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Game Theory Based Model for Predictive Analytics Using Distributed Position Function

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Abstract

This paper presents a game theory-based method for predicting the outcomes of negotiation and group decision-making problems. We propose an extension to the BDM model to address problems where actors' positions are distributed over a position spectrum. We generalize the concept of position in the model to incorporate continuous positions for the actors, enabling them to have more flexibility in defining their targets. We explore different possible functions to study the role of the position function and discuss appropriate distance measures for computing the distance between the positions of actors. To validate the proposed extension, we demonstrate the trustworthiness of our model's performance and interpretation by replicating the results based on data used in earlier studies.

Keywords

Distributed Position Function, Game Theory, Group Decision Making, Predictive Analytics

1. Introduction

The ability to predict the outcome of a negotiation is an important topic in various fields, including finance, healthcare, and political sciences. Decision or policy makers can use these forecasts to take the necessary measures before the results are known or to evaluate the possible outcomes of different positions they may choose to take. While policy experts typically use intuition to predict the future outcome of such problems, a mathematical framework is required to better forecast the outcome of a group decision problem that is reproducible, explainable, and free of bias [1] [2] [3].

In the literature, game theory-based models suggest that rational agents can be used to model negotiation or group decision-making problems, as these agents can adapt their positions to achieve the Nash equilibrium in each round. This allows stakeholders to have a reasonable and explainable forecast of the problem, enabling them to allocate more resources to their initial position, block other actors, form alliances with critical actors, or take a more extreme position to achieve their goals.

One of the best-known game theory-based methods for prediction was introduced by Bruce Bueno de Mesquita (BDM) in 1980 [4]. Later in 1994, he described the model in more detail [5]. Soon it was clear that the extreme predictive accuracy of the model seems to be more than a claim.

Bruce Bueno de Mesquita (BDM) introduced a well-known game theory-based method for prediction in 1980 [4]. The model was later explained in detail in [6] and the model's predictive accuracy was tested by replicating results based on data used in previous studies. However, the model was limited to one-dimensional problems, where the actors' positions are evaluated on a single subject. This limitation makes it difficult to predict the outcome of complex negotiations involving multiple issues. Therefore, there is a need for models that can address multi-dimensional problems and provide more accurate predictions. To address this need, a recent work proposed an N-dimensional model to tackle more complex problems involving multiple issues [7].

To predict the outcome of group decision-making problems, other models have also been developed. For example, in [8], the expected utility model was used to predict the outcome of Iran-US conflicts over various issues such as Iran's nuclear program and its stance toward Israel. Eftekhari developed another game theory-based prediction tool called Preana, which is based on BDM's model and uses a reinforcement learning mechanism to model the players' reasoning ability with regard to taking risks [9].

In [10] [11], a method for collecting data on European Union legislative initiatives was established, including the 2004 working-time-proposal, called Decision Making in the European Union (DEU) datasets. These datasets, DEU I and DEU II, are based on expert interviews and contain information on 331 contentious issues, as well as the policy positions and importance levels for each issue. In addition, [12] utilized expert interviews, text analysis, and media coverage to determine issue salience in EU legislative politics. Furthermore, [13] studied interstate bargaining related to major reforms of the Economic and Monetary Union through the EMU Positions dataset, which covers problems that were negotiated between 28 EU member states and significant EU institutions from 2010 to 2015, including their position and salience. The use of these datasets allows for the analysis and prediction of the EU legislative decision-making process, as well as the identification of influential actors and contentious issues. These insights can help stakeholders better allocate resources and develop effective strategies to achieve their goals in EU policy-making. Additionally, the combination of various data collection methods, such as expert interviews and text analysis, can improve the accuracy and comprehensiveness of the datasets.

To further advance the research in modeling negotiation and group decision-making problems, another direction will be explored in this paper to incorporate the generalized form of the position attribute in the model. The current model assumes that actors' positions are single points in the position spectrum, limiting them to have discrete values. To address this problem, the concept of position in the model will be generalized so that the actors are able to have continuous targets distributed over the position spectrum. Moreover, different possible position functions will be explored to study the skewness of the position function. We aim to define appropriate case studies with the assistance of subject matter experts to evaluate the capability of the expanded model. The integration of generalized position functions into the model has the potential to broaden its applicability to real-world negotiation and group decision-making problems. Furthermore, it will enable actors to express a more nuanced range of positions and preferences, making the decision-making process more representative of the real world.

The rest of the paper will describe the game theory-based predictive analytics model using the distributed position function. Section II explains the structure of the model. The extended model's formulation is presented in Section III. Section IV provides the case studies and their analysis, and finally the last section concludes the paper and offers some suggestions for future research.

2. Background and Structure of the Model

This section provides an overview of the BDM model, including its structure and terminology. In this context, a *problem* is defined as a situation involving actors with varying positions, capabilities, and salience. An *issue* refers to a specific point of dispute or contention within a problem. An *actor* is any entity that utilizes its power to achieve a goal regarding an issue in a problem, and its *position* represents its stance on each issue in an N-dimensional problem. The actor's *capability* refers to its level of power, wealth, or influence, while *salience* indicates the importance of the problem to the actor. The *utility function* measures the desirability of each position based on the actor's supported position, and *risk* is a parameter used to assess an actor's willingness to take risks.

In the BDM model, each actor is assumed to have normalized attributes of capability, salience, and position. Capability and salience range from 0 to 1, and desired positions are taken along a single dimension. The normalization of these attributes ensures that all actors' metrics are on the same scale, resulting in a spectrum of possible solutions based on the stakeholders' points of view on each issue. For more details on the one-dimensional problem's definition of terms and formulation, please see [6] [7].

2.1. Expected Utility

The Expected Utility Model described in [1] [7] computes the expected utility of actor i against actor j as the sum of expected utilities in two different scenarios:

when i challenges j and when it does not. In the challenging case, the expected utility includes two possible outcomes, depending on whether j decides to challenge back. The probability that j does not challenge i is represented by $(1-s_j)$, while s_j represents the probability of a challenge. The utility gained in case of success and failure for actor i is denoted as U_{si}^i and U_{fi}^i, respectively, in (2).

When i is not challenging j, the expected utility includes two different scenarios as well, which are represented in (3). In the first scenario, actor i goes through the status quo with a probability of Q, where Q is either 0.5 or 1, depending on the setting. The utility gained in the status quo is U_{sq}^i. In the second scenario, actor i does not go through the status quo with a probability of $(1-Q)$. In this scenario, actor i can experience a better or worse situation with a probability of T or $(1-T)$, respectively. T is set to 1 when $dist(i,\mu) < dist(i,j)$, where μ is the median voter position, and 0 otherwise.

The structure of the model is visualized as a tree in Figure 1. The computation of the expected utility is shown in (1), where $(EU_{ij}^i)_c$ and $(EU_{ij}^i)_{nc}$ represent the expected utilities in the challenging and non-challenging cases, respectively.

It is worth noting that the definition of basic utilities U_{si}^i, U_{fi}^i, U_{sq}^i, U_{bi}^i, U_{wi}^i for a two-dimensional model are provided in the next section. For more details on the one-dimensional model, the reader can refer to [7] [9].

$$EU_{ij}^i = (EU_{ij}^i)_c - (EU_{ij}^i)_{nc} \tag{1}$$

$$(EU_{ij}^i)_c = s_j\left(p_{ij}^i U_{si}^i + (1-p_{ij}^i)U_{fi}^i\right) + (1-s_j)U_{si}^i \tag{2}$$

$$(EU_{ij}^i)_{nc} = QU_{sq}^i + (1-Q)\left(TU_{bi}^i + (1-T)U_{wi}^i\right) \tag{3}$$

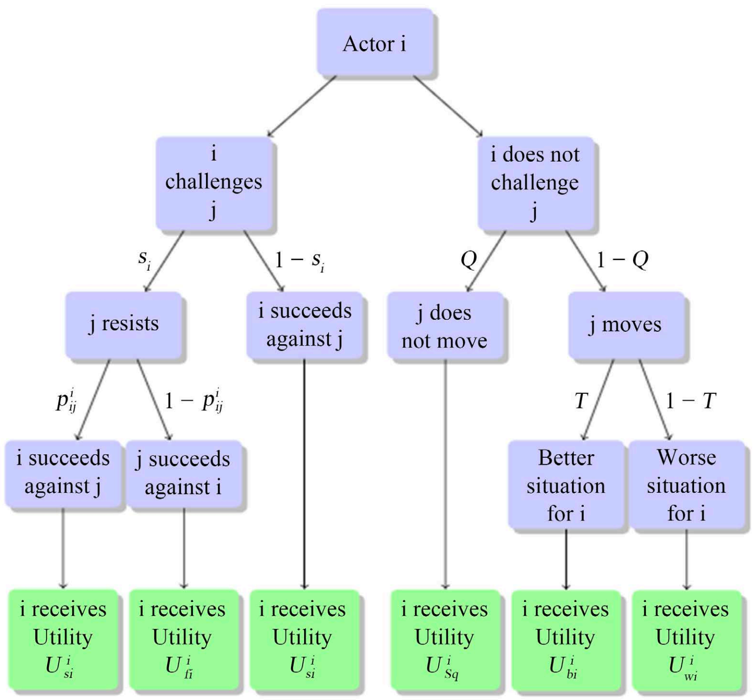

Figure 1. Game tree in expected utility model.

The basic utilities U_{si}^{i}, U_{fi}^{i}, U_{sq}^{i}, U_{bi}^{i}, U_{wi}^{i} are defined as follow. For more details on the formulation please see [1] [6] [7].

If actor *i* is contemplating a new position that differs from its current position, its preferred outcome, then its utility function can be defined as a decreasing function of the distance between the two positions. Hence, the utility function is highest for actor *i* when the alternative position is the same as its current position and is lowest when the two positions are at opposite ends of the position range, as viewed by actor *i*. The equation for this function is given below:

$$u_{ij}^{i} = f\left(-\left|x_i - x_j\right|\right) \tag{4}$$

where x_i is actor *i*'s position and *f* is any arbitrary descending function. The utility function u_{ij}^{i} demonstrates how much actor *i* attaches to his own policy portfolio. The specific function *f* that is used in our model is:

$$f(\sigma) = 1 - 2\sigma^{r_i} \tag{5}$$

where the risk parameter, denoted by r_i, ranges from 0.5 to 2 and will be defined mathematically in a subsequent section. This parameter is used to assess an actor's willingness to take risks in the decision-making process. Players who hold positions with less support from other players are considered more risk-seeking, ideological, or ambitious, and are likely to have lower values of r_i (minimum value of 0.5). Conversely, players who are risk-averse tend to hold positions that are closer to the current likely outcome and do not have to engage in conflicts with other players to reach an agreement and are likely to have higher values of r_i (maximum value of 2) [8]. Substituting (5) into (4), the utility function would become:

$$u_{ij}^{i} = 1 - 2\left|x_i - x_j\right|^{r_i} \tag{6}$$

where $u_{ij}^{i} \in [-1,1]$ and x_i and x_j are normalized so that $x_i, x_j \in [0,1]$. The figure depicted in **Figure 2** illustrates the alterations in the utility function of an actor as it moves away from its intended position, assumin that the actor's risk is given by $r_i = 1$. The application of the risk parameter is reflected in the utility functions of stakeholders, as shown in **Figure 3**. The introduction of risk preferences in stakeholder behavior leads to changes in their utility assessments. Stakeholders who are more willing to take risks exhibit utility functions with a convex shape, resulting in a rapid decline in their utility as they move further away from their preferred position. In contrast, stakeholders who are risk-averse exhibit utility functions with a concave shape, where their utility decreases slowly as they move away from their ideal position.

2.2. Median Voter Position

In a voting process to determine the median voter position, each actor's level of support for their preferred position must be assessed. Actor *i* casts a vote between positions *j* and *k*, indicating their preference for one over the other. The vote is calculated as follows:

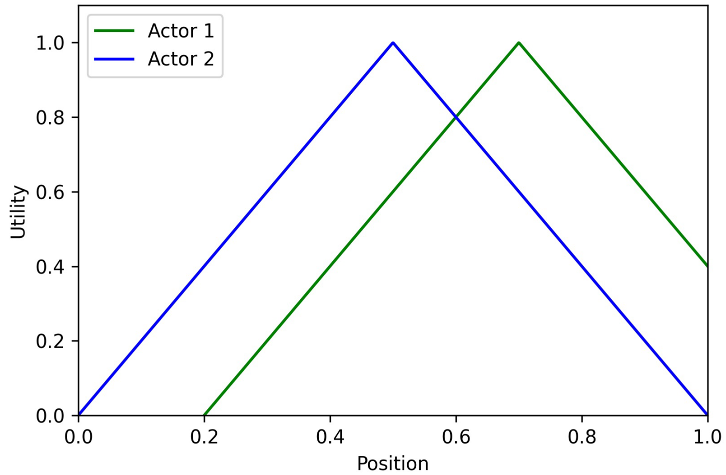

Figure 2. Utility function for two actors at desired positions of 0.7 and 0.5.

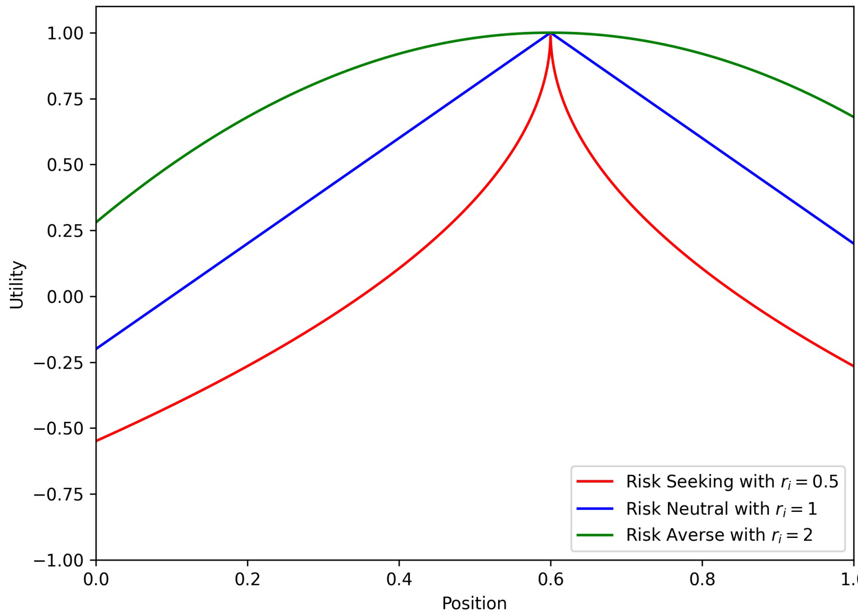

Figure 3. The impact of risk parameter on the shape of utility function for an actor at a position of 0.6. The degree of risk aversion or risk seeking behavior exhibited by the actor will determine its utility function's curvature.

$$v_{jk}^{i} = c_i s_i \left(u_{ij}^{i} - u_{ik}^{i} \right) \tag{7}$$

where u_{ij}^{i} is the utility function of actor i for challenging actor j from i's point of view, c_i and s_i are actor i's capability and salience, respectively, and v_{jk}^{i} is the resulting vote. By replacing (6) into (7), we can obtain the following expression:

$$v_{jk}^{i} = 2c_i s_i \left(\left| x_i - x_k \right| - \left| x_i - x_j \right| \right) \tag{8}$$

The Condorcet method of voting involves comparing the preference level of one position over another in a pair-wise voting system to determine which position receives the most votes. In this method, the position that receives the most votes is declared the winner. The preference level achieved by each position is determined by the vote cast by each actor, as described in the previous step. This process helps to identify the position that has the most support among the actors and is therefore most likely to be accepted as the median voter position. It is important to note that the median voter position may not necessarily be the optimal solution for the problem at hand, but rather a compromise that satisfies the preferences of the majority of actors. Using the preference level achieved by position *j* compared to position *k*, the median voter position μ can be obtained by finding the position with the most support. This calculation can be performed using the following equation:

$$v_{jk} = \sum_{i=1, i\neq j,k}^{n} v_{jk}^{i} \tag{9}$$

where v_{jk} is the votes cast for *j* versus *k* from all other players' point of view.

2.3. Probability of Success

When two actors *i* and *j* challenge each other, the probability of success p_{ij} can be expressed as the ratio of the total support received by one actor over the other:

$$p_{ij} = \frac{\sum_{k|u_{ki} > u_{kj}} v_{ij}^{k}}{\sum_{k=1}^{n} \left| v_{ij}^{k} \right|} \tag{10}$$

The probability of success, p_{ij}, can be interpreted as the amount of support received by actor *i* in comparison to actor *j* [14]. By substituting (8) into (10), the *i*'s probability of success over *j* can be achieved by:

$$p_{ij} = \frac{\sum_{k|arg>0} c_k s_k \left(\left| x_k - x_j \right| - \left| x_k - x_i \right| \right)}{\sum_{k=1}^{n} c_k s_k \left| \left(\left| x_k - x_j \right| - \left| x_k - x_i \right| \right) \right|} \tag{11}$$

where c_k and s_k are the capability and salience of player *k* in the issue. The numerator of (11) sums up the preference votes of all actors who prefer actor *i* over actor *j*. The denominator sums up the support for both actors, resulting in a probability value between 0 and 1.

2.4. Risk

The risk value is used to measure an actor's willingness to take risks in order to diverge from the median voter position. Actors with desired positions closer to the median are less likely to require intense negotiations and are therefore less likely to take risks, resulting in a risk-averse strategy. Conversely, actors with weaker support and weaker alliances are more likely to take risks, resulting in a risk-seeking strategy. The risk term for actor *i*, denoted as r_i involves the calculation of risk value R_i given below:

$$R_i = \frac{2 \sum_{j=1, j\neq i}^{n} EU_{ji}^{i} - \max_i \sum_{j=1, j\neq i}^{n} EU_{ji}^{i} - \min_i \sum_{j=1, j\neq i}^{n} EU_{ji}^{i}}{\max_i \sum_{j=1, j\neq i}^{n} EU_{ji}^{i} - \min_i \sum_{j=1, j\neq i}^{n} EU_{ji}^{i}} \tag{12}$$

where $R_i \in [-1,1]$. Once R_i is obtained, the risk term r_i can be computed using a transformation in (13) which guarantees $0.5 < r_i < 2$. Initially, all actors are considered to be risk neutral and assigned a risk term of 1, and it is recalculated for each round.

$$r_i = \frac{1 - R_i/3}{1 + R_i/3} \tag{13}$$

2.5. Utility of Success and Failure

In this section, we introduce the basic utilities consisting of U_{si}^i, U_{fi}^i, U_{sq}^i, U_{bi}^i, and U_{wi}^i. Given the assumption that actor *i* either defeats or wins over actor *j*, the corresponding utilities of success U_{si} and failure U_{fi} are derived by actor *i* as follows:

$$U_{si}^i = 2 - 4\left[0.5 - 0.5\left|x_i - x_j\right|\right]^{r_i} \tag{14}$$

$$U_{fi}^i = 2 - 4\left[0.5 + 0.5\left|x_i - x_j\right|\right]^{r_i} \tag{15}$$

where $2 - 4(0.5)^{r_i} \le U_{si}^i \le 2$ and $-2 \le U_{fi}^i \le 2 - 4(0.5)^{r_i}$. Referring to **Figure 1**, it can be observed that if actor *i* chooses not to challenge actor *j*, actor *j* may either maintain their current position as status quo or move, which may result in a better or worse situation for actor *i*, and can be associated with the utilities U_{bi} and U_{wi}, respectively. Therefore, the utilities U_{bi}^i, U_{wi}^i, and U_{sq}^i are defined as follows to satisfy the condition $U_{wi}^i \le U_{qi}^i \le U_{bi}^i$:

$$U_{bi}^i = 2 - 4\left[0.5 - 0.25\left(\left|x_i - \mu\right| + \left|x_i - x_j\right|\right)\right]^{r_i} \tag{16}$$

$$U_{wi}^i = 2 - 4\left[0.5 + 0.25\left(\left|x_i - \mu\right| + \left|x_i - x_j\right|\right)\right]^{r_i} \tag{17}$$

where $2 - 4(0.5)^{r_i} \le U_{bi}^i \le 2$ and $-2 \le U_{wi}^i \le 2 - 4(0.5)^{r_i}$, and μ is the median voter position which can be calculated using (9) to find the position with the most support. In a case where *i* does not challenge *j* and *j* does not move, the status quo utility is realized and is defined as:

$$U_{sq}^i = 2 - 4\left[\frac{4-0}{8}\right]^{r_i} = 2 - 4(0.5)^{r_i} \tag{18}$$

The value of parameter Q in (3) has been set to different values (0.5 or 1) in various research papers. For instance, in [15], Q is considered as 1, while in [16], it is taken as 0.5. However, in our study, we have assumed Q to be 1. On the other hand, parameter T determines whether the situation gets better for actor *i* or not. If $dist(i,\mu) < dist(i,j)$, then T is set to 1, indicating that the situation has improved for actor *i*. Otherwise, if $dist(i,\mu) \ge dist(i,j)$, then T is set to 0.

2.6. Offer Categories

The expected utilities are utilized to determine the negotiation outcomes among actors, which can fall into one of four categories: conflict, capitulation, compromise, or stalemate. **Figure 4** illustrates these scenarios and the corresponding expected utilities for each actor. Further details on each scenario are defined below.

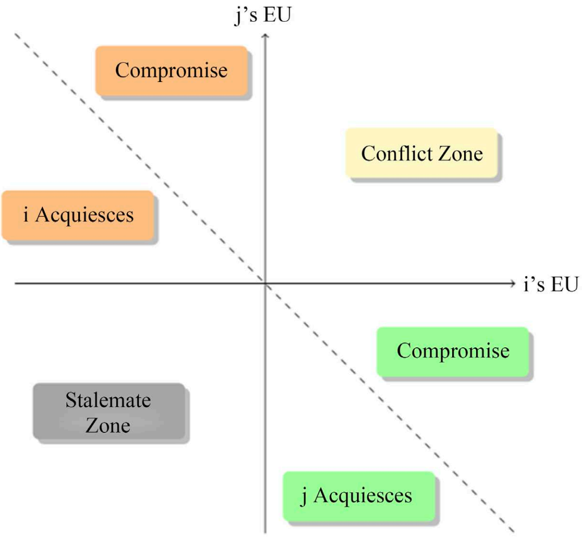

Figure 4. Expected utility model scenarios from actor *i*'s viewpoint.

• Conflict: When actors *i* and *j* both believe that they have the upper hand, a conflict is likely to occur, and the outcome of the conflict is determined based on the following conditions:

- If $EU_{ij}^{i} > EU_{ji}^{j}$, actor *j* moves to *i*'s position.
- If $EU_{ij}^{i} < EU_{ji}^{j}$, actor *i* moves to *j*'s position.
- Actor *j* moves to *i*'s position when $EU_{ij}^{i} > EU_{ji}^{j}$.
- Actor *i* moves to *j*'s position when $EU_{ij}^{i} < EU_{ji}^{j}$.

• Capitulate: If $EU_{ij}^{i} > 0$, $EU_{ji}^{j} < 0$, and $\left|EU_{ij}^{i}\right| < \left|EU_{ji}^{j}\right|$, actor *j* is expected to capitulate and accept actor *i*'s current position.

$$proposal_{ij}^{i} = x_i \tag{19}$$

• Compromise: If actor *i* has the upper hand and actor *j* is willing to agree to an acceptable offer, they compromise in favor of actor *i*, and the offer is closer to *i*'s position. This scenario occurs when $EU_{ij}^{i} > 0$, $EU_{ji}^{j} < 0$, and $\left|EU_{ij}^{i}\right| > \left|EU_{ji}^{j}\right|$.

$$\Delta x = \left(x_i - x_j\right)\left|\frac{EU_{ij}^{j}}{EU_{ji}^{i}}\right| \tag{20}$$

$$proposal_{ij}^{i} = x_j + \Delta x \tag{21}$$

• Stalemate: If both actors believe that they cannot beat the other, then they do not move from their current positions, resulting in a stalemate. This scenario happens when $EU_{ij}^{i} < 0$, $EU_{ji}^{j} < 0$.

2.7. Offer Selection

At the end of each round, actors receive offers from others, and the question is which offer to accept to determine each actor's position for the next round. Mesquita [17] and Baranick [18] suggested that the actor should choose the offer that requires the smallest possible movement. We implemented this idea and assumed that the minimum move should be greater than zero unless all received offers are the same as the actor's current position. Future studies may investigate for a more efficient offer selection algorithm.

3. Model Extension

Our main focus in this section is to update the formulation, extend the concept of position, and introduce corresponding distance measures. In the current model, the position of actors is assumed to be a single point or a discrete value in the position spectrum, which limits the flexibility of the model. To address this limitation, we propose an approach to replace the discrete or spike-like positions with more flexible alternatives.

One way to realize this idea is to consider each actor's position function as a Gaussian distribution [19] [20], the average (mean value) of which is assumed to be the initial position and its variance could be considered proportional to the inverse of salience. This approach would allow the position of actors to be represented by a continuous distribution rather than a discrete value. The Gaussian distribution is a popular choice for modeling position functions in various fields, such as signal processing, machine learning, and statistics. It offers several advantages over the current model, such as improved accuracy, better handling of uncertainty, and greater flexibility.

To incorporate this modification into the existing model, we need to update the formulation accordingly. Specifically, we need to redefine the position of actors as a continuous function rather than a discrete value. We also need to introduce a corresponding distance measure that accounts for the uncertainty in the position function. One possible distance measure is the Bhattacharyya distance [21], which is commonly used for comparing Gaussian distributions. This distance measure takes into account both the mean and variance of the position functions, and provides a more accurate measure of the similarity between two positions. For a comprehensive understanding of the position function concept, the reader is referred to Mousavi's dissertation (2023) [22].

3.1. Position Function

In this subsection, we discuss distributions that can be defined on an interval [0, 1] and are suitable for modeling actors' position function. The position of actors in a dynamic network is a critical factor that determines their behavior and influence. While in the current model, the position of actors is represented by a single point or a discrete value, a more realistic approach is to model it as a probability distribution $p_i(x)$, where actor i can take a continuous position towards the position spectrum. This allows us to capture the uncertainty and variability of actors' positions [22].

In the context of modeling actors' position functions, certain distributions can be utilized that are defined on the interval [0, 1] and exhibit skewness towards either direction. Among the candidate distributions, truncated normal, beta family, and triangular distributions are particularly suitable for this purpose. The upcoming sections will introduce each of these distributions in detail.

3.1.1. Truncated Normal Distribution

One way to realize this idea is to consider each actor's position function as a Gaussian distribution. This distribution has a bell-shaped curve and can be skewed towards either direction. It is bounded within the interval [0, 1] and can be truncated at any point within this range. The parameters of the distribution, namely the mean, variance, and truncation points, can be estimated from the data or set based on prior knowledge.

In order to incorporate Gaussian distribution into the existing dataset, we can assign the initial position as the mean of the distribution and the variance can be set to be proportional to the inverse of salience. The mathematical formulation of the Gaussian distribution is presented below:

$$p_i = p_i(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - m_x)^2}{2\sigma^2}\right) \tag{22}$$

where m_x represents the mean value and σ^2 represents the variance of the Gaussian distribution.

The position distribution of actors is illustrated in **Figure 5** using Gaussian functions with varying means and variances. The figure reveals that actors with higher salience, and thus lower variance, are less likely to share a common position with other actors.

By applying lower and upper bounds to the variable in the normal distribution, we can obtain the truncated normal distribution. The mathematical representation of the truncated normal distribution is as follows:

$$f(x;\mu,\sigma,A,B) = \frac{1}{\sigma} \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{B-\mu}{\sigma}\right) - \Phi\left(\frac{A-\mu}{\sigma}\right)} \tag{23}$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \tag{24}$$

$$\Phi(x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(x/\sqrt{2}\right)\right) \tag{25}$$

where μ and σ represent the mean and standard deviation of the normal distribution, respectively, and A and B are the lower and upper bounds of the truncated distribution. $\phi(.)$ and $\Phi(.)$ denote the probability density function and cumulative distribution function of the standard normal distribution, respectively.

By setting $A = -\infty$, we get $\Phi\left(\frac{A-\mu}{\sigma}\right) = 0$. Similarly, if $B = +\infty$, we obtain $\Phi\left(\frac{B-\mu}{\sigma}\right) = 1$. Thus, we can use the truncated normal distribution to represent one-sided position functions.

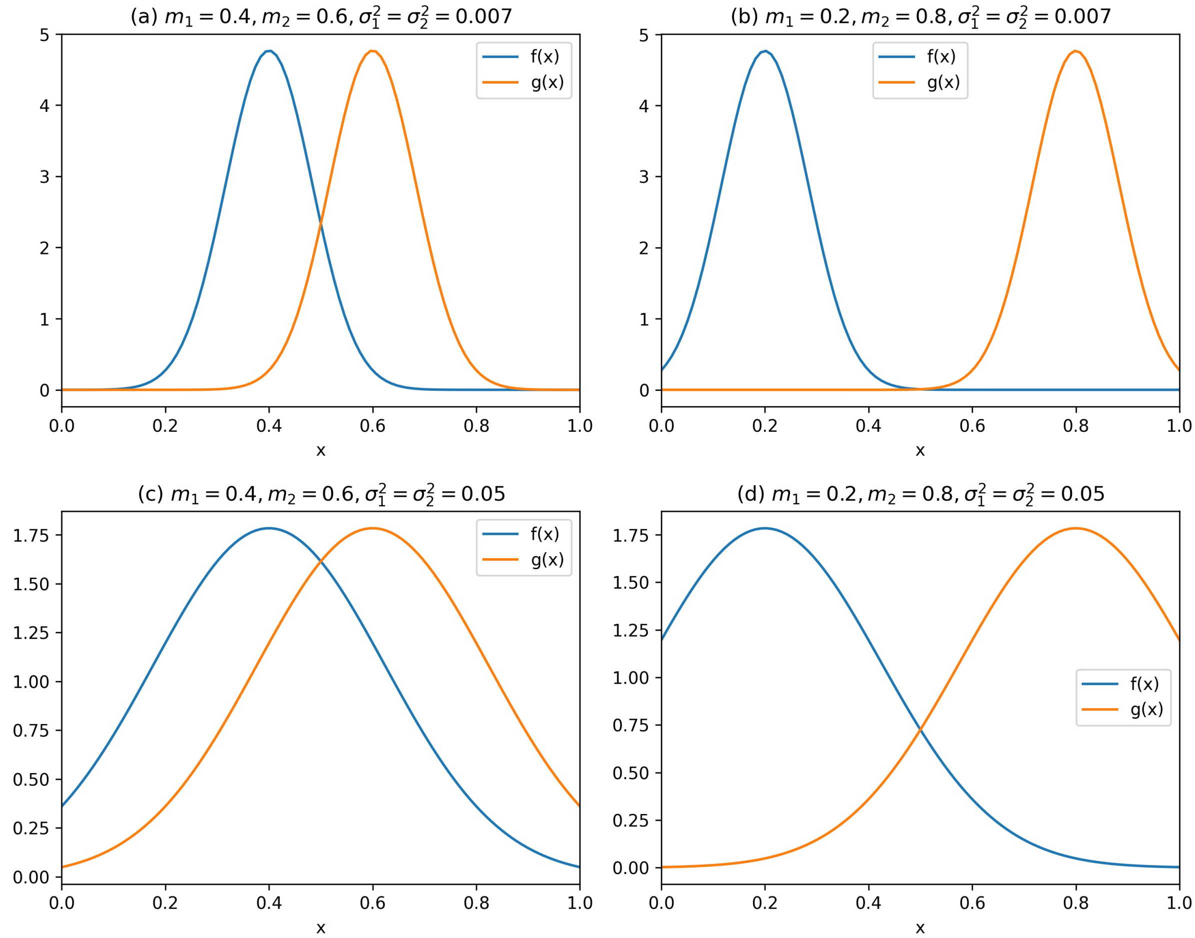

Figure 5. Two actors' position as Gaussian distribution functions with different means and variances.

Figure 6 shows an example of the truncated normal distribution with different lower and upper bounds. We can observe that the probability density function of the truncated normal distribution is zero outside the given bounds.

3.1.2. Beta Distribution

Another family of distributions that can be used for modeling position functions is the beta distribution. This distribution can be bounded within the interval [0, 1], and its shape is flexible, allowing it to be either skewed or symmetric. The beta distribution has two shape parameters that control the skewness and kurtosis of the distribution, and these parameters can be estimated using maximum likelihood or Bayesian methods.

The beta family of distributions can be defined over a general interval (A, B) by the following equation:

$$f(x;\alpha,\beta,A,B) = \frac{1}{B-A}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\left(\frac{x-A}{B-A}\right)^{\alpha-1}\left(\frac{B-x}{B-A}\right)^{\beta-1} \tag{26}$$

where α and β are the shape parameters, and A and B denote the interval endpoints. However, if the interval is confined to [0, 1], using $A = 0$ and $B = 1$ in (26) yields the standard Beta distribution:

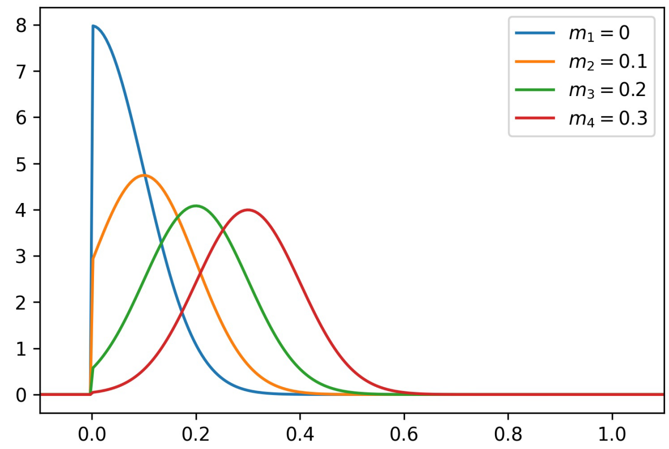

Figure 6. Example of truncated normal distribution with varying mean values. The variance (σ^2) for all curves is constant at 0.1.

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases} \tag{27}$$

where μ and σ can be calculated as:

$$\mu=\frac{\alpha}{\alpha+\beta} \tag{28}$$

$$\sigma^2=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \tag{29}$$

An alternative formulation of the beta distribution involves reparameterizing it in terms of $\mu=\dfrac{\alpha}{\alpha+\beta}$ and $\phi=\alpha+\beta$ [23]. The distribution can then be expressed as:

$$f(x;\mu,\phi)=\frac{\Gamma(\phi)}{\Gamma(\mu\phi)\cdot\Gamma((1-\mu)\phi)}x^{\mu\phi-1}(1-x)^{(1-\mu)\phi-1},\ 0\le x\le 1 \tag{30}$$

where $0<\mu<1$ and $\phi>0$. Using this formulation, μ can be used as the mean parameter, and ϕ can be interpreted as the precision parameter. Greater precision yields a smaller variance if μ is fixed. Figure 7 illustrates the beta distribution for various combinations of μ and ϕ. The figure presents the probability density function of the standard beta distribution for different combinations of its shape parameters, μ and ϕ. The beta distribution is a flexible distribution that can model a variety of data types and is particularly useful for modeling data that are bounded between 0 and 1. In subplot (a), we have fixed $\phi=5$ and varied μ from 0.05 to 0.95 in increments of 0.25. In subplots (b), (c), and (d), we have the same setup, but with ϕ fixed at 15, 50, and 100, respectively. As μ increases, the peak of the distribution shifts towards the right and the tails become thinner. As ϕ increases, the distribution becomes more concentrated around the mean and the tails become even thinner. The figure provides insights into

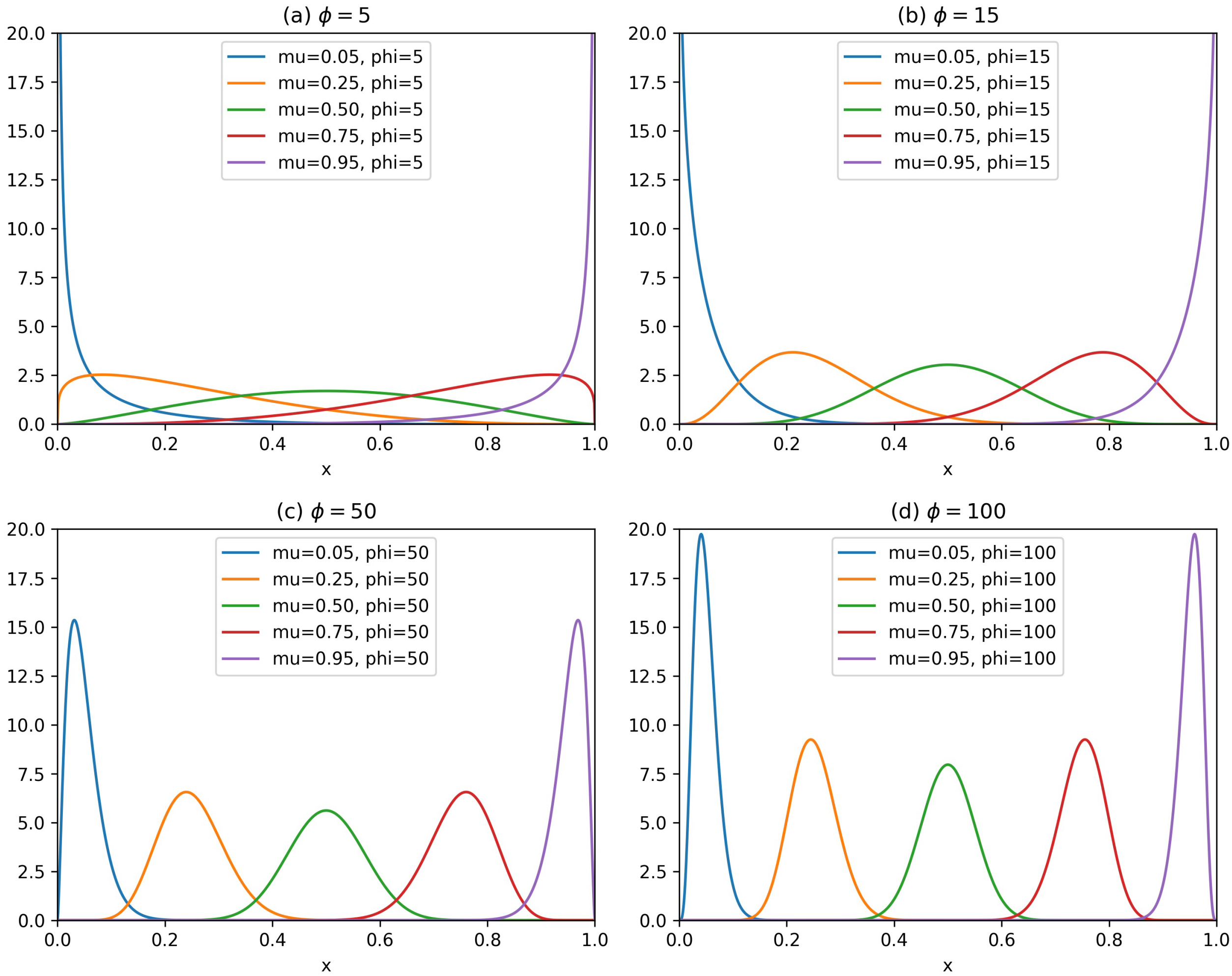

Figure 7. Variations in the Beta distribution function with changes in the mean and precision parameters μ and ϕ, respectively.

how the beta distribution changes with varying values of its shape parameters and can aid in selecting appropriate parameter values for modeling a position for a given actor. The utilization of beta distribution as a position function allows for the interpretation of the highest point of the distribution as the desired position, while the degree of sharpness in its tail can be regarded as an indication of the level of significance (salience) of the issue to the actor.

3.1.3. Triangular Distribution

In addition to the beta distribution, the triangular distribution is another candidate for modeling actors' position function. This distribution is particularly useful when prior knowledge about the range and mode of the position function is available. The triangular distribution can be bounded within the interval [0, 1] and is characterized by three parameters: the lower and upper bounds, and the mode. The skewness of the position can be adjusted by controlling the values of the lower and upper bounds.

The triangular distribution is a continuous probability distribution over the interval [a, b] and can be defined as follows:

$$f(x;a,b,c)=\begin{cases}0 & x<a,\\ \frac{2(x-a)}{(b-a)(c-a)} & a\le x<c,\\ \frac{2}{b-a} & x=c,\\ \frac{2(b-x)}{(b-a)(b-c)} & c<x\le b,\\ 0 & b<x.\end{cases} \tag{31}$$

where $a\in(-\infty,\infty)$ is the lower limit, $b>a$ is the upper limit, and $a\le c\le b$. Parameter c is the mode of the distribution and can be considered as the central position taken by the actor. To utilize this distribution as a position function limited within [0, 1], it is necessary to select the values of parameters a and b from the interval $a\in[-1,1]$ for our specific purposes. The skewness of the position can be set by parameters a and b. Figure 8 displays the triangular distribution for various combinations of parameters a, b, and c over the interval [0, 1].

Using either the beta or the triangular distribution as a position function, the peak of the distribution can be considered as the desired position, and the thinness of its tail can represent the level of salience of the issue to the actor. The choice of which distribution to use depends on the availability of prior knowledge about the position function and the desired level of control over the skewness of the distribution. Figure 8 illustrates the flexibility of the triangular distribution in this regard. We can observe that as the difference between the mode and the midpoint of the interval $(a+b)/2$ increases, the distribution becomes more skewed.

3.2. Distance Measure

Various techniques can be employed to quantify the distance between two stochastic variables and can be applied in our model to evaluate the distance between position functions. The choice of distance metric can depend on the nature of the data and the research question being addressed. In the following, we will examine a few of them.

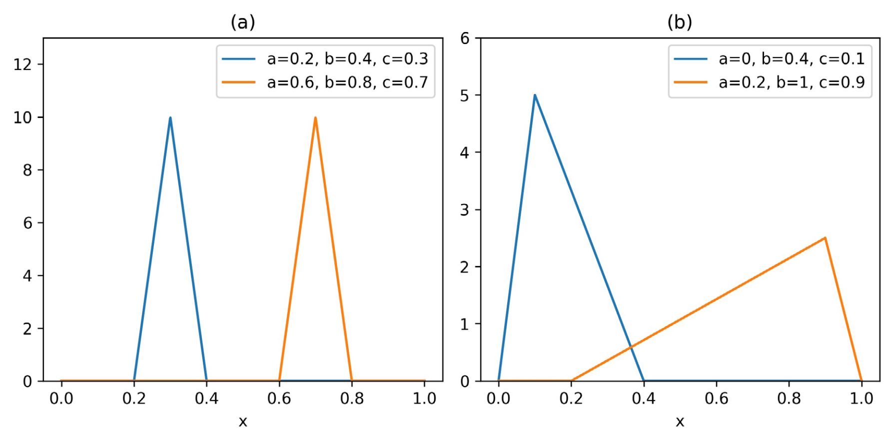

Figure 8. Triangular distribution function with varying parameters *a*, *b*, and *c*.

3.2.1. Lukaszyk-Karmowsky Distance Function

One approach to measure the statistical distance between two random variables is using Lukaszyk-Karmowsky function [24]. The distance between two variables X and Y, denoted by D, can be obtained using the following equation:

$$D(X,Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x-y|\, f(x)g(y)\,\mathrm{d}x\mathrm{d}y \tag{32}$$

where $f(x)$ and $g(y)$ represent the probability density functions of X and Y, respectively.

3.2.2. Bhattacharyya Distance Function

The Bhattacharyya distance function is another approach to calculate the statistical distance between two random variables [21]. With $p(x)$ and $q(y)$ representing probability density functions of random variables X and Y, respectively, the Bhattacharyya distance can be defined as:

$$D(X,Y)=D_B(p,q)=-\ln\left(BC(p,q)\right) \tag{33}$$

where the Bhattacharyya coefficient can be achieved by:

$$BC(p,q)=\int\sqrt{p(x)q(x)}\,\mathrm{d}x \tag{34}$$

It can be seen that $0\le BC\le 1$ and $0\le D_B\le\infty$. Note that the Bhattacharyya coefficient ranges from 0 to 1, with a value of 0 indicating that the two distributions have no overlap, and a value of 1 indicating that the two distributions are identical. Therefore, the Bhattacharyya Distance is defined as the negative logarithm of the Bhattacharyya coefficient, and it ranges from 0 to infinity, with a value of 0 indicating that the two distributions are identical, and larger values indicating greater dissimilarity between the distributions.

3.2.3. Cosine Similarity Based Distance Function

The cosine similarity method can be used to determine the distance between two distribution functions. It measures the angle between two vectors to determine their similarity. To compute the cosine similarity based distance between two random variables X and Y, the sum of squares for variable X, the sum of squares for variable Y, and the sum of the cross-product for X and Y are needed [25]:

$$D(X,Y)=1-\frac{A\cdot B}{A.B}=1-\frac{\sum_{i=1}^{n}A_iB_i}{\sqrt{\sum_{i=1}^{n}A_i^2}\sqrt{\sum_{i=1}^{n}B_i^2}} \tag{35}$$

where A and B are the vectors representing the two functions defined over two variables X and Y, respectively, and n is the number of data points the functions are sampled with.

The cosine similarity and distance measures have some limitations. They assume that the data are normalized and that the vectors are of equal length. Furthermore, they do not consider the shape of the distributions and may not be appropriate for certain types of data, such as ordinal or categorical data. In such cases, other distance measures, such as the Bhattacharyya distance or the Lukaszyk-Karmowsky distance, may be more appropriate.

3.2.4. Normalized Similarity Measure

One potential distance measure that has been proposed in the literature is based on the minimum of two functions [26] [27]. This measure calculates the distance between two probability density functions by integrating the minimum of the two functions over their common domain. The resulting distance measure is a non-negative value that ranges from 0 to 1, where 0 indicates identical distributions and 1 indicates completely dissimilar distributions.

Formally, let $p(x)$ and $q(x)$ be two probability density functions with a common domain $[a,b]$. The minimum-based distance measure $D(X,Y)$ between $p(x)$ and $q(x)$ is given by:

$$D(X,Y)=1-\frac{1}{b-a}\int_a^b \min\{p(x)q(x)\}\mathrm{d}x \tag{36}$$

where p and q representing probability density functions of random variables X and Y, respectively.

3.3. BDM Model Explanation Using the Proposed Model

By incorporating the Lukaszyk-Karmowsky distance function into the BDM model and treating the discrete position representation as a position function with a Dirac delta function, the same results can be achieved. The position function for actor i can be defined as:

$$p_i = p_i(x)=\delta(x-x_i) \tag{37}$$

where x_0 is the initial position of actor i. **Figure 9** represents two actors in the BDM position model using Dirac functions. Previously, the distance between the positions of two actors i and j was obtained using:

$$D(x_i,x_j)=|x_i-x_j| \tag{38}$$

However, using (32) to calculate the distance between two positions, we get the same result:

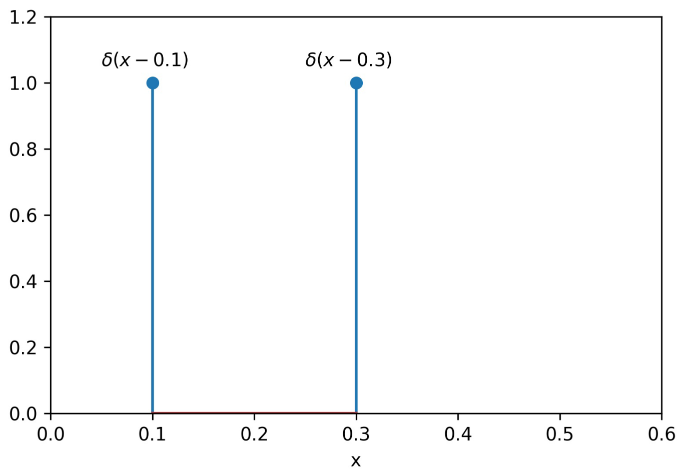

Figure 9. Position of BDM model as Dirac delta functions.

$$D(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|\,\delta(x-x_0)\,\delta(y-y_0)\,\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty} |y-x_0|\,\delta(y-y_0)\,\mathrm{d}y = |y_0 - x_0| \quad (39)$$

where x_0 and y_0 are actors *i* and *j*'s initial positions (discrete positions), respectively.

3.4. Distance of Median Voter Position to Actors

The median voter position with the maximum support can be calculated using (9). However, to calculate U_{bi}^{i} and U_{wi}^{i}, the distance between the median voter position (μ_0) and the distributed position function, p_i, needs to be determined. In order to calculate this distance, we can consider a position function for the median voter position as a truncated normal distribution with standard deviation of $\sigma = 0.5$ and average of $\mu = \mu_0$. The distance between the actor's position and the median voter position can then be calculated using a distance measure, such as the Lukaszyk-Karmowsky Distance Function.

3.5. Complexity Analysis

In dealing with problems that involve large number of issues (dimensions) and/or large number of actors per issue/dimension, the computational complexity of the problem becomes increasingly important. To demonstrate the analysis of the model's time complexity, a simple pseudo code is provided in Algorithm 10. Table 1 displays the complexity of different parts of the algorithm. The overall computational complexity of the algorithm is $O(M^3)$, where M is the number of actors. The main calculation in the for loops involves calculating the distance functions using the Lukaszyk-Karmowsky Distance Function. This involves calculating dual integral to find the distance between two position functions. To reduce computational time, we can pre-calculate the pairwise distance between all actors at the beginning of each round and store them in a lookup table for later retrieval. This optimization can reduce the time complexity remarkably, which can now be estimated as $O(M^2)$. However, the space complexity remains unchanged at $O(M^2)$.

4. Experimental Results

In this section, some experiments have been conducted to illustrate the capability of the proposed model. We fed the proposed model with a one-dimensional test data to see if it can produce reasonable results, similar to the one-dimensional model. In all experiments, the positions are modeled using truncated normal distribution with $\mu = x_i$ and $\sigma = 1/s_i$. In the following, we will discuss the process of selecting a distance measure followed by an overview of the dataset used in this study. Subsequently, we will present the outcomes of our proposed method, and finally, a case study will be presented to demonstrate the explanability of our approach.

Table 1. Complexity of different parts of the position function based model.

Component	Complexity
v_{jk}^{i}	$O(M^2)$
v_{jk}	$O(M)$
μ	$O(M)$
$U_s(i,j)$, $U_f(i,j)$, $U_b(i,j)$, $U_w(i,j)$	$O(M^2)$
$U_{sq}(i)$	$O(M)$
p_{ij}	$O(M^2)$
$EU_{i,j,d}^{i}$	$O(M^2)$
$EU_{i,j}^{i}$	$O(M^2)$
R_i	$O(M^2)$
r_i	$O(M)$
Offer categories and proposals	$O(M^2)$
Choosing proposals	$O(M^2)$

Algorithm 1 Distributed Position Function Model

Input: $[s_i]_{n\times 1}, [c_i]_{n\times 1}, [p_i]_{n\times 1}$

$r_i \leftarrow 1$ for all actors
while $threshold < 2.5\%$ **do**
  for $i, j, k \leftarrow 1$ to n **do**
    Calculate v_{jk}^{i}
  end for
  for $i \leftarrow 1$ to n **do**
    Calculate v_{jk}
  end for
  $votes = zeros(n, n)$
  for $k \leftarrow 1$ to n **do**
    $votes = votes + v_{jk}$
  end for
  $index = max(vote)$
  $\mu = p(:, index)$
  for $i, j \leftarrow 1$ to n **do**
    Calculate $U_s(i,j)$, $U_f(i,j)$, $U_b(i,j)$, $U_w(i,j)$
  end for
  for $i \leftarrow 1$ to n **do**
    Calculate $U_{sq}(i)$
  end for
  for $i, j, k \leftarrow 1$ to n **do**
    Calculate p_{ij}
  end for
  for $i, j \leftarrow 1$ to n **do**
    Calculate $EU_{i,j}^{i}$
  end for
  Calculate $EU_{i,j}^{i}$
  Calculate R_i and r_i
  Calculate offer categories and proposals
  Choose proposal
end while

4.1. Selecting Distance Function

In this subsection, we compare four different distance functions to find the most suitable one for measuring the distance between truncated normal distributions as position functions in our model. The four distance measures we considered are the Lukaszyk-Karmowsky Distance Function, the Bhattacharyya Distance Function, the Cosine Similarity-based Distance Function, and the Normalized Similarity Measure based Distance Function. It is worth noting that the suitability of each distance function may vary depending on the specific characteristics of the distributions being compared. Therefore, it is recommended to perform a comprehensive comparison of different distance functions for any given problem to select the most appropriate one.

To conduct the comparison, we generated four scenarios of truncated normal distributions with different parameters. These scenarios can be seen in **Figure 10**. For each subplot, we calculated the distance between the two truncated normal distributions using each of the four distance functions. The results of our comparison are summarized in **Table 2** below. The first row in the table shows the parameters of the truncated normal distributions used depicted in **Figure 10** subplots, where (0.2, 0.1) vs (0.3, 0.1) represents the comparison between the truncated normal distributions in subplot (a) with means 0.2 and 0.3, and standard deviations 0.1, respectively.

In our proposed model, we hypothesized that the degree of overlap between position functions would inversely affect their distance, *i.e.*, the distance would be higher when there is no overlap and lower when the functions share a common area. Additionally, when comparing two-position functions with similar μ but low σ (**Figure 10(b)** and **Figure 10(d)**), we expect the distance to approximate the difference between their mean values. Our analysis using **Table 1** reveals that only the Lukaszyk-Karmowsky distance function satisfies this expectation, and we have thus selected it for our model. However, it is important to note that for some case studies with similar positions like those in **Figure 10(b)** and **Figure 10(d)**, maximum distance may be preferred, and for such situations, the cosine similarity or normalized similarity measure based distance function could be considered.

Table 2. Comparison of distance functions for truncated normal distributions as position functions.

Distance Measure	(0.2, 0.1) vs (0.3, 0.1)	(0.2, 0.01) vs (0.3, 0.01)	(0.3, 0.1) vs (0.7, 0.1)	(0.3, 0.01) vs (0.7, 0.01)
Lukaszyk-Karmowsky	0.135	0.100	0.399	0.400
Bhattacharyya	0.119	12.500	1.999	200.000
Cosine similarity	0.220	1.000	0.982	1.000
Normalized similarity measure	0.377	1.000	0.954	1.000

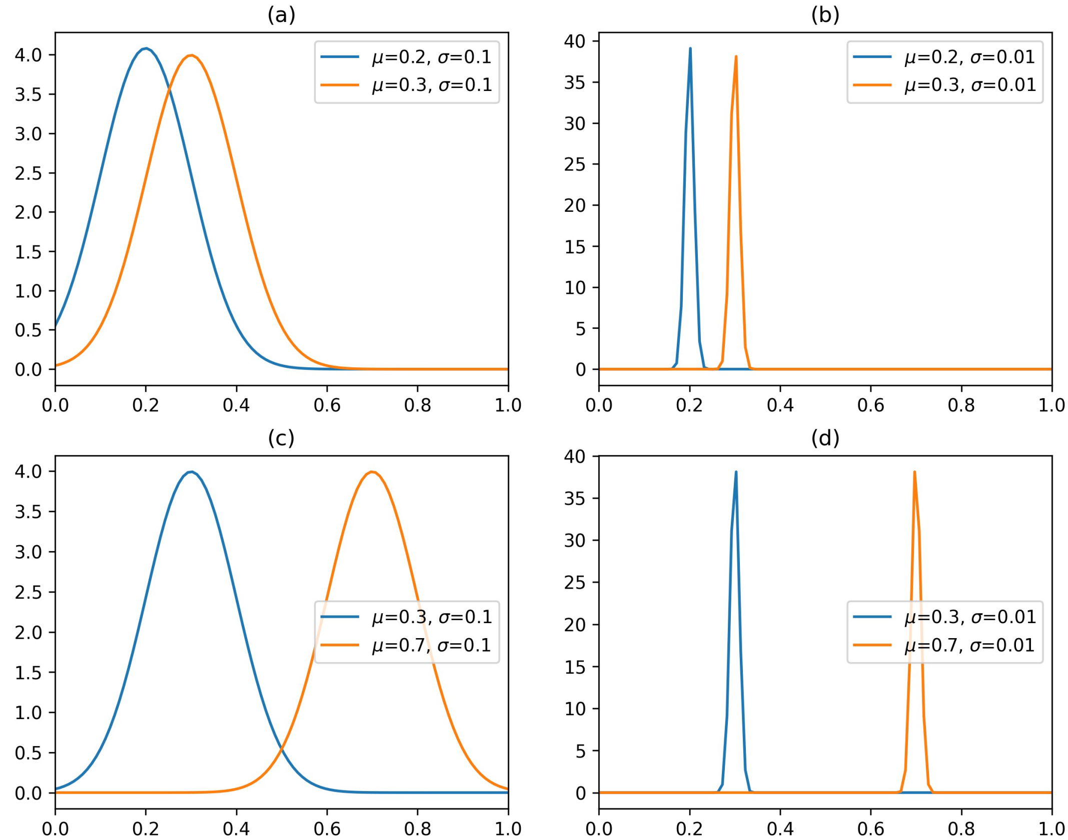

Figure 10. Truncated normal functions used for the distance measure selection.

4.2. Dataset

The EMU Positions dataset is a collection of data related to the major reforms of the Economic and Monetary Union that took place between 2010 and 2015. It covers 47 contested policy issues that were negotiated between 28 EU member states and significant EU institutions, including their position and salience. The dataset was created using various data collection methods, including expert interviews and text analysis, in order to improve the accuracy and comprehensiveness of the data [13].

The dataset contains information about the positions taken by different actors during the negotiation process, as well as the salience of these positions. In the online appendix and the codebook, they provide the data and detailed information on it. By analyzing this data, researchers can gain insights into the decision-making process of the EU legislative body, as well as the factors that influence decision-making.

It is important to note that the dataset had missing position or salience information for some actors in certain issues. Therefore, those actors were eliminated from the analysis of those issues. To estimate the actors' capability in the issues, their GDP for the year 2021 was used as a proxy for their level of power. This assumption is considered reasonable given that the issues being analyzed are related to financial and monetary policies. Although the actors' GDP may have been different at the time the dataset was created, we assume that their relative power remains almost the same. Table 3 shows the capability values used for the actors present in the issues.

4.3. Proposed Model Performance

To evaluate the model's capability in predicting negotiation problems, we ran the model on the EMU position dataset and compared its performance with the BDM model and BiLSTM model. In our model, we used a truncated normal distribution with $\mu = x_i$ and σ proportional to $1/s_i$ as the position function for each actor. Equation (40) was used to calculate the sigma value used in the position function. We considered $p_s = 1$, $m_s = 100$, and $\varepsilon = 0.01$ as the parameters.

$$\sigma = \frac{1}{m_s \left(s_i\right)^{p_s} + \varepsilon} \tag{40}$$

where s_i is actor *i*'s salience. The parameter ε is used as denominator offset to avoid division by zero, and p_s and m_s are reserved as degrees of freedom in the implementation of the position function. For this experiment, the BiLSTM architecture had 10 neurons in the first layer and 5 neurons in the second layer.

The evaluation metric used was the mean absolute error (MAE). Table 4 shows the comparison of MAE for the proposed model, BDM model, and the BiLSTM model. Our position function based model outperformed both the BiLSTM model and BDM model with an MAE of 0.2122, indicating its effectiveness in predicting negotiation outcomes.

Table 3. Capability of European Union countries used in EMU position dataset Evaluation.

Country	GDP (in million USD)	Country	GDP (in million USD)
Austria	480368.40	Italy	2107702.84
Belgium	594104.18	Latvia	39853.50
Bulgaria	84056.31	Lithuania	66445.26
Croatia	68955.08	Luxembourg	85506.24
Cyprus	28407.87	Malta	17364.04
Czechia	281777.89	Netherlands	1012846.76
Denmark	398303.27	Poland	679444.83
Estonia	37191.17	Portugal	253663.14
Finland	297301.88	Romania	284087.56
France	2957879.76	Slovakia	116527.10
Germany	4259934.91	Slovenia	61748.59
Greece	214873.88	Spain	1427380.68
Hungary	181848.02	Sweden	635663.80
Ireland	504182.60	UK	3131377.76

4.4. Model's Explainability

To present the explainability of the proposed model, we employed a one-dimensional case study from [5]. The issue is to predict the number of years it would take for the introduction of emission standards for medium-sized automobiles. Table 5 shows the initial capabilities, positions, and salience of the players involved in the problem. Figure 11 depicts the positions of the players after 10 rounds. The results obtained are in agreement with the findings reported in [6]. According to [6], the expected utility model predicted the outcome to be 7.0 years, while the actual delay was 8.83 years. According to the median voter position result, our model predicts the outcome to be 7.0 years. From Figure 11, it can be seen that the actors are negotiating during each round until they reach an agreement in the final round. For instance, the UK is gradually changing its position, while Italy is changing its position at a faster pace. Although the capability of the UK and Italy is considered equal towards the issue, it can be inferred that the UK has more salience towards the issue, which is why it changed its position at a slower pace than Italy. Another instance could be a comparison between Denmark and Germany. Since Denmark's capability is lower than Germany's, it left its initial position earlier.

Table 4. Comparison of results in the evaluation stage.

Method	MAE
BiLSTM Model	0.3351
BDM model	0.2244
Position function based model	0.2122

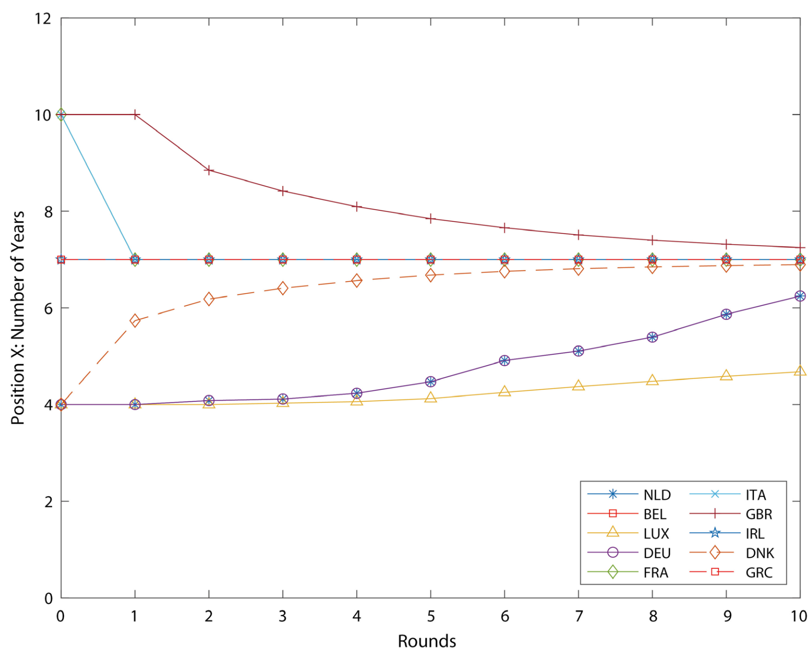

Figure 11. Players' positions along x-axis over different rounds in case study I. Position X shows the number of years that would need to pass before the introduction of emission standards for medium sized automobiles.

Table 5. Input data for case study I.

Players	Capability	Salience	Position
Netherlands	0.08	0.8	4
Belgium	0.08	0.4	7
Luxembourg	0.03	0.2	4
Germany	0.16	0.8	4
France	0.16	0.6	10
Italy	0.16	0.6	10
UK	0.16	0.9	10
Ireland	0.05	0.1	7
Denmark	0.05	1.0	4
Greece	0.08	0.7	7

5. Conclusions and Future Work

In this paper, we introduce a game theory-based approach for forecasting outcomes of negotiation and group decision-making problems. We propose an extension to the BDM model that addresses situations where actors' positions span a spectrum, thus offering increased flexibility in defining their targets.

We provide comprehensive explanations for the formulation of position functions and distance functions. Utilizing the EMU Positions dataset, we illustrate the model's notable capability to produce superior results when compared to existing models. By replicating the findings of previous studies, we demonstrate the model's interpretability and its proficiency in providing clear and comprehensible explanations.

Looking forward, our future research will delve into various facets. We plan to explore offer selection and the potential for alliance formation. Moreover, we intend to employ natural language processing techniques to estimate actors' desired position functions, which could potentially enhance prediction accuracy. Further, conducting additional case studies will be essential to continually assess the efficacy of our proposed model.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Karimi, M.M. and Rahimi, S. (2021) A Two-Dimensional Model for Game Theory

Based Predictive Analytics. 8*th Annual International Conference on Computational Science Computational Intelligence* (*CSCI*'21), Las Vegas, 15-17 December 2021, 510-515. https://doi.org/10.1109/CSCI54926.2021.00036

[2] Nagahisarchoghaei, M., Nur, N., Cummins, L., Nur, N., Karimi, M.M., Nandanwar, S., Bhattacharyya, S. and Rahimi, S. (2023) An Empirical Survey on Explainable AI Technologies: Recent Trends, Use-Cases, and Categories from Technical and Application Perspectives. *Electronics*, **12**, Article No. 1092. https://doi.org/10.3390/electronics12051092

[3] Nagahisarchoghaei, M., Karimi, M.M., Rahimi, S., Cummins, L. and Ghanbari, G. (2023) Generative Local Interpretable Model-Agnostic Explanations. *The International FLAIRS Conference Proceedings*, **36**. https://doi.org/10.32473/flairs.36.133378

[4] Bueno De Mesquita, B. (1980) An Expected Utility Theory of International Conflict. *American Political Science Review*, **74**, 917-931. https://doi.org/10.2307/1954313

[5] Bueno De Mesquita, B. (1994) Political Forecasting: An Expected Utility Method. Yale University Press, New Haven, 71-104. https://doi.org/10.12987/9780300158113-007

[6] Scholz, J.B., Calbert, G.J. and Smith, G.A. (2011) Unravelling Bueno de Mesquita's Group Decision Model. *Journal of Theoretical Politics*, **23**, 510-531. https://doi.org/10.1177/0951629811418142

[7] Mousavi Karimi, M., Rahimi, S., Nagahisarchoghaei, M. and Luo, C. (2022) A Multidimensional Game Theory-Based Group Decision Model for Predictive Analytics. *Computational and Mathematical Methods*, **2022**, Article ID: 5089021. https://doi.org/10.1155/2022/5089021

[8] Jesse, E. (2011) Forecasting the Future of Iran: Implications for U.S. Strategy and Policy. Ph.D. Thesis, RAND Corporation, Santa Monica.

[9] Eftekhari, Z. and Rahimi, S. (2014) Preana: Game Theory Based Prediction with Reinforcement Learning. *Natural Science*, **6**, 1108-1121. https://doi.org/10.4236/ns.2014.613099

[10] Thomson, R. and Stokman, F.N. (2006) Research Design: Measuring Actors' Positions, Saliences and Capabilities. In: Thomson, R., Stokman, F.N., Achen, C.H. and König, T., Eds., *The European Union Decides*, Cambridge University Press, Cambridge, 25-53. https://doi.org/10.1017/CBO9780511492082.003

[11] Thomson, R., Arregui, J., Leuffen, D., Costello, R., Cross, J., Hertz, R. and Jensen, T. (2012) A New Dataset on Decision-Making in the European Union before and after the 2004 and 2007 Enlargements (DEUII). *Journal of European Public Policy*, **19**, 604-622. https://doi.org/10.1080/13501763.2012.662028

[12] Warntjen, A. (2012) Measuring Salience in EU Legislative Politics. *European Union Politics*, **13**, 168-182. https://doi.org/10.1177/1465116511428495

[13] Wasserfallen, F., Leuffen, D., Kudrna, Z. and Degner, H. (2019) Analysing European Union Decision-Making during the Eurozone Crisis with New Data. *European Union Politics*, **20**, 3-23. https://doi.org/10.1177/1465116518814954

[14] De Mesquita, B.B. (1997) A Decision Making Model: Its Structure and Form. *International Interactions*, **23**, 235-266. https://doi.org/10.1080/03050629708434909

[15] Bueno De Mesquita, B. (1985) The War Trap Revisited: A Revised Expected Utility Model. *American Political Science Review*, **79**, 156-177. https://doi.org/10.2307/1956125

[16] De Mesquita, B.B. (2011) A New Model for Predicting Policy Choices: Preliminary

Tests. *Conflict Management and Peace Science*, **28**, 65-87. https://doi.org/10.1177/0738894210388127

[17] Bueno De Mesquita, B. (2002) Predicting Politics. The Ohio State University Press, Columbus.

[18] Baranick, M., Brian Efird, J.K. and Abdollahian, M. (2004) Stability and Regime Change in Iraq: An Agent Based Modeling Approach. Military Operations Research Society (MORS), Arlington.

[19] Bishop, C.M. (2006) Pattern Recognition and Machine Learning. Springer, Berlin.

[20] Weisstein, E.W. (2003) Gaussian Distribution. https://mathworld.wolfram.com/GaussianDistribution.html

[21] Bhattacharyya, A. (1943) On a Measure of Divergence between Two Statistical Populations Defined by Their Probability Distributions. *Bulletin of the Calcutta Mathematical Society*, **35**, 99-109.

[22] Mousavi Karimi, M. (2023) Expanding One-Dimensional Game Theory-Based Group Decision Models: Extension to n-Dimension and Integration of Distributed Position Function. Ph.D. Dissertation, Mississippi State University, Starkville.

[23] Ferrari, S. and Cribari-Neto, F. (2004) Beta Regression for Modelling Rates and Proportions. *Journal of Applied Statistics*, **31**, 799-815. https://doi.org/10.1080/0266476042000214501

[24] Lukaszyk, S. (2004) A New Concept of Probability Metric and Its Applications in Approximation of Scattered Data Sets. *Computational Mechanics*, **33**, 299-304. https://doi.org/10.1007/s00466-003-0532-2

[25] Connor, R. (2016) A Tale of Four Metrics. *International Conference on Similarity Search and Applications*, Tokyo, 24-26 October 2016, 210-217. https://doi.org/10.1007/978-3-319-46759-7_16

[26] Cha, S.-H. and Srihari, S.N. (2002) On Measuring the Distance between Histograms. *Pattern Recognition*, **35**, 1355-1370. https://doi.org/10.1016/S0031-3203(01)00118-2

[27] Cha, S.-H. (2007) Comprehensive Survey on Distance/Similarity Measures between Probability Density Functions. *City*, **1**, 1.