Scientific Research Publishing

International Journal of Astronomy and Astrophysics, 2024, 14, 1-19
https://www.scirp.org/journal/ijaa
ISSN Online: 2161-4725
ISSN Print: 2161-4717

The Criteria for Reducing Centrally Restricted Three-Body Problem to Two-Body Problem

Bijay Kumar Sharma

Department of Electronics and Communication, National Institute of Technology, Patna, India
Email: bksharma@nitp.ac.in

How to cite this paper: Sharma, B.K. (2024) The Criteria for Reducing Centrally Restricted Three-Body Problem to Two-Body Problem. *International Journal of Astronomy and Astrophysics*, **14**, 1-19.
https://doi.org/10.4236/ijaa.2024.141001

Received: December 7, 2023
Accepted: March 15, 2024
Published: March 18, 2024

Abstract

Our Solar System contains eight planets and their respective natural satellites excepting the inner two planets Mercury and Venus. A satellite hosted by a given Planet is well protected by the gravitational pertubation of much heavier planets such as Jupiter and Saturn if the natural satellite lies deep inside the respective host Planet Hill sphere. Each planet has a Hill radius a_H and planet mean radius R_P and the ratio $R_1 = R_P/a_H$. Under very low R_1 (less than 0.006) the approximation of CRTBP (centrally restricted three-body problem) to two-body problem is valid and planet has spacious Hill lobe to capture a satellite and retain it. This ensures a high probability of capture of natural satellite by the given planet and Sun's perturbation on Planet-Satellite binary can be neglected. This is the case with Earth, Mars, Jupiter, Saturn, Neptune and Uranus. But Mercury and Venus has $R_1 = R_P/a_H = 0.01$ and 5.9862×10^{-3} respectively hence they have no satellites. There is a limit to the dimension of the captured body. It must be a much smaller body both dimensionally as well masswise. The qantitative limit is a subject of an independent study.

Keywords

Hill's Radius, Two-Body Problem, Fixed-Point Solution, Lagrange Points, Earth-Moon-Test Particle, CRTBP

1. General Three-Body Problem (TBP) and Its Current Status

Isaac Newton (1643-1727) with his publication of *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*) in 1687 [1], opened the door to the investigation of Three-Body Problem (TBP). Two-Body Problem was first formulated by Johannes Kepler [2] (Kepler 1609) and solved by Newton in 1687 [1]. Gottfried Leibiz further elucidated and published Kepler's Law in 1690 [3]. Newton took the first step in defining and analyzing the movement of three massive bodies subject to their mutually gravitational attractions formally. He examined the perturbation of Moon's orbit under the gravitational influence of Earth and Sun. This formally laid the ground work for TBP but with approximations and its exact solution preoccupied and eluded the Mathematicians and Scientists from mid 1700 to the early 1900s.

AmerigoVespucci (1454-1512) (Room 2004, Encyclopædia Britannica Online, *Amerigo Vespucci*) addressed the physical problem of TBP in connection with determining the longitude while conducting the sea voyages to Brazil as the chief navigator of Spain and boldly conjectured the existence of the New World which was named Americas after his first name. Vespucci used the Moon path as a guidance during his exploratory voyages to South America and John Harrison (1693-1776) [4] invented the marine Chronometer for determining the longitude during long Sea Voyages.

Moon's orbit is perturbed by the strong gravitational field of Sun, Venus and Jupiter on different time scales [5]:

1) Moon orbital parameters repeat over synodic month of 29.53 days.

2) Strong signals in the time series of Moon's orbital parameters. These have a frequency of 6 months. This corresponds to the bi-annual impact of Earth's orbit about the Sun.

3) Orientation of Moon's orbit to the Sun cycles over the course of the year as well as the distance variation to the sun due to ellipticity of the orbit has a direct effect on Moon's orbit.

4) In addition to the solar perturbation, Venus and Jupiter also perturbs the Lunar Orbit.

5) Earth's asymmetric gravitational field also perturbs the Lunar's orbit.

All these had to be accounted for while designing the Marine Chronometer and this is precisely why TBP became a chief concern in mercantile trade era.

Various exact results were obtained, notably the existence of stable equilateral triangle configurations corresponding to so-called Lagrange points but in restricted framework. Heinrich Burns had shown in 1887 [6] and Henri Poincare had concluded in 1890 [7] that no analytical solution could be obtained for generalized TBP. Hence approximations were made which is known as Circular Restricted Three-Body Problem (CRTBP) approach. But even CRTBP remained insoluble except for some special cases.

From 1890 to 1930, George Darwin [8] [9], George Hill [10], Henry Plummer [11], Forest Moulton [12], Elis Stromgren [13] and their colleagues contributed to the discovery of the first known periodic orbits in CRTBP. In next 40 years from 1930 to 1970, 150 periodic orbits were computed [14]. In 1968 Roger Bourcke published a large catalog of families of planar periodic orbits that exist in CRTBP with Earth-Moon masses [14].

1960 onward Digital computers were used for numerical calculations and analysis which generated 3-D periodic orbits [15]-[24]. Halo and quasi Halo orbits were discovered [25] [26] [27] [28]. After 1975 a significant number of periodic orbits were discovered which can be classified in three distinct families of periodic orbits namely:

1) Lagrange-Euler Solution in 18th century which led to the discovery of librating points in CRTBP system. This was supplemented by Moore (1993) [29];

2) The Broucke-Henon-Hadjidemetrion family of periodic orbits in mid 1970s [30] [31] [32] [33] [34];

3) Figure 8 Family of orbits [29] [35]-[40] http://suki.ipb.ac.rs/3body.

In 1980, David Richardson generated halo orbits around the three collinear Lagrange points L1, L2 and L3 [41]. Towards the end of 20th century Lissajous and other quasi-halo orbits were generated [26] [42] [43] [44]. Libration orbits have been used for practical spacecraft missions and scientific missions such as WMAP, SOHO, communication relays [45] [46] [47] [48], as transportation nodes [49] [50] and navigation services [51]-[56].

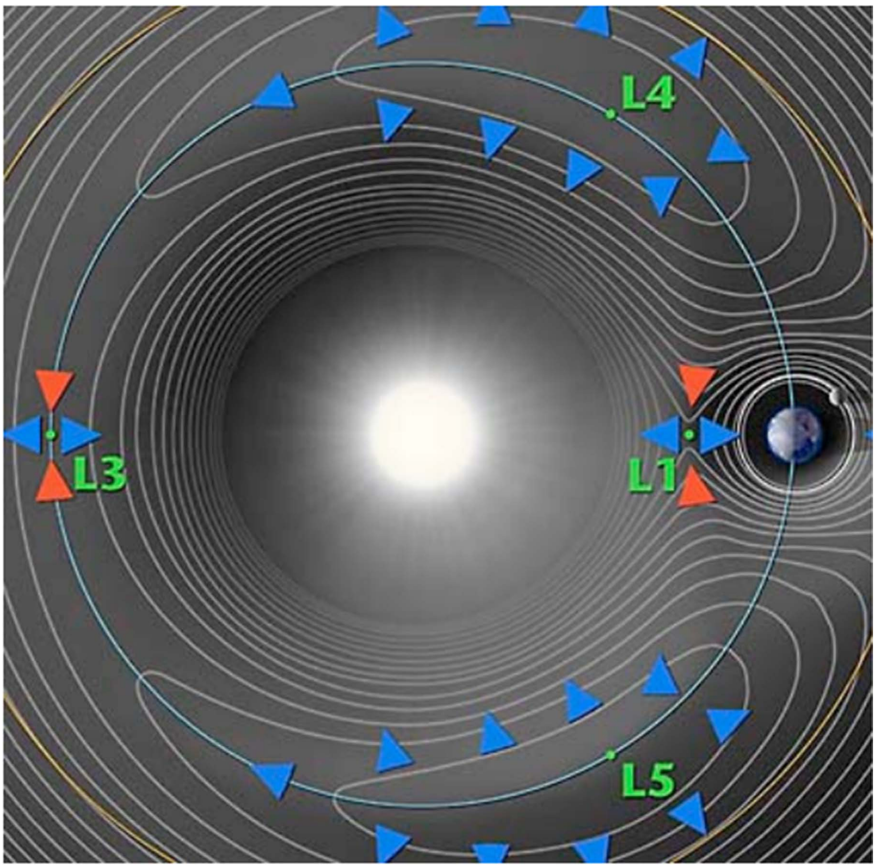
Through numerical search for periodic collisionless, planar solutions with zero-angular momentum in a two parameter sub-space of (the full four-dimension space of) scaled zero angular momentum initial conditions Šuvakov, M. and Dmitrašinović, V. (2013) [40] have discovered 13 new distinct equal mass zero-angular momentum planar, collisionless periodic 3-body orbits that can be classified as three new classes of orbits in addition to the existing class in 1975. The classes are sorted out on algebraic and geometric symmetry basis. No three-body system with equal masses and zero angular momentum have been observed by Astronomers hence these new solutions cannot be physically verified. Most of the observed TBP belong to Euler-Lagrange Class and to quasi Keplerian Broucke-Henon-Hadjidemetrion class of solutions.

2. Fixed Point Solutions of Circular Restricted Three-Body Problem

TBP had stymied the Scientists through out the 18th century so Joseph-Louis (Comte-de) Lagrange (1736-1813) reduced TBP to CRTBP and was the first to carry out the Fixed point solution of CRTBP and predicted two librating points L4 and L5 (Lagrange 1867-92) [57]. Leonhard Euler (a Swiss Mathematician 1707-1783) discovered L1, L2 and L3 subsequently [58]. Lagrange finally gave a comprehensive analysis of all the five points in his seminal paper "Essai sur le problem des trois corps" published in Euvres completes, tome 6, 229-331, Prix de l' Academie royale des sciences de Paris, tome IX, 1772. These five Librating points also known as Lagrange Points are stationary with respect to the synodic framework and they are stationary indefinitely. Hence they are known as Fixed-point solution. There are periodic orbit solutions also such as Lyapunov Orbits around LL1, LL2, LL3 (LL refers to the Lagrange points in Earth-Moon-Test particle system), Direct Prograde Orbits which fit between LL1 and LL2 are Lyapunov Orbits, Direct Retrograde Orbits, Halo Orbits, Vertical Lyapunov Orbits and Resonant Orbits.

CRTBP is a dynamical model used to describe the motion of a test particle in the presence of massive orbiting bodies such as Sun-Earth or Earth-Moon. It is convenient to describe the motion of the TBP in a synodic framework that rotates at the same rate as the two primaries around the barycenter of the binary. In effect the Frame of Reference is centered at the barycenter or center of mass (COM). The origin is at COM and X-axis is directed from COM to the smaller Primary. Z-axis extends perpendicular to the orbital plane and Y-axis completes the right handed coordinate fame-work. With respect to the synodic framework two primaries are stationary and test particle moves about in non-Keplerian motion. In case of Fixed Point Solution the Lagrange Points are stationary with respect to the two primaries and maintain the same relative position with respect to the two Primaries hence they are Fixed-Point Solution.

Roche Lobe or Hill Sphere of the primary and secondary meet at L1. Drop-let shaped figures in equi-potential plot define the boundaries of Roche's Lobe as shown in **Figure 1**. A critical equi-potential plot intersects itself at L1 forming a two-lobe figure of eight. One of the two components is at the center of each lobe. As the host fills up its respective Roche's lobe mass transfer takes place from via L1 to the second lobe. This is known as Roche's Lobe overflow. As shown in **Figure 1**, L1, L2 and L3 are saddle points (minima points) of potential and L4



A contour plot of the effective potential (not drawn to scale!)

Figure 1. A contour plot of effective potential of sun-earth along with the five Lagrange points (curtsey: https://map.gsfc.nasa.gov/mission/observatory_l2.html).

and L5 are the hill top(maxima point) of the potential. At these points forces are in equilibrium. The gravitational pull on the test particle is exactly balanced by the outward centrifugal force on the test particle. If m/M is less than 1/25 where m = mass of the secondary and M = mass of the primary then L4 and L5 become stable otherwise these equilibrium points are unstable [59].

In case of instability Coriolis force helps restore stability to satellites drifting at L4 and L5.

Asteroids at L4 (leading the secondary) and L5 (lagging the secondary) were first discovered in Sun-Jupiter System in early 20th century. These were called Trojan Asteroids named after Greek heroes who were victorious in capturing Troy and liberating their queen Helen. These Trojan asteroids were leading and lagging Jupiter in Jupiter's orbit itself. There are dozens of Trojan Asteroids trailing or heading Jupiter in its own orbit. Trojan Asteroids lie in Mars orbit also.

Tethys is a natural satellite of Saturn and Calypso is trailing Tethys and Telesto is leading Tethys in Tethys orbit. All three moons of Saturn are co-orbital with identical orbital period of 1.887802d and all are in synchronous orbit. That is all three are showing the same face to Saturn all the time [60].

In 1958, Polish astronomer Kazimierz Kordylewski discovered dust clouds at L4 and L5 librating points of Earth-Moon's system [61]. In 2010 NASA's WISE telescope (Wide Field Infra-Red Survey Explorer) after much speculation and doubts confirmed the first Trojan asteroid in Earth's Orbit at L4 [62]. Through archival search of infra-red data, Earth Trojan 2010TK7 was discovered librating around the leading Lagrange point L4 of Sun-Earth System. Lifetime is 10,000 years before it destabilizes. Table 1 tabulates the list of probes and telescopes which existed or are planned to be placed at Lagrange points of Sun-Earth System and of Earth-Moon System.

2.1. Analysis and Calculation of Lagrange Points L4 and L5 in Sun-Earth-Test Particle and in Earth-Moon-Test Particle Systems Is a Text Book Exercise in CRTBP

Figure 2 gives the layout of the five Lagrange points in a synodic framework. Figure 2 is not drawn to the scale. Synodic Framework is centered at COM and rotating anticlockwise at orbital rotation angular frequency of Ω radians per second. The dark dashed orbit is the orbit of the smaller primary and light gray dashed orbit is the orbit of the larger primary. D is the distance between the two primaries.

In case of Sun-Earth-Test particle:

$$D = 1\ \text{AU} \quad \text{and} \quad \Omega = \frac{2\pi}{365.25d} \tag{1}$$

In case of Earth-Moon-Test particle:

$$D = 384400\ \text{km} \quad \text{and} \quad \Omega = \frac{2\pi}{27.32d} \tag{2}$$

Table 1. List of Astronomical probes placed or planned to be placed at the five Lagrange Points.

		Sun-Earth System		
L1	**L2 (Ideal for Obs.)**	**L3**	**L4**	**L5**
SOHO*	WMAP†	None	Asteroid 2010TK7	Asteroid2010SO16*****
	PLANCK‡			
	WEBB Tel.††			
	Chang'e 2(2011-12)**			
	Herschal Space Obs. (ESA2013)***			
	Gaia Probe‡‡			
	PLATO†††			
	L1SA‡‡‡			
		Earth-Moon System		
L1	**L2**	**L3**	**L4**	**L5**
	ARTEMIS††††		Kordylewski Cloud	Proposed Space Colony
	Chang'e 5-T1****		Future location of TDRS style Communication satellite to support L2 satellites	A low energy trajectory for Lunar Orbit.
			Exploratory Gateway Platform	

†WMAP—Wilkinson Microwave Anisotropy Probe. ††Webb-James Webb Space Telescope to be launched in October 2018. It has 6.5 m mirror. Hubble has 2.4 m mirror. †††PLATO—Planetary Transits and Oscillations of Stars—it will search Earth-like rocky planets. ††††ARTEMIS—It is Berkeley Mission. It will launch a space craft on Liossajous orbit around Moon. ‡PLANCK—an advanced version of WMAP. ‡‡GAIA—It will generate 3D map of our galaxy. ‡‡‡LISA—LASER Interferometer Space Antenna for Gravitational Wave detection. To be launched in 2034. *SOHO—Solar and Heliospheric Observatory Satellite. **Chang'e 2-Chinese Lunar Unmanned Probe (2011-12). ***Herschal Space Observatory-2013 It was lauched by ESA and has 3.5 m mirror. ****Chang'e 5T1-Experimental Chinese unmanned Lunar Probe. *****Asteroid 2010SO16-Horse shoe companion of Earth in Sun-Earth System

Figure 3 gives the geometrical configuration of L4, L5 and the two primaries for calculation of the location of L4 and L5.

From Figure 3 we obtain the following:

$$x = \frac{q}{1+q} D \text{ and } (D-x) = \frac{1}{1+q} D \text{ where } q = \frac{m}{M} \tag{3}$$

The centrifugal force on the test particle is exactly balanced by the centripetal force provided by the gravitational forces on test particle due to Primary and smaller Primary. This is true in all five cases hence the five points are equilibrium points but unstable. In case of L4 and L5 if m/M is less than 1/25 then the two points become stable.

$$\text{Centrifugal force on the test particle} = \mu \times \frac{v_{\text{tangential}}^2}{\text{distance from COM}} \tag{4}$$

Let distance of the test particle from COM = d.

Then trigonometry of Figure 3 leads to:

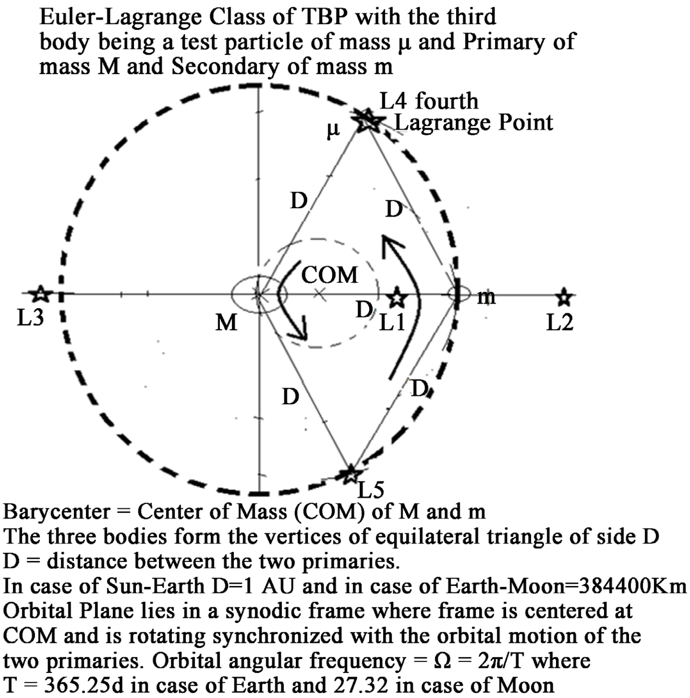


Figure 2. Lay-out of fixed point solution of CRTBP.

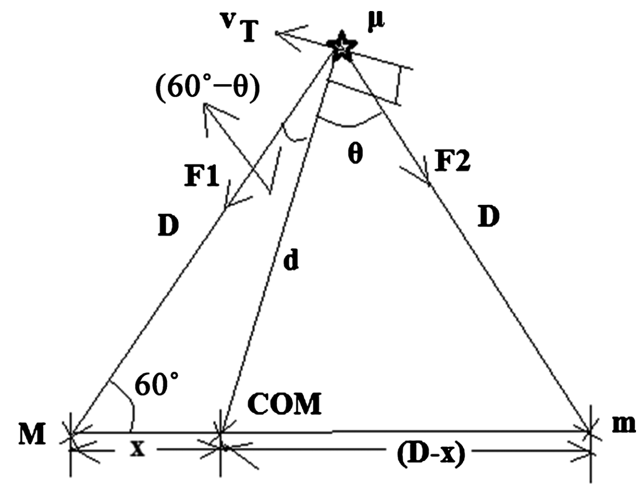


Figure 3. Geometry of TBP from which the equilibrium of forces will be calculated and position of L4 and L5 located.

$$d^2 = D^2\left[\frac{1+q+q^2}{1+2q+q^2}\right] = D^2 \times \frac{A}{B} \tag{5}$$

where $A = 1+q+q^2$ and $B = 1+2q+q^2$ or $\sqrt{B} = 1+q$

From equilibrium of forces at Lagrangian Point L4 marked by star, angular velocity of rotation is calculated to be:

$$\Omega^2 = \frac{GM}{D^3}\sqrt{\frac{B}{A}}\left[\left(\frac{1}{2}+q\right)\sqrt{\left[1-\frac{3}{4}\times\frac{1}{A}\right]}+\frac{3}{4}\times\frac{1}{\sqrt{A}}\right] \tag{6}$$

From (6) Ω is calculate and hence T (orbital period is calculated).

In Sun-Earth-Test particle system:

$$D = 1\ \text{AU},\ q = 3\times10^{-6},\ A = 1,\ B = 1.0001,\ \Omega = 1.99054\times10^{-7}\ \text{radians/s}$$

Therefore $T = 1.00027$ solar year

Therefore L4 and L5 lie at 1 AU on Earth's orbit from Earth leading as well as lagging.

In Earth-Moon-Test particle system:

$$D = 384400000\ \text{m},\ q = 1/81,\ A = 6643/6561,\ B = 6724/6561,$$
$$\Omega = 2.66457\times10^{-6}\ \text{radians/s}$$

Therefore $T = 27.3$d.

Therefore L4 and L5 lie at 384,400,000 m on Moon's orbit from Moon leading as well as lagging.

2.2. Analysis and Calculation of L1, L2 and L3 in Sun-Earth and Earth-Moon Systems

Figure 4 gives the geometrical layout of L1, L2 and L3 for any TBS (Sun-Earth-test particle or Earth-Moon-test particle).

The Lagrange's Points lie on a rotating frame of reference of rotation angular velocity Ω. Bigger primary is M, smaller primary is m and test particle is of mass μ. All three bodies are orbiting Center of Mass (COM) at orbital angular frequency = ω.

Lagrange's points L1, L2 and L3 are co-linear and orbiting around COM with orbital angular velocity = ω. Distance of separation between M and m is D. Distance of Lagrange point is L from m in case of OL1 and OL2 and L is the distance of L3 from M while calculating OL3.

The calculations have been taken from Merlyn Home Page, Astronomy & Astrophysics-Gravity4-lagrange points. System parameters have also been taken from Merlyn Home Page.

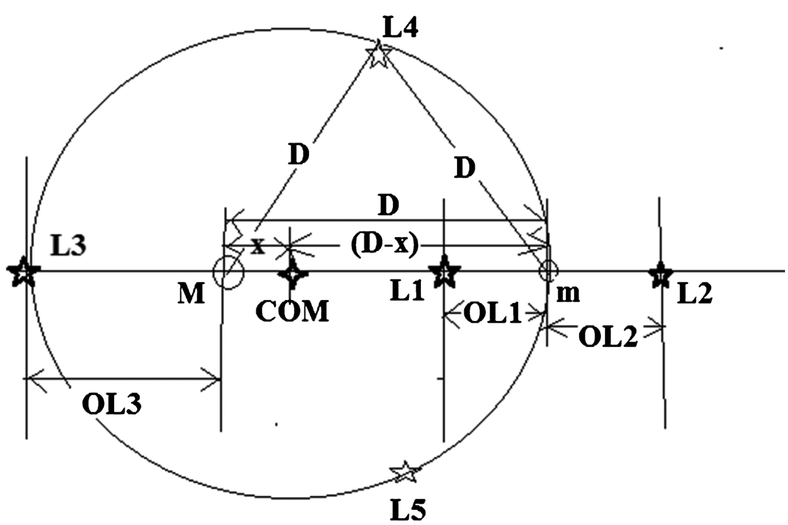


Figure 4. Geometrical layout of L1, L2, L3, L4 and L5 for a generalized CRTBP not to the scale.

http://www.merlyn.demon.co.uk/.

For L1:

$$\mu\omega^2 (D-L) = G\times\mu\left(\frac{M}{(D-L)^2} - \frac{m}{L^2}\right) \tag{7}$$

Equation (7) is an approximation since test particle is also orbiting COM hence radius of rotation is not (D-L) but infact it is (D-L-x) but for ease of calculation it has been taken as (D-L) which for practical cases quite valid.

Solving (7) for $D = 1$, $m = 1$, $M = 81$ for Earth-Moon-test particle gives:

$L = 0.151272$ which gives LL1 from Moon 58,149 Km for $D = 384{,}400$ Km (Earth Moon separation)

For Sun-Earth-Test particle: $D = 1$, $m = 1$, $M = 331{,}772.5755$ since Sun mass = 1.984×10^{30} Kg and Earth mass = 5.98×10^{24} Kg and D = 1AU = 149.46×10^6 Km we get $L = 0.009982$.

Hence EL1 from Earth is EL1 = 1,491,909 Km (1/100AU)

For L2:

$$\mu\omega^2 (D+L) = G\times\mu\left(\frac{M}{(D+L)^2} + \frac{m}{L^2}\right) \tag{8}$$

Solving (8) for $D = 1$, $m = 1$, $M = 81$ for Earth-Moon-test particle gives:

$L = 0.168327$ which gives LL2 from Moon 64,705 Km for $D = 384{,}400$ Km.

For Sun-Earth-Test particle: $D = 1$, $m = 1$, $M = 331{,}772.5755$ since Sun mass = 1.984×10^{30} Kg and Earth mass = 5.98×10^{24} Kg and D = 1AU = 149.46×10^6 Km we get $L = 0.010049$.

Hence EL2 from Earth is EL2 = 1,501,921 Km (1/100AU)

For L3:

$$\mu\omega^2 (L) = G\times\mu\left(\frac{M}{L^2} + \frac{m}{(L+D)^2}\right) \tag{9}$$

Solving (9) for $D = 1$, $m = 1$, $M = 81$ for Earth-Moon-test particle gives:

$L = 1.001029$ which gives LL3 from Earth 384,795.5 Km for $D = 384{,}400$ Km.

For Sun-Earth-Test particle: $D = 1$, $m = 1$, $M = 331{,}772.5755$ since Sun mass = 1.984×10^{30} Kg and Earth mass = 5.98×10^{24} Kg and D = 1AU = 149.46×10^6 Km we get $L = 1.000$.

Hence EL3 from Sun is 149,460,000 Km = 1AU.

Table 2 tabulates the Lagranges points for Sun-Earth-test particle and for Earth-Moon-test particle systems. Lagranges points have an intimate connection with Hill Radius discussed in next section. Hill Radius decides the gravitational sphere of influence of the given astronomical object.

From the Table 2, a very simple picture emerges about L1, L2 and L3.

Figure 5 gives the layout of L1, L2 and L3 in Earth-Moon system and, as shown in Figure 5, L1 is the Roches Overflow point. When Primary body surface in Primary Roches Lobe extends out beyond its respective Roche's Lobe then the material outside the Primary Roche's Lobe falls into the Roches Lobe of

Table 2. Lagranges point (L1, L2 and L3) in CRTBP framework.

	Sun-Earth-Test particle	Earth-Moon-Test particle
L1 from the smaller primary as in Figure 4	1,491,926 Km(~1/100AU) = Hill's Radius of Earth	58,149 Km~Hill's Radius of Moon
L2 from the smaller primary as in Figure 4	1,501,921 Km(~1/100AU) = Hill's Radius of Earth	64,705 Km~Hill's Radius of Moon
L3 from the bigger primary as in Figure 4	149,460,000 Km (1AU) = Earth's orbital Radius around Sun	384,795.5 Km~ Lunar Orbital Radius

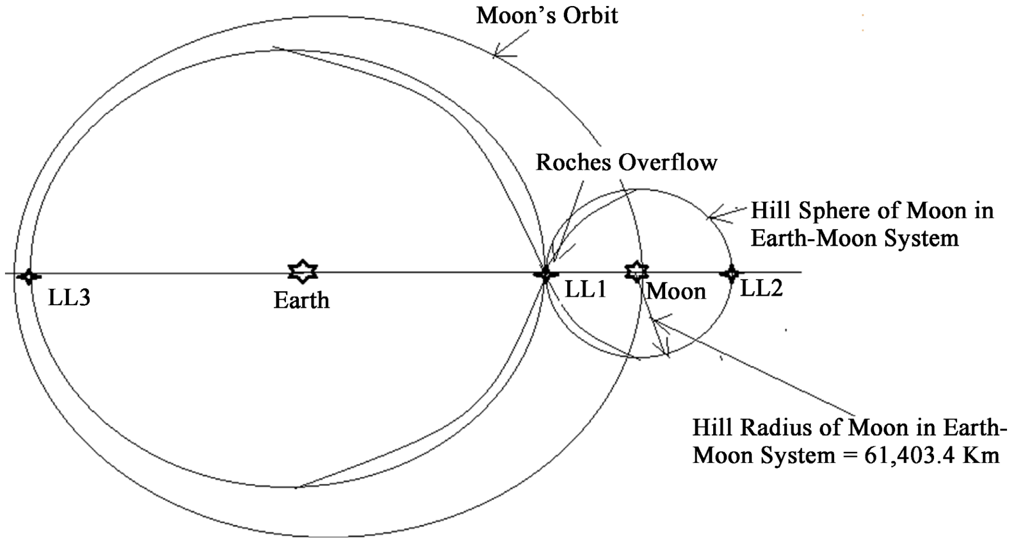


Figure 5. Earth's Orbital Radius = 1AU and Earth's Hill Radius = (0.01AU) decide EL1, EL2 and EL3.

the secondary.

In Table 3, the Cartesian coordinates of the Five Lagrange's Points in Sun-Earth-Test particle CRTBP system and in Earth-Moon-Test particle system. (Here the Cartesian coordinate system is a synodic framework with COM centered at the origin of the synodic framework. L1, L2, L3 lie on X-axis and L4 and L5 lie on the vertices of equilateral triangle co-planar with XY plane.) and co-orbital with the smaller primary-L1 heading the smaller primary and L5 trailing the smaller primary.

3. Circular Restricted Three-Body Problem (CRTBP) and Its Approximation as Two-Body Problem for Planetary Satellites [63] (Kokubo *et al.* 2000)

Traditionally CRTBP is used for the study of Planetary Satellites. The Hill approximation describes the motion of two bodies orbiting a much more massive central body using a rotating coordinate system. The Hill coordinate system is defined so that the x axis points radially outward, the y axis is tangent to a circular orbit, and the z axis is normal to the orbital plane. The angular velocity of the coordinate system is just the Keplerian orbital frequency, $\Omega = \left(GM_C/a_0^3\right)1/2$, where a_0 is the reference orbital radius and M_C is the mass of the central body.

Table 3. Cartesian coordinates of five Lagrange's points in sun-earth and earth-moon systems ([5] Appendix Parker & Anderson 2013).

		X(km)	Y(Km)	Z(Km)
		Sun-Earth-Test Particle System		
L1	EL1	148,099,795		0
L2	EL2	151,105,019.2		0
L3	EL3	−149,598,060.2		0
L4	EL4	74,798,480.5	129,555,556.4	0
L5	EL5	74,798,480.5	−129,555,556.4	0
		Earth−Moon−Test Particle System		
L1	LL1	321,710.177		0
L2	LL2	444,244.222		0
L3	LL3	−386,346.081		0
L4	LL4	187,529.315	332,900.165	0
L5	LL5	187,529.315	−332,900.165	0

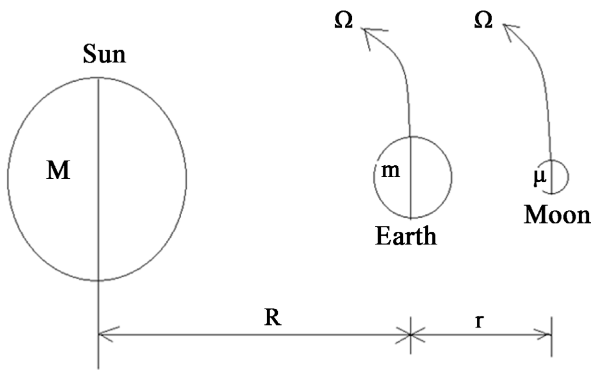


Figure 6. Sun-Earth-Moon in planar circular restricted three-body problem (CRTBP) framework.

The Hill coordinate system is shown in **Figure 6**.

Two important parameters are required namely Roche Limit (a_R) [63] and Hill Sphere with radius (a_H) in CRTBP approach. Roche Limit (a_R) sets the limit for accretion of dust particles near a celestial object. Within this limit dust particles cannot accrete to form a solid body and if a solid body does enter this limit it will be pulverized. Hydrodynamic static equilibrium spherical state can be achieved only beyond this limit. This limit is given as follows:

$$\text{Roche's Limit} = a_R = \left(16\frac{\rho_E}{\rho_M}\right)^{1/3} \cdot R_E \tag{10}$$

Equation (10) is applicable to Earth-Moon system where ρ_E and ρ_M are the mean densities of Earth and Moon and R_E is the globe radius of the Earth.

An *astronomical body*'s *Hill sphere* is the region in which it dominates the attraction of satellites. The outer shell of that region constitutes a zero-velocity surface. To be retained by a planet, a moon must have an orbit that lies within the planet's *Hill sphere*.

Hill Sphere defines the gravitational sphere of influence of the given astronomical body and it shields the captured satellites from the perturbative action of the central massive primary namely Sun in our case. Calculation of the Hill Sphere referring to **Figure 6**:

$$a_H = a\left(\frac{m}{3M}\right)^{1/3} \tag{11}$$

Equation (11) gives the Hill Radius of the Earth in presence of Sun and "*a*" is the semi-major axis of the Earth, *m* is mass of Earth and *M* is the mass of Sun.

From **Table 4**, gravitational sphere of influence of Sun is 2.87ly = 181,501 AU.

Hence Heliosphere extends up to 181,501 AU which is the extent of Oort's cloud which lies from 50,000 AU to 100,000 AU.

Gravitational extent of Earth is 0.01 AU = 1,496,280 Km. Moon Orbital Radius is 384,400 Km which is well within Earth's Hill radius hence Moon is held captive by Earth.

It is shown in Kokubo *et al.* (2000) [63] that:

$$\frac{R_E}{a_{H-EARTH}} = 0.579 \times \frac{a_{R-E-S}}{a_E} \tag{12}$$

For $a_E \sim a_R$, R_E (physical size of Earth) becomes comparable to the Hill Sphere Radius of Earth. Under such circumstances Earth's Hill Sphere is occupied by the physical size of Earth and Earth cannot accommodate a satellite. If there is a satellite then it will have to be studied in a framework of Circular Restricted Three-Body Problem (CRTBP) as given in **Figure 6**.

For $a_E \gg a_R$, R_E becomes insignificant as compared to Hill Sphere Radius of Earth hence Earth has a spacious Hill Sphere in which it can accommodate a satellite with high probability which it does as our Moon proves and Earth-Moon can be treated as 2-Body Problem. Hence CRTBP reduces to 2-Body Problem and perturbing effect of Sun on E-M system can be ignored.

We derive the ratio (Planet globe radius/Hill radius of the Planet) and see if Hill Sphere of the Planet is spacious enough to capture and accommodate a natural

Table 4. Calculation of hill radius of sun and P. centauri in sun-proxima centauri system and hill radius of earth in sun-earth system.

Celestial Body	Mass (Kg)	Orbital radius of the secondary	Hill Radius	Radius of the Gravitational Sphere of influence of the celestial object
Sun	1.99^{30}	4.37ly	Of Sun w.r.t. P.Centauri is 2.87 ly	2.87 ly
Proxima Centauri	2.446^{29}	4.37ly	Of P.Centauri w.r.t. Sun is 1.5 ly	1.5 ly
Earth	5.9736^{24}	1AU	Of Earth w.r.t. Sun 0.01 AU = 1,496,280 Km	Earth's gravitational extent is 0.01 AU hence Moon is within Earth's gravitational influence.

satellite. We carry this exercise for the terrestrial planets, Jupiter and satellites and tabulate them in Table 5 and Table 6.

For the calculation of the ratio R1/R2, all the system parameters are given in Appendix I.From the Table 6 it is evident that parameter R1 = Planet Radius/Hill Radius is highly correlated with probability of natural satellite capture. Once satellite capture has taken place it qualifies for two-Body Problem analysis.

Satellite capture implies that host has a spacious Hill sphere and once a satellite is hosted it is gravitationally shielded from the perturbative effect of Sun, Jupiter

Table 5. Roche's limit of the sun-planet system or planet-satellite system as the case may be, hill radius of the planet/satellite, planet/satellite globe radius.

	Roche's limit (a_R) × 10^6 Km	Hill Radius × 10^6 Km	a_P (×10^6Km)	R_p (Km)
Sun-Mercury	1.11809	220,559 Km	57.9	2448.5
Sun-Venus	1.13102	1.01099	108.2	6052
Sun-Earth	1.11218	1.496	149.6	6378
Sun-Mars	1.24477	982,465 Km	206.6	3396
Sun-Jupiter	1.78847	53.1397	778.57	71,492
Earth-Moon	18,973.8 Km	Of Moon-61,403.4 Km	384,400 Km	1737.5
Mars-Phobos	10,905 Km	Of Phobos-165,574 Km	9378 Km	11
Mars-Deimos	11,209 Km	Of Deimos-2,524,142 Km	23,459 Km	6.892
Sun-P.Centauri		Of P.Centauri-94,861 AU	4.37 ly	100,900
Sun-P.Centauri		Of Sun-181,501 AU	4.37 ly	695,700

Table 6. (Planet Radius/Hill Radius) ratio = R1 and (a_R(Sun-Planet)/a_P) ratio = R2, R1/R2 and comment on Planet's acceptability of natural satellite or on satellite's acceptability of a sub-satellite.

	R1	R2	R1/R2	Comment
Sun-Mercury	0.01	0.0193	0.5181	Mercury can accept satellites with low probability
Sun-Venus	5.9862×10^{-3}	0.01045	0.5728	Venus can accept satellites with low probability
Sun-Earth	4.26×10^{-3}	7.434×10^{-3}	0.57	Earth has a satellite
Sun-Mars	3.4566×10^{-3}	6.025×10^{-3}	0.5737	Mars has two satellites
Sun-Jupiter	1.345×10^{-3}	2.297×10^{-3}	0.5855	Much higher probability of satellites It has 67 natural satellites
Earth-Moon	0.0283	0.04936	0.5733	Moon cannot accept a sub-satellite
Mars-Phobos	0.66	1.16	0.5689	Phobos cannot accept sub-satellite
Mars-Deimos	0.2735	0.477815	0.5723	Deimos cannot accept a sub-satellite

and Saturn. Once the shielding has taken place two-body problem analysis can be done. So in nutshell parameter R1 decides the approximation of TBP to 2-body problem

Ratio R1 comprehensively explains why Mercury and Venus lack a moon. The reason is simple. Mercury and Venus do not have a spacious enough Hill Sphere to capture and retain natural satellites. Ratio R1 must be less than 0.006 in order to qualify as a natural satellite host.

4. Discussion

This study has looked into three-body problems and its reduction to Circular-restricted-three-body problem. The derivation of Five Lagrange's Points is a text book exercise in Circular Restricted Three-Body Problem (CRTBP). In case of Sun-Earth System we treat Sun-Earth-Test Particle as the CRTBP and in case of Earth-Moon System we consider Earth-Moon-Test Particle as CRTBP.

Whenever the third body is deep in the Hill Sphere of the second body, the third body is gravitationally shielded due to the gravitational sphere of influence of the second primary from the gravitational perturbative effects of the most massive first primary then such three-body problems can be approximated as two-body problem containing the second primary and the secondary. This is the case with Sun-Earth-Moon as well as with Sun-Mars-Phobos. Hence these general three-body problems can be approximated as Earth-Moon and Mars-Phobos or Mars-Deimos two-body problem without any loss of generality or accuracy.

This study has also established that in three-body problem whenever the Hill Sphere of the second primary is spacious enough that is (Second Primary Radius/Hill Radius) < 0.006, the second primary is receptive to a natural satellite. Mercury and Venus do not have a spacious enough Hill Sphere hence they do-not host a natural satellite.

There is a limit to the dimension of the captured body. It must be a much smaller body both dimensionally as well masswise. The qantitative limit is a subject of an independent study.

5. Conclusion

This paper gives the theoretical justification for treating S-E-M as E-M system and S-Mars-Phobos or S-Mars-Deimos as Mars-Phobos or Mars-Deimos 2-body problem. Just as we have electro-magnetic shielding (metallic shielding) of electrical system and protection of electrical system from powerful electromagnetic waves in exactly the same manner the celestial bodies are protected from gravitational perturbations by the Hill's sphere. The Hill's sphere acts as the gravitational sphere of influence. Earth's Hill's sphere protects our Moon from gravitational perturbations of much heavier bodies such as Sun, Jupiter and Saturn.

Acknowledgements

I acknowledge the cooperation of IIT, Director, for utilizing the infrastructure of

IIT, Patna, in preparing this manuscript. I also acknowledge the cooperation of Prof. Preetam Kumar for letting me utilize the resources of Communication System Lab. I am indebted to my D.Sc supervisor Prof. Bhola Ishwar for his guidance in developing this Three-Body Problem. I also acknowledge the financial support given to me by University Grants Commission India as Emeritus Fellow under the Fellowship Scheme EMERITUS/2012-13-GEN-855/.

Dedication

This paper is dedicated to "Aditya-L1 the Solar Probe launched by Indian Space Research Organization".

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Cohen, I.B. and Whitman, A. (1687) Newton, Isaac—The Principia. Mathematical Principles of Natural Philosophy, Assisted by Julia Budenz, 5th July.

[2] Kepler (1609). https://www.nytimes.com/1990/01/23/science/after-400-years-a-challenge-to-kepler-he-fabricated-his-data-scholar-says.html?pagewanted=1

[3] Porter, R. (1992) The Scientific Revolution in National Context. Cambridge University Press, Cambridge, 102.

[4] Carter, W.E. and Carter, M.S. (2012) The British Longitude Act Reconsidered. *American Scientist*.

[5] Parker, J.S. and Anderson, R.L. (2013) Low Energy Lunar Trajectory Design. Jet Propulsion Lab, Pasadena.

[6] Bruns, E.H. (1887) Uber dia Integrale desVielkorper-Problem. *Acta Mathematica*, **11**, 25-96. https://doi.org/10.1007/BF02418042

[7] Poincare, H. (1987) Chapter XXIX, No. 341, Les methods nouvelles de la mecanique celeste, tomes I (1892) and III (1899). Re-Edition Blanchard, Paris.

[8] Darwin, G.H. (1897) Periodic Orbits. *Acta Mathematica*, **21**, 99-242. https://doi.org/10.1007/BF02417978

[9] Darwin, G.H. (1911) Periodic Orbits. Scientific Papers, Vol. 4, Cambridge University Press, Cambridge.

[10] Hill, G.W. (1898) Review of Darwin's Periodic Orbits. *Astronomical Journal*, **18**, 120. https://doi.org/10.1086/102833

[11] Plummer, H.C. (1903) On Oscillating Satellites-1. *Monthly Notices of the Royal Astronomical Society*, **63**, 436-443. https://doi.org/10.1093/mnras/63.8.436

[12] Moulton, F.R. (1920) Priodic Orbits. Carnegie Institute of Washington Publications, Washington.

[13] Stromgren, E. (1935) Connaissance actuelle des orbites dans le probleme des trios corps. Copenhagen Observatory Publications, No. 100.

[14] Broucke, R.A. (1968) Periodic Orbits in the Restricted Three-Body Problem with Earth-Moon Masses. Tech. Rep. 32-1168, Jet Propulsion Laboratory, California In-

stitute of Technology, Pasadena.

[15] Du Problenon, M.H. (1965) Exploration Numeme des Trois Corps, (I), Masses Egales, Orbites Periodiques. *Annales d'Astrophysique*, **28**, 499-511.

[16] Du Problenon, M.H. (1965) Exploration Numéme des Trois Corps, (II), Masses Egales, Orbites Periodiques. *Annales d'Astrophysique*, **28**, 992-1007.

[17] Du Problenon, M.H. (1966) Exploration Numeme des Trois Corps, (III), Masses Egales, Orbites Non Periodiques. *Bulletin Astronomique*, **1**, 57-80.

[18] Henon, M. (1966) Exploration Numerique du Probleme des Trois Corps, (IV), Masses Egales, Orbites Non Periodiques. *Bulletin Astronomique*, **1**, 49-66.

[19] Henon, M. (1969) Numerical Exploration of the Restricted Problem. V., Hill's Case: Periodic Orbits and Their Stability. *Astronomy & Astrophysics*, **1**, 223-238.

[20] Arenstorf, R.F. (1963) Existence of Periodic Solutions Passing near both Masses of the Restricted Three-Body Problem. *AIAA Journal*, **1**, 238. https://doi.org/10.2514/3.1516

[21] Goudas, C.L. (1963) Three Dimensional Periodic Orbits and Their Stability. *Icarus*, **2**, 1-18. https://doi.org/10.1016/0019-1035(63)90003-4

[22] Bray, T.A. and Goudas, C.L. (1967) Doubly Symmetric Orbits about the Collinear Lagrange Points. *The Astronomical Journal*, **72**, 202-213. https://doi.org/10.1086/110218

[23] Bray, T.A. and Goudas, C.L. (1967) Three Dimensional Periodic Oscillations about L1, L2, L3. *Advances in Astronomy and Astrophysics*, **5**, 71-130. https://doi.org/10.1016/B978-1-4831-9923-8.50007-3

[24] Kolenkiewicz, R. and Carpenter, L. (1968) Stable Periodic Orbits about the Sun Perturbed Earth-Moon Triangular Points. *AIAA Journal*, **6**, 1301-1304. https://doi.org/10.2514/3.4738

[25] Farquhar, R.W. (1968) The Control and Use of Libration-Point Satellites. PhD Thesis, Department of Aeronautics and Astronautics, Stanford University, Stanford.

[26] Farquhar, R.W. and Kamel, A.A. (1973) Quasi-Periodic Orbits about the Translunar Libration Point. *Celestial Mechanics*, **7**, 458-473. https://doi.org/10.1007/BF01227511

[27] Breakwell, J.V. and Brown, J.V. (1979) The Halo Family of 3-Dimensional Periodic Orbits in the Earth-Moon Restricted 3-Body Problem. *Celestial Mechanics*, **20**, 389-404. https://doi.org/10.1007/BF01230405

[28] Howell, K.C. (1984) Three-Dimensional, Periodic, "Halo" Orbits. *Celestial Mechanics*, **32**, 53-71. https://doi.org/10.1007/BF01358403

[29] Moore, C. (1993) Braids in Classical Dynamics. *Physical Review Letters*, **70**, 3675-3679. https://doi.org/10.1103/PhysRevLett.70.3675

[30] Broucke, R. and Boggs, D. (1975) Periodic Orbits in the Planar General Three Body Problem. *Celestial Mechanics*, **11**, 13-38. https://doi.org/10.1007/BF01228732

[31] Hadjidemetrion, J.D. and Christides, Th. (1975) Families of Periodic Orbits in Three Body Problem. *Celestial Mechanics*, **12**, 175-187. https://doi.org/10.1007/BF01230210

[32] Hadjidemetrion, J.D. (1975) The Continuation of Periodic Orbits from the Restricted to the General Three Body Problem. *Celestial Mechanics*, **12**, 255-276. https://doi.org/10.1007/BF01228563

[33] Henon, M. (1976) A Family of Periodic Solutions of Planar Three Body Problem and Their Stability. *Celestial Mechanics*, **13**, 267-285.

https://doi.org/10.1007/BF01228647

[34] Henon, M. (1977) Stability of Interplay Motions. *Celestial Mechanics*, **15**, 243-261. https://doi.org/10.1007/BF01228465

[35] Chenciner, A. and Montgomery, R. (2000) A Remarkable Periodic Solution of the Three Body Problem in the Case of Equal Masses. *Annals of Mathematics*, **152**, 881-901. https://doi.org/10.2307/2661357

[36] Simo, C. (2002) Celestial Mechanics. *Contemporary Mathematics*, **292**, 209.

[37] Chenciner, A., Fejoz, J. and Montgomery, R. (2005) The Rotating Eights: 1. The Three Families. *Nonlinearity*, **18**, 1407-1424. https://doi.org/10.1088/0951-7715/18/3/024

[38] Broucke, R., Elipe, A. and Riagus, A. (2006) On the Figure-8 Periodic Solutions in the Three Body Problem. *Celestial Chaos*, *Solitons & Fractals*, **30**, 513-520. https://doi.org/10.1016/j.chaos.2005.11.082

[39] Nauenberg, M. (2007) Continuity and Stability of Families of Figure Eight Orbits with Finite Angular Momentum. *Celestial Mechanics*, **97**, 1-15. https://doi.org/10.1007/s10569-006-9044-7

[40] Šuvakov, M. and Dmitrašinović, V. (2013) Three Classes of Newtonian Three Body Periodic Orbits. *Physical Review Letters*, **110**, Article ID: 114301. https://doi.org/10.1103/PhysRevLett.110.114301

[41] Richardson, D.L. (1980) Analytical Construction of Periodic Orbits about the Collinear Points. *Celestial Mechanics*, **22**, 241-253. https://doi.org/10.1007/BF01229511

[42] Howell, K.C. and Pernicka, H.J. (1988) Numerical Determination of Lissajous Trajectories in the Restricted Three-Body Problem. *Celestial Mechanics*, **41**, 107-124. https://doi.org/10.1007/BF01238756

[43] Gomez, G., Masdemont, J. and Simo, C. (1998) Quasihalo Orbits Associated with Libration Points. *The Journal of the Astronautical Sciences*, **46**, 135-176. https://doi.org/10.1007/BF03546241

[44] Gomez, G., Jorba, A., Llibre, J., Martinez, R., Masdemont, J. and Simo, C. (2001) Dynamicsand Mission Design near Libration Points. Vol. I-IV. World Scientific Publishing Co., Singapore. https://doi.org/10.1142/9789812794635

[45] Clarke, A.C. (1950) Interplanetary Flight. Temple Press Books Ltd., London.

[46] Farquhar, R.W. (1966) Station-Keeping in the Vicinity of Collinear Libration Points with an Application to a Lunar Communications Problem. In: *Space Flight Mechanics*, Vol. 11 of Science and Technology Series, American Astronautical Society, New York, 519-535.

[47] Farquhar, R.W. (1967) Lunar Communications with Libration-Point Satellites. *Journal of Spacecraft and Rockets*, **4**, 1383-1384. https://doi.org/10.2514/3.29095

[48] Hill, K., Born, G.H. and Lo, M.W. (2005) Linked, Autonomous, Interplanetary Satellite Orbit Navigation (LiAISON) in Lunar Halo Orbits. *Proceedings of the AAS/AIAA Astrodynamics Specialist Conference*, South Lake Tahoe, 7-11 August 2005, Vol. 123, Paper AAS 05-400.

[49] Bond, V.R., Sponaugle, S.J., Fraietta, M.F. and Everett, S.F. (1991) Cislunar Libration Point as a Transportation Node for Lunar Exploration. *Proceedings of the 1st AAS/AIAA Spaceflight Mechanics Meeting*, Houston, 11-13 February 1991, Vol. 75, Paper AAS 91-103.

[50] NASA Review (2009) Seeking a Human Spaceflight Program Worthy of a Great Nation. Review of U.S. Human Spaceflight Plans Committee, National Aeronautics and Space Administration, September 2009.

[51] Hill, K. (2007) Autonomous Navigation in Libration Point Orbits. Ph.D. Thesis, University of Colorado, Boulder.

[52] Hill, K., Lo, M.W. and Born, G.H. (2006) Linked, Autonomous, Interplanetary Satellite Orbit Navigation (LiAISON). *Proceedings of the AAS/AIAA Astrodynamics Specialist Conference*, South Lake Tahoe, 7-11 August 2005, Vol. 123, Paper AAS 05-399.

[53] Parker, J.S., Anderson, R.L., Born, G.H. and Fujimoto, K. (2012) Linked Autonomous Interplanetary Satellite Orbit Navigation (LiAISON) between Geosynchronous and Lunar Halo Orbits. Tech. Rep. D-72688 (Internal Document), Jet Propulsion Laboratory, California Institute of Technology, Pasadena.

[54] Villac, B., Chow, C., Lo, M., Hintz, G. and Nazari, Z. (2010) Dynamic Optimization of Multi-Spacecraft Relative Navigation Configurations in the Earth-Moon System. *Proceedings of the AAS George H. Born Symposium*, Boulder, 13-14 May 2010.

[55] Hill, K., Parker, J.S., Born, G.H. and Demandante, N. (2006) A Lunar L2 Navigation, Communication, and Gravity Mission. *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference*, Keystone, August 2006, Paper AIAA 2006-6662. https://doi.org/10.2514/6.2006-6662

[56] Hill, K. and Born, G.H. (2007) Autonomous Interplanetary Orbit Determination Using Satellite-to-Satellite Tracking. *AIAA Journal of Guidance*, *Control*, *and Dynamics*, **30**, 679-686. https://doi.org/10.2514/1.24574

[57] Lagrange, J.-L. (1867-92) Tome 6, Chapitre II: Essai sur le problème des trois corps. In: *Œuvres de Lagrange*, Gauthier-Villars, Paris, 229-334. (In French)

[58] Koon, W.S., Lo, M.W., Marsden, J.E. and Ross, S.D. (2006) Dynamical Systems, the Three-Body Problem, and Space Mission Design. 9.

[59] Liu, C. and Dong, L. (2019) Stabilization of Lagrange Points in Circular Restricted Three-Body Problem: A Port-Hamiltonian Approach. *Physics Letters A*, **383**, 1907-1914. https://doi.org/10.1016/j.physleta.2019.03.033

[60] Williams, D.R. (2011) Saturnian Satellite Fact Sheet.

[61] Laufer, R., Tost, W., Zielie, O., *et al.* (2007) The Karodylewski Clouds—An Example for a Cruise Phase Observation during the Lunar Mission BW1. 11*th ISU Annual International Symposium*, Strasbourg, 21-23 February 2007, 1-5.

[62] Connors, M., Weigert, P. and Veillet, C. (2011) Earth's Trojan Asteroid. *Nature*, **475**, 481-483. https://doi.org/10.1038/nature10233

[63] Kokubo, E., Canup, R.M. and Ida, S. (2000) Lunar Accretion from an Impact Generated Disk. In: Canup, R. and Righter, K., Eds., *Origin of Earth and Moon*, University of Arizona Press, Tucson, 145-163. https://doi.org/10.2307/j.ctv1v7zdrp.14

Appendix I.

Fact Sheet of Earth-Moon: http://nssdc.gsfc.nasa.gov/planetary/factsheet/moonfact.html

parameters	Earth	Moon
Mass (Kg)	5.9726×10^{24}	0.07342×10^{24}
GM (Km^3/s^2)	0.3986×10^6	0.0049×10^6
Volumetric Mean Radius Or Median Radius ($\times 10^3$ m)	6371	1737
Flattening(ellipticity)	0.00335	0.0012
Mean Density (Kg/m^3)	5514	3344
Moment of Inertia ($I/(MR^2)$)	0.33086	0.394
Sidereal Spin period	23.9344 h	27.322d
Sidereal Orbital period (d)	-	655.7208 h (27.3217d)
a* (semi-major axis) ($\times 10^8$ m)	-	3.84400
Lunar Orbit eccentricity	-	0.0549
Lunar Orbital inclination w.r.t. Ecliptic	-	5.145 degrees
$B = \sqrt{G(M+m)}$ ($m^{3/2}/s$)		2.00873×10^7

*Mean Orbital Distance from the center of Earth.