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Output Feedback Stabilization for a 1-D Conservative Wave Equation with General Corrupted Boundary Observation

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Abstract

In this paper, we consider the output feedback stabilization of a 1-D conservative wave equation, where the boundary velocity observation is subjected to a general disturbance. We first consider using only the output of the system to online estimate the disturbance by active disturbance rejection control (ADRC). The observer is designed in terms of the disturbance estimator. Then we present an observer-based output feedback law to achieve stabilization. The estimated disturbance is proved to be convergent to the unknown disturbance and the velocity signal can be asymptotically recovered when time tends to infinity. At the same time, the asymptotic stability of the closed-loop system can be verified. Finally, some simulations are given to illustrate the theoretical conclusions.

Keywords

Conservative Wave Equation, Stabilization, Boundary Control, Disturbance Rejection

1. Introduction

The stabilization of strings and flexible beams is always an important research direction in recent decades, see [1] [2] [3] [4], to name just a few. When actuators and sensors are collocated, system can be stabilized by utilizing passive principle [5] [6]. Compared with the collocated case, the non-collocated stabilization problem is more difficult because the passivity principle can not be used. However, non-collocated case is more widely used than collocated case in engineering (see, e.g., [7] [8]). With the proposal of the backstepping approach, this method has been extensively used in stabilization problem for parabolic equations [9] [10] [11], first-order hyperbolic equations [12] [13] [14], wave equations [15] [16] [17] and other partial differential equations [18] [19] [20]. In [21], in order to stabilize an unstable wave equation, using the backstepping method, not only the collocated Dirichlet boundary control but also the non-collocated Neumann boundary control is considered. The strong stabilization of unstable wave equation by using non-collocated boundary displacement can be found in [22]. Then in [23], the stabilization of unstable wave equation with Neumann boundary control can be achieved by using only collocated boundary displacement. Good progress has been made in [24], where the finite-time stabilization of 1-D wave equation by using only non-collocated boundary displacement is considered.

When there exist external disturbances or unknown internal nonlinear uncertainties, there are some methods to achieve stabilization. Sliding mode control is applied in [25] to stabilize a 1-D wave equation with nonlinear van der Pol type boundary condition that covers the anti-stable boundary, and subject to boundary control matched disturbance on the other side. Adaptive method is used in [26] to study the stabilization problem of an unstable wave equation, in which the boundary observation is suffered with a harmonic disturbance. However, the above methods are not applicable when considering the stabilization of a 1-D wave equation with corrupted boundary observation by general disturbance. ADRC plays an important role in solving the stabilization problem with general corrupted boundary disturbance. In [27], ADRC is the first time adopted to set up an ordinary differential equation disturbance estimator to estimate the disturbance, in which the designed disturbance estimator is only dependent on the output of the original system.

In this paper, we consider the stabilization of the following 1-D conservative wave equation

$$\begin{cases} y_{tt}(s,t) = y_{ss}(s,t), \ s \in (0,1), \ t > 0, \\ y(0,t) = 0, \ t \geq 0, \\ y_s(1,t) = U(t), \ t \geq 0, \\ y(s,0) = y_0(s), \quad y_t(s,0) = y_1(s), \ 0 \leq s \leq 1, \\ y_m(t) = \{y(1,t), y_t(1,t) + d(t)\}, \ t \geq 0, \end{cases} \tag{1.1}$$

where and henceforth y' or y_s is the derivative of y with respect to s and $\dot{y}$ or y_t the derivative with respect to t. $U(t)$ is the boundary input, $y_m(t)$ is the boundary output, y_0 and y_1 are initial values and suppose that $d(t)$ is a differentiable external disturbance. The major concern for this kind of output is that the velocity is relatively difficult to measure. If there is no disturbance in the velocity measurement, it is easy to see that system (1.1) can be exponentially stabilized directly by an output feedback controller.

The purpose of this paper is to use ADRC approach to stabilize (1.1) through the output of (1.1). Our method is more general than [28], where the stabilization of (1.1) with an infinite dimensional exosystem periodic disturbance is studied. Accurately speaking, in [28], the boundary velocity measurement with the disturbance has the following form

$$d(t)=\sum_{i=1}^{\infty}a_i\cos\frac{(2i-1)\pi}{2l}t, \tag{1.2}$$

in which $a_i(i=1,2,\cdots)$ denote Fourier coefficients and period $T=4l(l>0)$. In [28], $d(t)$ is an output of an exosystem, and then the stabilization of the system coupled by the original system and the exosystem is considered. However, this method can only be used to solve the periodic disturbance which can be written as an output of external system. It may not be suitable for more general periodic disturbance. In addition, when $l=\left\{\frac{2j+1}{2i}\mid j,i\in Z\right\}$, the method is not applicable.

The organization in this paper is as follows. In Section 2, a disturbance estimator is designed to online estimate disturbance, then we verify the convergence of the error system. In Section 3, an observer-based law is designed and the closed-loop system is verified to be asymptotically stable. In Section 4, some numerical simulations are provided.

2. Estimator and Observer Design

In this section, we design a disturbance estimator to estimate the disturbance $d(t)$, then establish an observer in terms of the designed disturbance estimator. Suppose $d(t)\in H_{loc}^1[0,\infty)$ and

$$\lim_{t\to\infty}\frac{|\dot d(t)|+|d(t)|}{p(t)}=0, \tag{2.1}$$

in which $p(t)\in C^1[0,\infty)$, and for any $t\ge 0$ satisfies

$$\begin{cases} p(t)>0,\ \dot p(t)>0, \\ p(t)\to\infty \text{ as } t\to\infty,\ \sup\limits_{t\in[0,\infty)}\left|\dfrac{\dot p(t)}{p(t)}\right|<\infty. \end{cases} \tag{2.2}$$

Same as the disturbance estimator in [27], we design the disturbance estimator as

$$\begin{cases} \dot q(t)=-p(t)q(t)+\left[y_t(1,t)+d(t)+p(t)y(1,t)\right], \\ \dot r(t)=\hat d(t)-p(t)r(t), \\ \dot{\hat d}(t)=p^2(t)\left[q(t)-y(1,t)-r(t)\right], \\ q(0)=q_0,\ r(0)=r_0,\ \hat d(0)=\hat d_0, \end{cases} \tag{2.3}$$

where $(q_0,r_0,\hat d_0)$ is the initial value of estimator, $\hat d(t)$ is an approximation of $d(t)$. Then, the observer of (1.1) is designed according to disturbance estimator (2.3) as follows

$$\begin{cases} \hat y_{tt}(s,t)=\hat y_{ss}(s,t), \\ \hat y(0,t)=0, \\ \hat y_s(1,t)=U(t)+k\left[y_t(1,t)+d(t)-\hat y_t(1,t)-\hat d(t)\right], \end{cases} \tag{2.4}$$

where $k>0$ is a constant. We consider the observer (2.3) and (2.4) in the space $\mathbb{R}^3\times\mathcal{H}=\mathbb{R}^3\times H_L^1(0,1)\times L^2(0,1)$, $H_L^1(0,1)=\{f\in H^1(0,1)\,|\,f(0)=0\}$. Let

$$\begin{cases}\xi(s,t)=y(s,t)-\hat{y}(s,t),\\ z(t)=p(t)\left[r(t)-q(t)+y(1,t)\right],\\ \tilde{d}(t)=d(t)-\hat{d}(t).\end{cases}\tag{2.5}$$

It is easy to see that the error system is governed by

$$\begin{cases}\dot{z}(t)=-p(t)\left[z(t)+\tilde{d}(t)\right]+\dfrac{\dot{p}(t)}{p(t)}z(t),\\ \dot{\tilde{d}}(t)=p(t)z(t)+\dot{d}(t),\\ \xi_{tt}(s,t)=\xi_{ss}(s,t),\\ \xi(0,t)=0,\\ \xi_s(1,t)=-k\xi_t(1,t)-k\tilde{d}(t).\end{cases}\tag{2.6}$$

It can be known that the ODE-part of (2.6) is independent of its PDE-part and its well-posedness and convergence has been proved in [27]. Therefore, for brevity, the proof of the well-posedness and convergence of the ODE-part is omitted in this paper, we only need to consider the well-posedness and convergence of the PDE-part. We consider the PDE-part of (2.6) in the space $\mathcal{H}=H_L^1(0,1)\times L^2(0,1)$ with the normal inner product.

Define an operator $\mathbf{C}:D(\mathbf{C})(\subset)\to\mathcal{H}$ as

$$\begin{cases}\mathbf{C}(f,g)=(g,f''),\\ D(\mathbf{C})=\left\{(f,g)\in H_L^1(0,1)\times L^2(0,1)\,|\,f'(1)=-kg(1)\right\}.\end{cases}\tag{2.7}$$

As we all know, $\mathbf{C}$ generates an exponentially stable C_0-semigroup $e^{\mathbf{C}t}$ on $\mathcal{H}$. The dual operator is

$$\begin{cases}\mathbf{C}^*(\phi,\psi)=(-\psi,-\phi''),\\ D(\mathbf{C}^*)=\left\{(\phi,\psi)\in H_L^1(0,1)\times L^2(0,1)\,|\,\phi'(1)=k\psi(1)\right\}\end{cases}\tag{2.8}$$

Take the inner product of $(\phi,\psi)\in D(\mathbf{C}^*)$ with the PDE-part of (2.6) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\begin{pmatrix}\xi\\ \xi_t\end{pmatrix},\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle=\left\langle\begin{pmatrix}\xi\\ \xi_t\end{pmatrix},\mathbf{C}^*\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle+\left\langle\begin{pmatrix}0\\ \delta(s-1)\end{pmatrix}h(t),\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle,\tag{2.9}$$

in which $\delta(\cdot)$ being Dirac distribution and

$$h(t)=-k\tilde{d}(t).\tag{2.10}$$

Hence, the PDE-part of (2.6) is equivalent to

$$\begin{cases}\xi_{tt}(s,t)=\xi_{ss}(s,t)+\delta(s-1)h(t),\\ \xi(0,t)=0,\\ \xi_s(1,t)=-k\xi_t(1,t),\\ \xi(s,0)=\xi_0(s),\ \ \xi_t(s,0)=\xi_1(s)\end{cases}\tag{2.11}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi(\cdot,t)\\ \xi_t(\cdot,t)\end{pmatrix}=\mathbf{C}\begin{pmatrix}\xi(\cdot,t)\\ \xi_t(\cdot,t)\end{pmatrix}+\begin{pmatrix}0\\ \delta(s-1)\end{pmatrix}h(t)$$
$$=\mathbf{C}\begin{pmatrix}\xi(\cdot,t)\\ \xi_t(\cdot,t)\end{pmatrix}+\mathbf{D}h(t), \tag{2.12}$$

where $\mathbf{D}=(0,\delta(s-1))^{\mathrm{T}}$.

Theorem 2.1. *For arbitrary initial datum* $(\xi_0,\xi_1)\in\mathcal{H}$, *the PDE-part of* (2.6) *has a unique solution* $(\xi,\xi_t)\in C(0,\infty;\mathcal{H})$, *and for arbitrary* $T>0$, *there is a* $D_T>0$ *depending on T only such that*

$$\left\|(\xi(\cdot,T),\xi_t(\cdot,T))\right\|_H^2\le D_T\left\{\left\|(\xi_0,\xi_1)\right\|_H^2+\int_0^T|h(\tau)|^2\,\mathrm{d}\tau\right\}.$$

Proof. By [29], it only needs to verify that $\mathbf{D}^*$ is admissible for $e^{\mathbf{C}^*t}$, which is equal to say (a) $\mathbf{D}^*\mathbf{C}^{*-1}$ is bounded from $\mathcal{H}$ to $\mathbb{C}$, and (b) for arbitrary $T>0$, there is a $L_T>0$ only depends on *T*, which makes

$$\begin{cases}\xi_{tt}^*(s,t)=\xi_{ss}^*(s,t),\\ \xi^*(0,t)=0,\\ \xi_s^*(1,t)=-k\xi_t^*(1,t),\\ y_{\xi^*}(t)=\xi_t^*(1,t),\end{cases} \tag{2.13}$$

satisfies

$$\int_0^T\left|\xi_t^*(1,t)\right|^2\mathrm{d}t\le L_TE_{\xi^*}(0),$$

where

$$E_{\xi^*}(t)=\frac{1}{2}\int_0^1\left[\left|\xi_s^*(s,t)\right|^2+\left|\xi_t^*(s,t)\right|^2\right]\mathrm{d}s.$$

A simple calculation obtains

$$\begin{cases}\mathbf{C}^{*-1}(\phi,\psi)=\left(\left(k\phi(1)+\int_0^1\psi(l)\,\mathrm{d}l\right)s-\int_0^s(s-l)\psi(l)\,\mathrm{d}l,-\phi(s)\right),\\ \mathbf{D}^*\mathbf{C}^{*-1}(\phi,\psi)=(0,\phi(1)).\end{cases} \tag{2.14}$$

Thus, $\mathbf{D}^*\mathbf{C}^{*-1}$ is bounded on $\mathcal{H}$. Then, by differentiating $E_{\xi^*}(t)$ we get

$$\dot{E}_{\xi^*}(t)=-k\left(\xi_t^*(1,t)\right)^2\le 0,$$

integrating both sides yields $\int_0^T\left[\xi_t(1,t)\right]^2\mathrm{d}t\le\frac{1}{k}E_{\xi^*}(0)$. □

Theorem 2.2. *Suppose that* $p(t)\in C^1[0,\infty)$ *and* $d(t)\in H^1_{loc}[0,\infty)$. *Then, the observers* (2.3) *and* (2.4) *are well-posed, i.e., for arbitrary initial date* $\left(\hat{y}(\cdot,0),\hat{y}_t(\cdot,0),q_0,r_0,\hat{d}_0\right)\in H\times\mathbb{R}^3$, (2.3) *and* (2.4) *have a unique solution* $\left(\hat{y},\hat{y}_t,q,r,\hat{d}\right)\in C\left(0,\infty;H\times\mathbb{R}^3\right)$. *Furthermore, if we also suppose that* $p(t)$ *and* $d(t)$ *satisfy* (2.2) *and* (2.1), *respectively, then the solution of* (2.3) *and* (2.4) *satisfy*

$$\lim_{t\to\infty}\left\|\left(y-\hat{y},y_t-\hat{y}_t,q-y(1,\cdot),r,\hat{d}-d\right)\right\|_{H\times\mathbb{R}^3}=0. \tag{2.15}$$

Proof. Based on Theorem 2.1, the solution of (2.12) depends on the initial date and $\tilde{d} \in L^2_{loc}(0,\infty)$. For arbitrary $\sigma > 0$, for some $t_1 > 0$, we can assume that $|h(t)| = |k\tilde{d}(t)| \le \sigma$ for all $t > t_1$. Thus, we can write the solution of (2.12) as

$$\left(\xi(\cdot,t),\xi_t(\cdot,t)\right)^{\mathrm{T}} = e^{\mathbf{C}t}\left(\xi(\cdot,t),\xi_t(\cdot,t)\right)^{\mathrm{T}} + \int_0^t e^{\mathbf{C}(t-l)}\mathbf{D}h(l)\,\mathrm{d}l. \tag{2.16}$$

It follows from the admissibility of $\mathbf{D}$ and [29] that

$$\left\|\int_{t_1}^t e^{\mathbf{C}(t-l)}\mathbf{D}\tilde{d}(l)\,\mathrm{d}l\right\|_H \le \left\|\int_0^t e^{\mathbf{C}(t-l)}\mathbf{D}\left(0 \underset{t_1}{\Diamond} h\right)(l)\,\mathrm{d}l\right\|_H \le L\|h\|_{L^2(t_1,\infty)} \le L\sigma, \tag{2.17}$$

in which L is a constant independent of $h(t)$, and

$$\left(w \underset{\varsigma}{\Diamond} u\right)(t) = \begin{cases} w(t), & 0 \le t \le \varsigma, \\ u(t), & t > \varsigma. \end{cases}$$

Since $e^{\mathbf{C}t}$ is exponential stable, there are two constants $M > 0$ and $\mu > 0$, hold in

$$\left\|e^{\mathbf{C}t}\right\|_H \le Me^{-\mu t}, \quad \forall\, t \ge 0.$$

Thus, we have

$$\begin{aligned}\left\|e^{\mathbf{C}(t-t_1)}\int_0^{t_1} e^{\mathbf{C}(t_1-l)}\mathbf{D}h(l)\,\mathrm{d}l\right\|_H &\le \left\|e^{\mathbf{C}(t-t_1)}\right\|\left\|\int_0^{t_1} e^{\mathbf{C}(t_1-l)}\mathbf{D}h(l)\,\mathrm{d}l\right\|_H \\ &\le Me^{-\mu(t-t_1)}\left\|\int_0^{t_1} e^{\mathbf{C}(t_1-l)}\mathbf{D}h(l)\,\mathrm{d}l\right\|_H.\end{aligned} \tag{2.18}$$

Rewriting (2.16) as

$$\begin{aligned}&\left(\xi(\cdot,t),\xi_t(\cdot,t)\right)^{\mathrm{T}} \\ &= e^{\mathbf{C}t}\left(\xi(\cdot,0),\xi_t(\cdot,0)\right)^{\mathrm{T}} + e^{\mathbf{C}(t-t_1)}\int_0^{t_1} e^{\mathbf{C}(t_1-l)}\mathbf{D}h(l)\,\mathrm{d}l + \int_{t_1}^t \mathrm{e}^{\mathbf{C}(t-l)}\mathbf{C}h(l)\,\mathrm{d}l,\end{aligned} \tag{2.19}$$

according to (2.17) and (2.18), we obtain

$$\begin{aligned}&\left\|\left(\xi(\cdot,t),\xi_t(\cdot,t)\right)\right\|_H \\ &\le Me^{-\mu t}\left\|\left(\xi(\cdot,0),\xi_t(\cdot,0)\right)\right\|_H + Me^{-\mu(t-t_1)}\left\|\int_0^{t_1} e^{\mathbf{C}(t_1-l)}\mathbf{D}h(l)\,\mathrm{d}l\right\|_H + L\sigma.\end{aligned} \tag{2.20}$$

Take $t \to \infty$ on both sides of (2.20) to get

$$\lim_{t\to\infty}\left\|\left(\xi(\cdot,t),\xi_t(\cdot,t)\right)\right\|_H \le L\sigma. \tag{2.21}$$

Because σ is arbitrarily selected, we get

$$\lim_{t\to\infty}\left\|\left(\xi,\xi_t\right)\right\|_H = 0. \tag{2.22}$$

Combining (2.5), we have

$$\lim_{t\to\infty}\left\|\left(y-\hat{y}, y_t-\hat{y}_t\right)\right\|_H = 0. \tag{2.23}$$

(2.15) then can be obtained by (2.23) and [27]. □

Remark 2.1. *If we only consider using displacement to stabilize* (1.1), *i.e., use* $y(1,t)$ *only. Inspired by* [24], *let*

$$v(s,t) = y(1-s,t), \tag{2.24}$$

thus, $v(s,t)$ *is determined by*

$$\begin{cases} v_{tt}(s,t) = v_{ss}(s,t), \\ v_s(0,t) = -U(t), \\ v(1,t) = 0, \\ y_m(t) = v(0,t). \end{cases} \tag{2.25}$$

The observer of (2.25) *is constructed as*

$$\begin{cases} \hat{v}_{tt}(s,t) = \hat{v}_{ss}(s,t), \\ \hat{v}_s(0,t) = -U(t), \\ \hat{v}(1,t) = \dfrac{1}{2}\left[\hat{v}(0,t-1) - v(0,t-1)\right]. \end{cases} \tag{2.26}$$

Let $\tilde{v}(s,t) = \hat{v}(s,t) - v(s,t)$ *be the error. Therefore, the error* $\tilde{v}(s,t)$ *is determined by*

$$\begin{cases} \tilde{v}_{tt}(s,t) = \tilde{v}_{ss}(s,t), \\ \tilde{v}_s(0,t) = 0, \\ \tilde{v}(1,t) = \dfrac{1}{2}\tilde{v}(0,t-1). \end{cases} \tag{2.27}$$

As we all know, system (2.27) *is well-posed and can be finite-time stable:* $(\tilde{v}(s,t), \tilde{v}_t(s,t)) \equiv 0$ *as* $t \geq 3$ ([24]), *i.e.,* $(\hat{v}(s,t) - v(s,t), \hat{v}_t(s,t) - v_t(s,t)) \equiv 0$ *as* $t \geq 3$. *Therefore, if the controller is presented like* $U(t) = -k\hat{v}_t(0,t)$, *then when* $t \geq 3$, *we have* $\hat{v}_t(0,t) \equiv v_t(0,t)$, *which will use the velocity measurement* $v_t(0,t)$ *of system* (2.25). *With* (2.24), *we will use the velocity measurement* $y_t(1,t)$ *of system* (1.1). *Thus, the stabilization of* (1.1) *can't use only the displacement measurement* $y(1,t)$.

3. Well-Posedness and Stability of Closed-Loop System

The closed-loop system consists of system (1.1), observer (2.3) and (2.4) in the state space $\mathcal{H}^2 \times \mathbb{R}^3$. Based on the observer we designed, we can apply the same controller as in [30]

$$U(t) = -\tanh(m)\hat{y}_t(1,t), \tag{3.1}$$

where m is a normal constant. Therefore, the closed-loop system is

$$\begin{cases} y_{tt}(s,t) = y_{ss}(s,t), \\ y(0,t) = 0, \\ y_s(1,t) = -\tanh(m)\hat{y}_t(1,t), \\ \dot{q}(t) = -p(t)q(t) + \left[y_t(1,t) + d(t) + p(t)y(1,t)\right], \\ \dot{r}(t) = \hat{d}(t) - p(t)r(t), \\ \dot{\hat{d}}(t) = p^2(t)\left[q(t) - y(1,t) - r(t)\right], \\ \hat{y}_{tt}(s,t) = \hat{y}_{ss}(s,t), \\ \hat{y}(0,t) = 0, \\ \hat{y}_s(1,t) = -\tanh(m)\hat{y}_t(1,t) + k\left[y_t(1,t) + d(t) - \hat{y}_t(1,t) - \hat{d}(t)\right]. \end{cases} \tag{3.2}$$

Theorem 3.1. *Assume that* $p(t) \in C^1[0,\infty)$ *and* $d(t) \in \mathcal{H}^1_{loc}[0,\infty)$ *satisfy*

(2.2) *and* (2.1), *respectively. For arbitrary initial date* $\left(y(\cdot,0),y_t(\cdot,0),\hat{y}(\cdot,0),\hat{y}_t(\cdot,0),q(0),r(0),\hat{d}(0)\right)\in\mathcal{H}^2\times\mathbb{R}^3$, *system* (3.2) *has a unique solution* $\left(y,y_t,\hat{y},\hat{y}_t,q,r,\hat{d}\right)\in C\left(0,\infty;\mathcal{H}^2\times\mathbb{R}^3\right)$ *and* (3.2) *is asymptotically stable, i.e.*,

$$\lim_{t\to\infty}\left\|\left(y,y_t,\hat{y},\hat{y}_t,q,r,\hat{d}-d\right)\right\|_{\mathcal{H}^2\times\mathbb{R}^3}=0. \tag{3.3}$$

Proof. By the invertible transformation

$$\begin{pmatrix} y\\ y_t\\ \xi\\ \xi_t\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix} y\\ y_t\\ \hat{y}\\ \hat{y}_t\end{pmatrix}, \tag{3.4}$$

then the PDE-part of (3.2) is equivalent to

$$\begin{cases} y_{tt}(s,t)=y_{ss}(s,t),\\ y(0,t)=0,\\ y_s(1,t)=-\tanh(m)y_t(1,t)+\tanh(m)\xi_t(1,t),\\ \xi_{tt}(s,t)=\xi_{ss}(s,t),\\ \xi(0,t)=0,\\ \xi_s(1,t)=-k\xi_t(1,t)-k\tilde{d}(t),\end{cases} \tag{3.5}$$

then system (3.5) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y(\cdot,t),y_t(\cdot,t),\xi(\cdot,t),\xi_t(\cdot,t)\right)=\mathcal{C}\left(y(\cdot,t),y_t(\cdot,t),\xi(\cdot,t),\xi_t(\cdot,t)\right)+\mathcal{B}h(t), \tag{3.6}$$

where

$$\mathcal{B}=\left(0,0,0,\delta(s-1)\right), \tag{3.7}$$

and the operator $\mathcal{C}:D(\mathcal{C})\subset\mathcal{H}^2\to\mathcal{H}^2$ is defined by

$$\begin{cases}\mathcal{C}(f,g,\phi,\psi)=(g,f'',\psi,\phi''),\ \forall(f,g,\phi,\psi)\in D(\mathcal{C}),\\ D(\mathcal{C})=\left\{\mathcal{C}(f,g,\phi,\psi)\in\mathcal{H}^2\mid f'(1)=-\tanh(m)g(1)+\tanh(m)\psi(1),\right.\\ \qquad\left.\phi'(1)=-k\psi(1)\right\}.\end{cases} \tag{3.8}$$

As we all know, $\mathcal{C}$ generates an exponential stable C_0-semigroup $e^{\mathcal{C}t}$ on $\mathcal{H}^2$. Along the same line for (2.7) to (2.23), we can obtain that $\mathcal{B}$ in (3.9) is admissible for $e^{\mathcal{C}t}$ and

$$\lim_{t\to\infty}\left\|\left(y(\cdot,t),y_t(\cdot,t),\xi(\cdot,t),\xi_t(\cdot,t)\right)\right\|_{\mathcal{H}^2}=0. \tag{3.9}$$

Combining (3.4), we have

$$\lim_{t\to\infty}\left\|\left(y,y_t,\hat{y},\hat{y}_t\right)\right\|_{\mathcal{H}^2}=0. \tag{3.10}$$

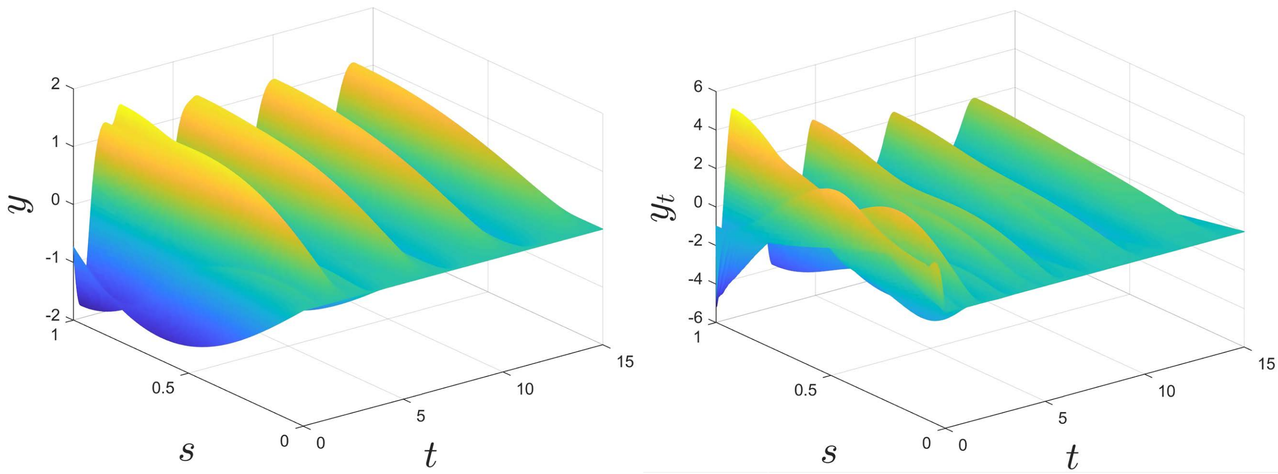
Then, the stability of the closed-loop system is obtained by Theorem 2.1, Theorem 2.2 and (3.10). □

4. Simulation Results

In this section, some simulations are carried out for open-loop (1.1) and closed-loop (3.2). In (1.1) and (3.2), the initial date is selected as follows

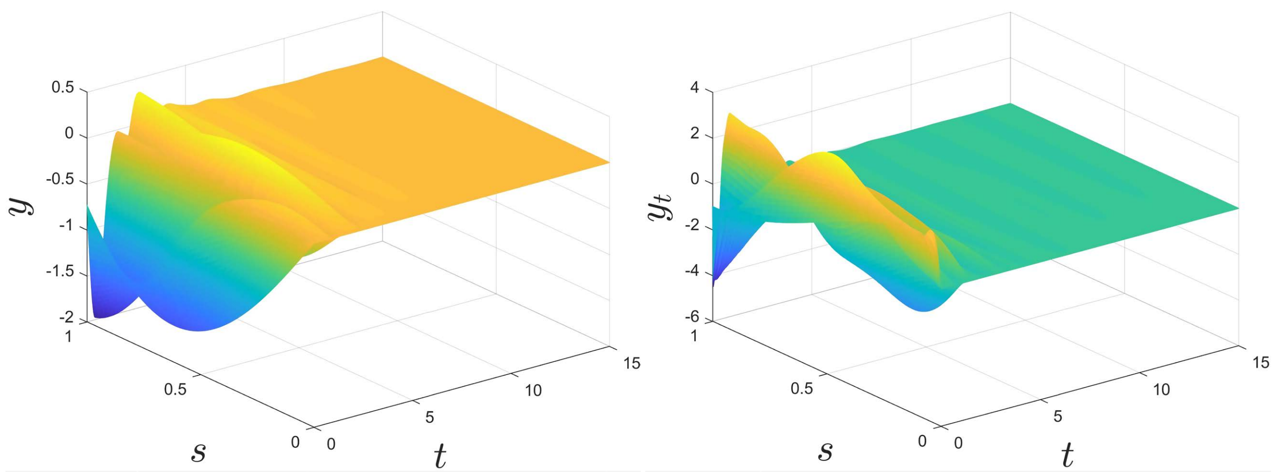
$$\begin{cases} y_0(s,0) = 2s - \sin(2s), \ y_1(s,0) = s + 2\cos(3s), \\ \hat{y}_0(s,0) = -2s + \sin(2s), \ \hat{y}_1(s,0) = 2s + 3\cos(\pi s), \\ q(0) = -1, \ r(0) = 1, \ \hat{d}(0) = -2, \\ p(t) = 1 + 30t, \ k = 2, \ m = 5. \end{cases} \tag{4.1}$$

For system (1.1) and system (3.2), we use the finite element method to calculate their solutions. The system (1.1) is conservative, which is presented in **Figure 1**. If we give a (3.1) controller at the $s = 1$ endpoint, the system will be asymptotically stable, it can be seen in **Figure 2**. We can see from **Figure 3** that the designed observer is convergent. In **Figure 4**, $\hat{d}(t)$ can approximate converge to $d(t)$ well, and when time tends to infinity, both $q(t)$ and $r(t)$ converge to 0, which indicates that the established disturbance estimator has satisfactory convergence.



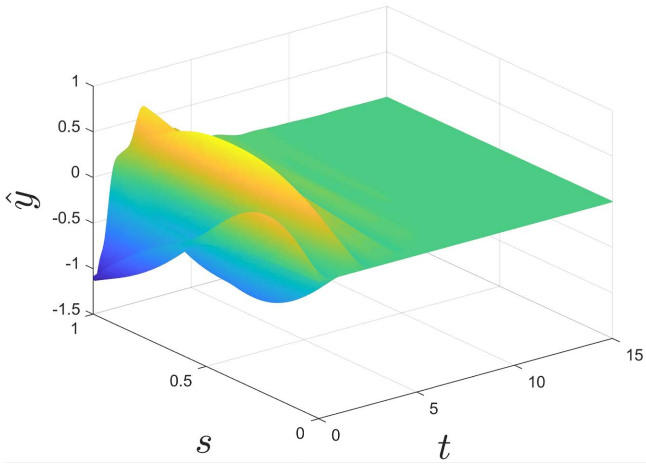
(a) (b)

Figure 1. Displacement of open-loop system (1). (a) Displacement of $y(s,t)$; (b) Displacement of $y_t(s,t)$.

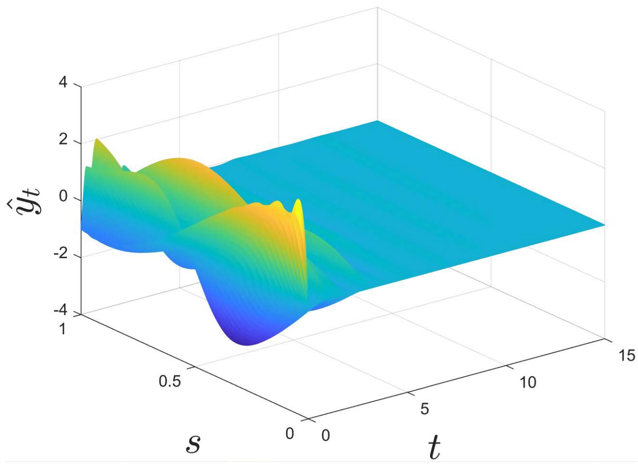


(a) (b)

Figure 2. The *y*-part displacement of closed-loop system (31). (a) Displacement of $y(s,t)$; (b) Displacement of $y_t(s,t)$.



(a)



(b)

Figure 3. The $\hat{y}$ -part displacement of closed-loop system (31). (a) Displacement of $\hat{y}(s,t)$; (b) Displacement of $\hat{y}_t(s,t)$.

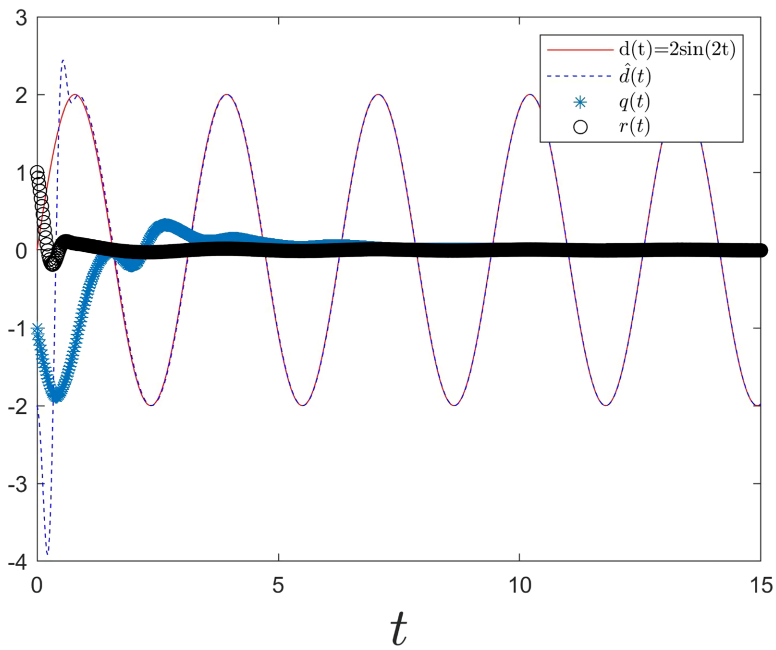


Figure 4. Trajectory of the ODE-part of (31).

5. Concluding Remarks

In this paper, the problem of stabilization for a 1-D conservative wave equation is studied. The difficulty in this paper is that the boundary velocity observation is affected by a general disturbance. The merit of our method lies in that the boundary velocity observation is subjected to a general disturbance, including constant disturbance and periodic disturbance as its special cases. If there is no collocated boundary displacement measurement, it seems that only using the corrupted collocated boundary velocity measurement cannot estimate the disturbance, which is a disadvantage of this paper. In future studies, we will extend this method to the Schrödinger equation and the Euler-Bernoulli equation.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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