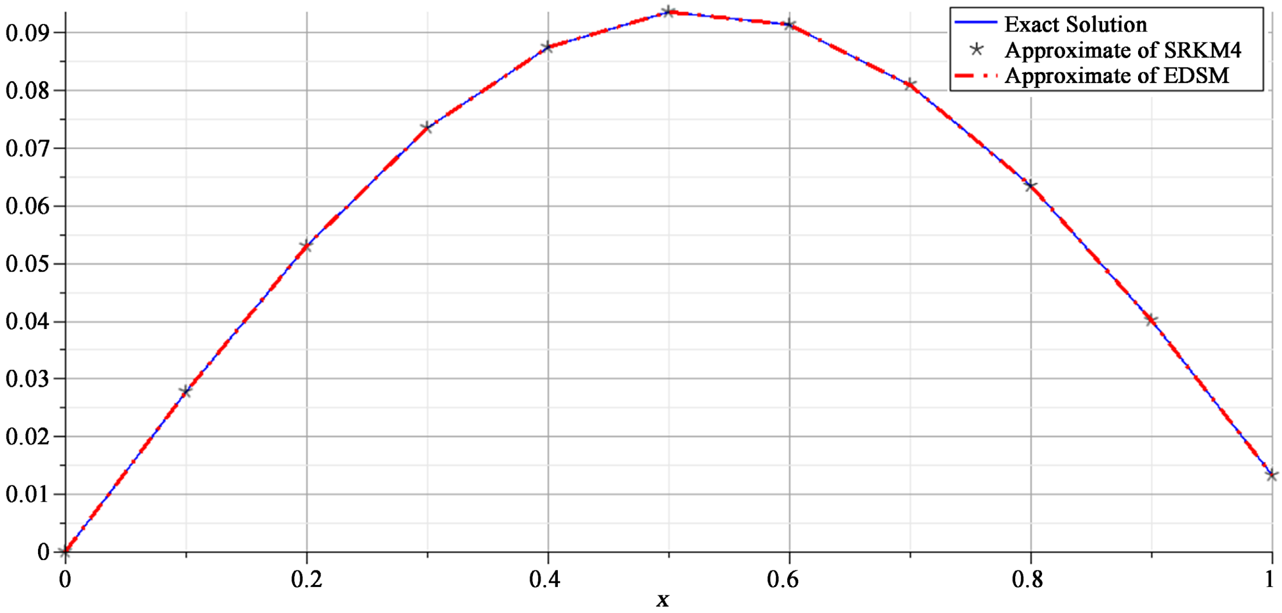

Figure 4. Graphical comparative depictions of the exact and approximate solutions with $m = 10$ and $\epsilon = \frac{1}{32}$.

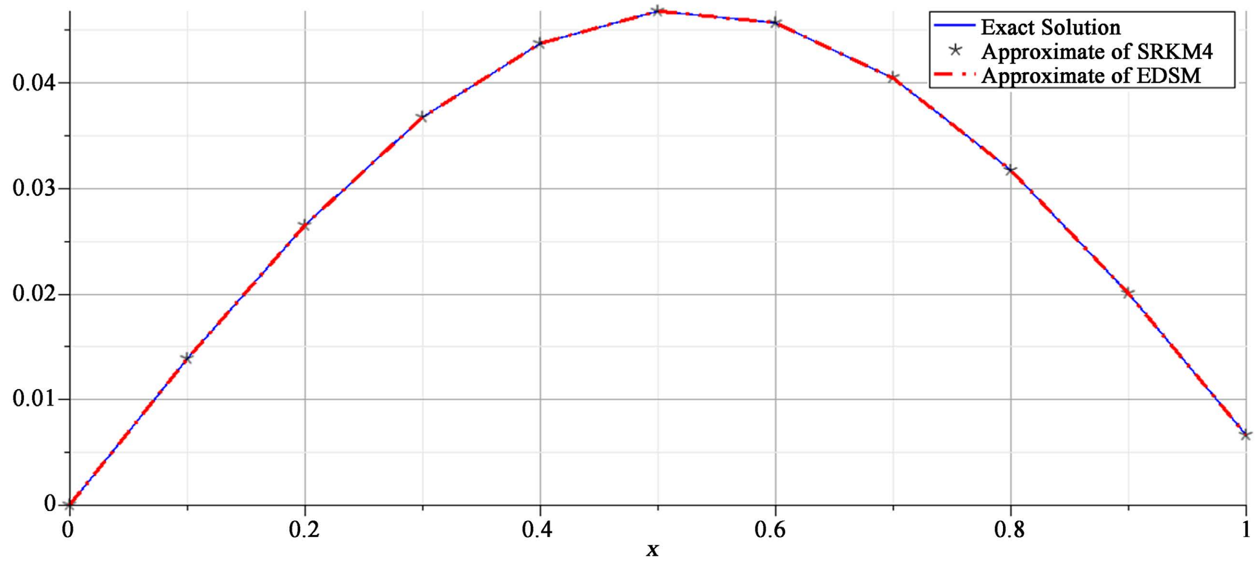

Figure 5. Graphical comparative depictions of the exact and approximate solutions with $m = 10$ and $\epsilon = \frac{1}{64}$.

$$z(x_k) = u(x_k) + \frac{-u(1)}{v(1)} v(x_k).$$

In **Table 7**, we report the absolute error difference between the exact analytical solution and, the proposed solution E_{EDSM} and, further validated with E_{SRKM4}.

From **Table 8**, we can see that EDSM is the most efficient method for solving example 4, comparison with the results of the all methods in [24] [27] [28] and SRKM4 when $m = 10$. Also, the EDSM was more accurate when compared with the same methods at $m \geq 20$, where we obtained the absolute error value of zero

Table 7. The absolute error for our method when $m = 10$ and different values ϵ.

x	$\epsilon = \frac{1}{16}$		$\epsilon = \frac{1}{32}$		$\epsilon = \frac{1}{64}$	
	E_{SRKM4}	E_{EDSM}	E_{SRKM4}	E_{EDSM}	E_{SRKM4}	E_{EDSM}
0.0	0	0	0	0	0	0
0.1	7.0×10^{-6}	4.8×10^{-31}	3.3×10^{-6}	4.0×10^{-28}	1.5×10^{-6}	2.8×10^{-25}
0.2	1.0×10^{-5}	1.9×10^{-30}	4.5×10^{-6}	1.6×10^{-27}	1.7×10^{-6}	1.1×10^{-24}
0.3	1.2×10^{-5}	4.4×10^{-30}	4.8×10^{-6}	3.6×10^{-27}	1.6×10^{-6}	2.6×10^{-24}
0.4	1.2×10^{-5}	7.9×10^{-30}	4.6×10^{-6}	6.6×10^{-27}	1.4×10^{-6}	4.9×10^{-24}
0.5	1.1×10^{-5}	1.2×10^{-29}	4.0×10^{-6}	1.1×10^{-26}	1.2×10^{-6}	8.1×10^{-24}
0.6	9.1×10^{-6}	1.8×10^{-29}	3.2×10^{-6}	1.6×10^{-26}	8.1×10^{-7}	1.3×10^{-23}
0.7	7.2×10^{-6}	2.6×10^{-29}	2.2×10^{-6}	2.3×10^{-26}	4.3×10^{-7}	1.9×10^{-23}
0.8	5.1×10^{-6}	3.5×10^{-29}	1.4×10^{-6}	3.3×10^{-26}	1.4×10^{-7}	2.9×10^{-23}
0.9	2.8×10^{-6}	4.6×10^{-29}	7.0×10^{-7}	4.4×10^{-26}	1.5×10^{-8}	4.2×10^{-23}
1.0	0	0	0	0	0	0

Table 8. Comparison between different methods when $m = 10$ and different values ϵ.

Numerical Methods	Maximum Error		
	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$
EDSM	4.6×10^{-29}	4.4×10^{-26}	4.2×10^{-23}
SRKM4	1.2×10^{-5}	4.8×10^{-6}	1.7×10^{-6}
RKT3s4	6.5×10^{-6}	2.8×10^{-6}	1.0×10^{-6}
ESM	1.0×10^{-6}	4.3×10^{-7}	1.8×10^{-7}
QNSM	6.9×10^{-6}	2.9×10^{-6}	1.2×10^{-6}

for all values of ϵ.

Again, we portray the exact analytical and approximate solutions in **Figures 6-8**; one would notice an ideal agreement between these solutions.

5. Conclusion

In conclusion, the present paper proposed a numerical method to treat a particular class of third-order BVPs based on the combination of the shooting method and, the Adomian decomposition method (EDSM). A complete derivation of the method has been provided, in addition to its numerical implementation and, validation with the help of the shooting method with the fourth-order Runge-Kutta method (SRKM4). The proposed method was further applied to certain test problems and turned out to outperform the SRKM4 and, other available

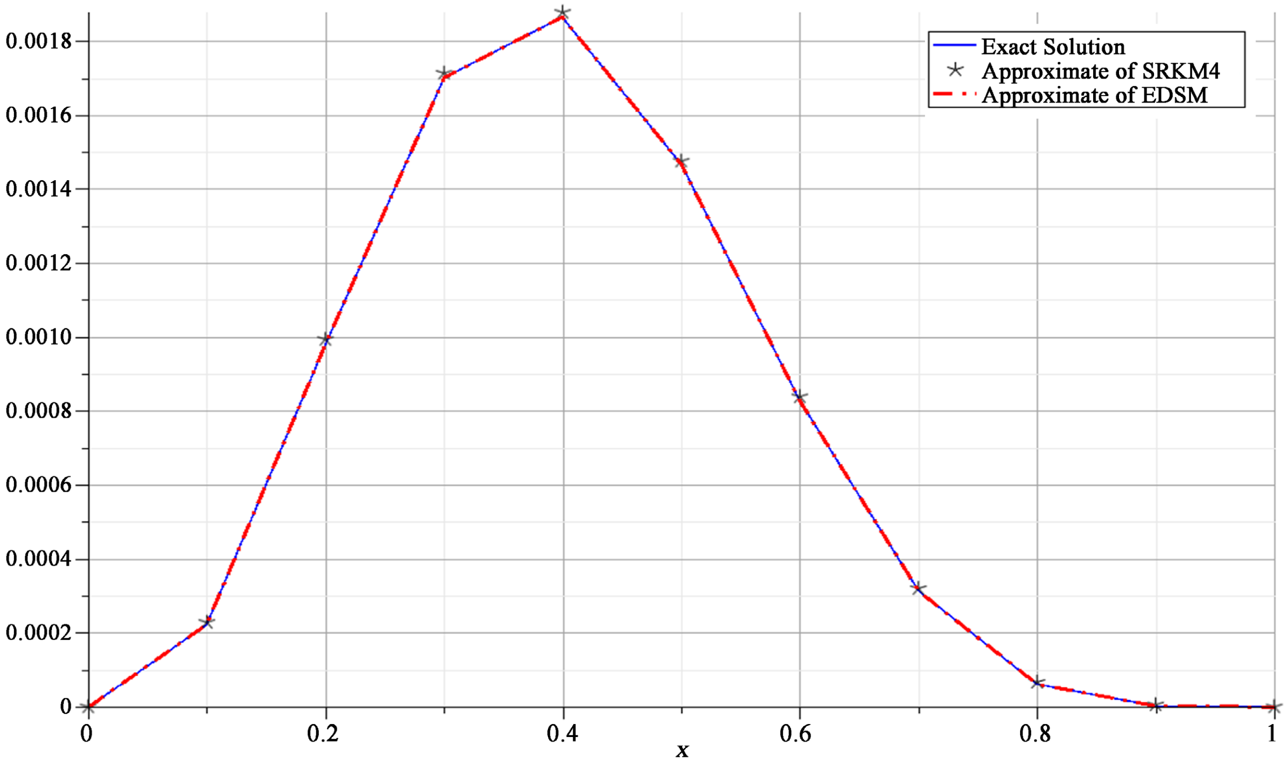

Figure 6. Graphical comparative depictions of the exact and approximate solutions with $m = 10$ and $\epsilon = \frac{1}{16}$.

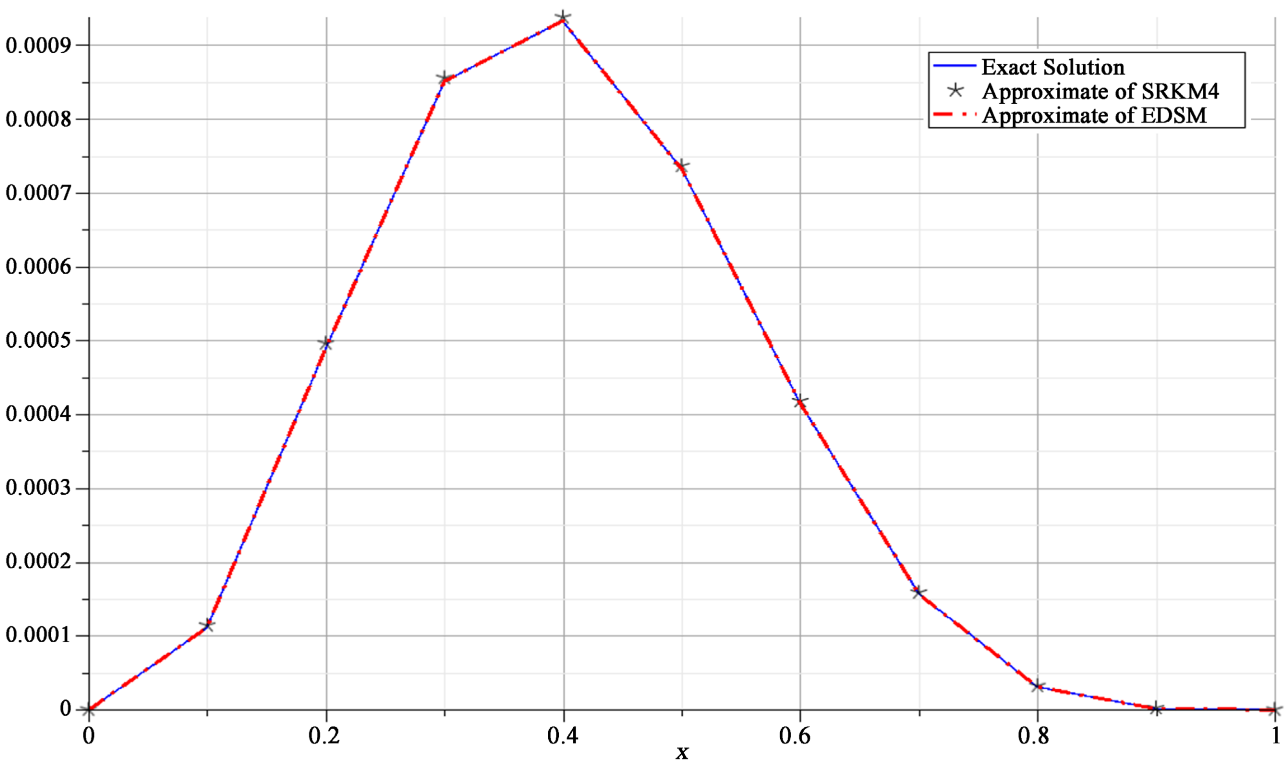

Figure 7. Graphical comparative depictions of the exact and approximate solutions with $m = 10$ and $\epsilon = \frac{1}{32}$.

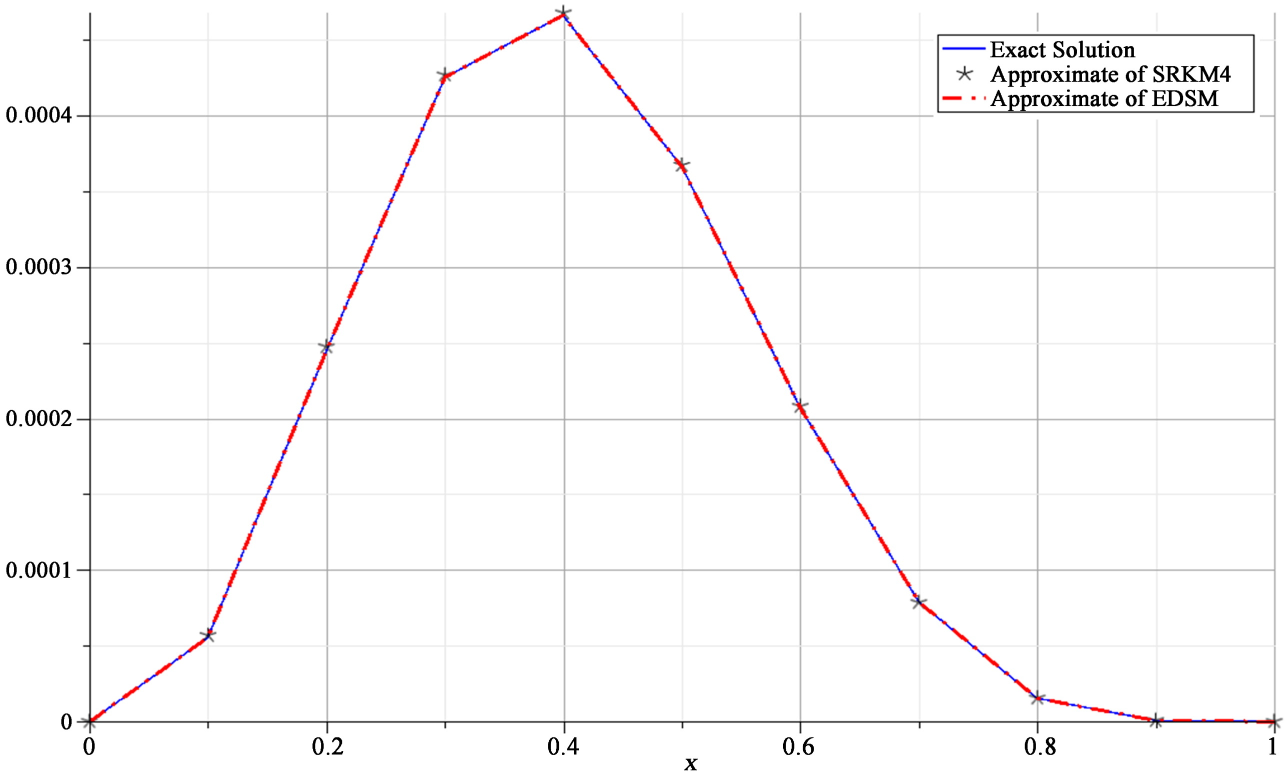

Figure 8. Graphical comparative depictions of the exact and approximate solutions with $m = 10$ and $\epsilon = \frac{1}{64}$.

methods in the literature. Lastly, we reported the simulated numerical results via graphical illustrations and, comparison tables.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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