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A Priori Error Analysis for NCVEM Discretization of Elliptic Optimal Control Problem

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Abstract

In this paper, we propose the nonconforming virtual element method (NCVEM) discretization for the pointwise control constraint optimal control problem governed by elliptic equations. Based on the NCVEM approximation of state equation and the variational discretization of control variables, we construct a virtual element discrete scheme. For the state, adjoint state and control variable, we obtain the corresponding prior estimate in H^1 and L^2 norms. Finally, some numerical experiments are carried out to support the theoretical results.

Keywords

Nonconforming Virtual Element Method, Optimal Control Problem, a Priori Error Estimate

1. Introduction

The main purpose of this paper is to discuss the prior error analysis of the NCVEM discretization for the elliptic optimal control problem. Consider the following optimal control problems with state constrained:

$$\min_{z\in Z_{ad}} J(q,z) := \frac{1}{2}\int_\Omega (q-q_d)^2\,\mathrm{d}x + \frac{\gamma}{2}\int_\Omega z^2\,\mathrm{d}s \tag{1.1}$$

subject to

$$\begin{cases} -\Delta q = f + z & \text{in } \Omega, \\ q = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $J(q,z)$ is the objective functional, q is the state variable, q_d is the desired state and γ is a positive constant parameter. The aim of the control is to make the state variable q as close as to the desired state q_d. Ω is a bounded polygon on R^2. The admissible set of the control is given by

$$Z_{ad}=\left\{z\in L^2(\Omega): z_a\le z(x)\le z_b \text{ a.e. in } \Omega \text{ with } z_a, z_b\in\mathbb{R} \text{ and } z_a\le z_b\right\}.$$

The above state constrained optimal control models perform an increasingly important role in many science and engineering fields. For this reason,the research on optimal control problems becomes meaningful. Different types of optimal control problems are solved by finite elements method (FEM) [1] [2], discontinuous Galerkin method [3], spectral method [4] so on. In [5] error estimates of approximate local optimal control for semi-linear elliptic equation with finite many state constraints were given. In [6] A control vector was used instead of a control function to establish a high order error estimate for similar Settings. In [7], these results were generalized to a less regular setting for the states and the convergence of FEM approximations for semilinear distribution and boundary control problems was obtained. In [8], a discretization concept was investigated, which used the relationship between adjoint state and control to discretized control variables. They discretized the equation of state using linear FEM and proved the convergence of the equation of state on L^2 norm. On the other hand, for the optimal control problems of elliptic equations, Stokes equations and parabolic equations, the corresponding posteriori error estimates of the conforming FEM were given in [9] [10] [11].

The basic driving force of the VEM comes from the processing of arbitrarily shaped polygons [12]. The traditional meaning of finite element or finite difference requires discretization of physical entities with significant geometric features to solve, which to some extent loses the "macro" description of entity geometric information. However, in practical engineering requirements, more and more calculations involve dealing with specific geometric structures, such as the deformation of non convex polygons and the contact of complex structures. Compared to the FEM, the VEM also requires discretization of the geometric space, approximating the actual problem by forming and solving a system of linear equations. The difference lies in: 1) The approximation function used in the finite element method is an explicit polynomial function; In the VEM, in addition to polynomial functions, approximate functions also have functions that are continuous polynomials at the boundary of the element and satisfy certain conditions inside the element (such as polynomials after Laplace operation). These functions are not explicitly expressed in the element domain; 2) In the FEM, the degrees of freedom are all values of approximate functions. The VEM avoids calculating the values of approximate functions inside the element when forming a stiffness matrix by defining reasonable degrees of freedom; 3) The stiffness matrix of the finite element method only contains one term, while the stiffness matrix of the virtual element method includes the coordination matrix and the stability matrix to ensure the convergence of the calculation.

Due to these advantages of VEM, the VEM for solving various partial differential equations has been proposed and developed, such as linear elasticity [13], Stokes or Navier-Stokes equations [14] [15], Cahn-Hilliard equations [16], and so on. Back to the NCVEM proposed in this paper. The NCVEM was first introduced for elliptic problems in [17]. In the past years, it has been successfully used to solve different models [18]-[23]. But so far, NCVEM has not been used to solve the elliptic optimal control problem. In this paper, NCVEM is introduced to approach the elliptic optimal control problem with pointwise control constraints. Based on NCVEM approximation (through VEM projection operator), a VEM discrete scheme is constructed. Then we obtain the corresponding priori estimate for three variables in H^1 and L^2 norms.

Throughout this paper, for an open bounded domain ω in Ω, standard notation $|\cdot|_{s,\omega}$ and $\|\cdot\|_{s,\omega}$ denote seminorm and norm, respectively, in the Sobolev space $H^s(\omega)$. when $m=0$, $L^P(\omega)=W^{0,p}(\omega)$ $(\cdot,\cdot)_{0,\omega}$ and $\|\cdot\|_{0,\omega}$ denote the inner product and the norm of $L^p(\omega)$. When ω is the whole domain Ω, the subscript can be omitted. Let $\mathcal{P}_k(E)$ denote the space of polynomials of degree at most $K\in\mathbb{N}_0$ on E. Usually, $\mathcal{P}_{-1}(E)=0$.

The aim of this paper is to construct a NCVEM for constrained optimal control problems. The rest of this paper is as follows. In the next section, we introduce some preliminaries knowledge about VEM. Then, the continuous first-order optimality system of elliptic optimal control problem are introduced. In Section 3, we derive the VEM discrete scheme, discrete first-order optimality condition. Then the priori error estimates are derived both for three variables in H^1 and L^2 norm. In Section 4, we show a numerical example to verify our theoretical analysis. Finally, we make some summaries in Section 5.

2. Preliminaries Knowledge

In this section, we mainly introduce local projection operators and the definitions of virtual element space.

Fristly, suppose $\mathcal{T}_h$ is a family of decompositions of the domain Ω divided into star-shaped polygons E. For any $E\in\mathcal{T}_h$, $h=\max_{E\in\mathcal{T}_h} h_E$ and $h_E=diam(E)$ The set of edges s of $\mathcal{T}_h$ is denoted by $\mathcal{S}_h$, which is subdivided into the set of boundary edges $\mathcal{S}_h^{bdry}:=\{s\in\mathcal{S}_h : s\subset\partial\Omega\}$ and the set of internal edges $\mathcal{S}_h^{int}=\mathcal{S}_h\mathcal{S}_h^{bdry}$. Finally, h_s denote the length of the edge s.

Before introducing the virtual element space, we first make the following assumptions about the grid.

Assumption 2.1 (See [24]) (mesh regularity) We assume that there exists a real number $\rho>0$ such that, for every $E\in\mathcal{T}_h$ satisfies the following two assumptions.

1) Every element E is star-shaped with respect to a circle with a radius $\geq\rho h_E$ ρh_E;

2) Every edge s of E has length $h_s\geq\rho h_E$;

Next we give the definition of the projection operator.

Definition 2.1 (See [24]) Define the L^2 projection operator $\Pi_1^0 : L^2(E)\to\mathcal{P}_1(E)$ as follows:

$$\left(\Pi_1^0 w_h - w_h, q\right)_{0,E} = 0 \quad \forall w_h \in L^2(E), \forall q \in \mathcal{P}_1(E). \tag{2.1}$$

Definition 2.2 (See [24]) Define the H^1 projection operator $\Pi_1^\nabla : H^1(E) \to \mathcal{P}_1(E)$ as *follows:*

$$\left(\nabla\left(\Pi_1^\nabla w_h - w_h\right), \nabla p\right)_{0,E} = 0 \quad \forall w_h \in H^1(E), \forall p \in \mathcal{P}_1(E)$$

plus

$$\int_{\partial E}\left(w_h - \Pi_1^\nabla w_h\right)\mathrm{d}s = 0.$$

A finite dimensional function space $W_h \subset H^1(\mathcal{T}_h)$, where

$$H^1(\mathcal{T}_h) := \left\{w \in L^2(\Omega) : w_{|E} \in H^1(E)\, \forall E \in \mathcal{T}_h\right\} \tag{2.2}$$

(not necessarily a subspace of $H_0^1(\Omega)$)

We denote by $\mathcal{M}_l^*$ the set of scaled polynomials

$$\mathcal{M}_l^*(E) := \left\{\left(\frac{x - x_E}{h_E}\right)^s \text{ with } s \in \mathbb{N}^d, |s| = l\right\},$$

The global virtual element space in each case is constructed from [18] as a subspace of an infinite dimensional space W, defined differently for the VEM and NCVEM. For the VEM we simply take $W = H_0^1(\Omega)$. For the NCVEM, we introduce the subspace $H_k^{1,nc}(\mathcal{T}_h)$ of the nonconforming broken Sobolev space $H^1(\mathcal{T}_h)$ defined in (3.1), by imposing certain weak inter-element continuity requirements such that

$$W := H_k^1(\mathcal{T}_h) = \left\{w \in H^1(\mathcal{T}_h) : \int_s [\![w]\!] \cdot \boldsymbol{n}_s q \mathrm{d}s = 0, \forall q \in \mathcal{P}_{k-1}(s), \forall s \in \mathcal{S}_h\right\}.$$

The jump operator $[\![\cdot]\!]$ across a mesh interface $s \in \mathcal{S}_h$ is defined as follows for $v \in H^1(\mathcal{T}_h)$.

If $s \in \mathcal{S}_h^{int}$, $[\![w]\!] = w^+ \boldsymbol{n}_s^+ + w^- \boldsymbol{n}_s^-$ there exist E^+ and E^- such that $S \subset \partial E^+ \cap \partial E^-$. Denote by $w^\pm$ the trace of $\boldsymbol{w}|_{E\pm}$ on *s* from within $E_\pm$ and by $\boldsymbol{n}_s^\pm$ the unit outward normal on s from $E^\pm$.

If $s \in \mathcal{S}_h^{bdry}$, $[\![w]\!] = w\boldsymbol{n}_s$, *w* representing the trace of *v* from within the element *E*, having *s* as an interface and n_s is the unit outward normal on *s* from *E*.

The modified virtual element space from is defined as

$$W_h := \left\{W_h \in W : W_h|_E, \forall E \in \mathcal{T}_h\right\}$$

where

$$W_h^E := \left\{w_h \in \mathcal{V}_h^E : \left(w_h - \Pi_1^\nabla w_h, q\right)_E = 0, \forall q \in \mathcal{M}_1^*(E) \cup \mathcal{M}_0^*(E)\right\}$$

$$\mathcal{V}_h^E := \left\{w_h \in H^1(E) : \Delta w_h \in \mathcal{P}_1(E) \text{ and } \frac{\partial w_h}{\partial \boldsymbol{n}} \in \mathcal{P}_0(s), \forall s \subset \partial E\right\}.$$

We can see that the space W_h is not a subspace of $H_0^1(\Omega)$, so we need to define a discrete norm

$$|w|_{1,h} = \left(\sum_{E \in \mathcal{T}_h} |w|_{1,E}^2\right)^{1/2}, \forall w \in W_h$$

Similar to reference [18], we define the degrees of freedom as

$$\frac{1}{h_e}\int_e wp\,\mathrm{d}s,\ \ \forall p\in\mathcal{P}_0(e),\ \forall e\subseteq\partial E \tag{2.3}$$

$$\frac{1}{|E|}\int_E wp\,\mathrm{d}x,\ \ \forall p\in\mathcal{P}_{-1}(E). \tag{2.4}$$

Using the above degrees of freedom (2.3) and (2.4), projection Π_1^0 and Π_1^∇ are exactly computable.

Next we introduce the continuous first-order optimality system

Theorem 2.1 Let (y,u) be the the solution of (1.1) and (1.2). Then the following first-order optimality system holds

$$\begin{cases}-\Delta q = f + z, & \text{in } \Omega,\\ q = 0, & \text{on } \Gamma,\end{cases}$$

and

$$\begin{cases}-\Delta r = q - q_d, & \text{in } \Omega,\\ r = 0, & \text{on } \Gamma,\end{cases}$$

where p is called the adjoint state variable and

$$\int_\Omega(\gamma z + r)(w - z)\mathrm{d}x \ge 0,\ \forall w\in Z_{ad}. \tag{2.5}$$

Let

$$R_{z_{ad}}(z) = \max\{z_a, \min\{z, z_b\}\} \tag{2.6}$$

denotes the pointwise projection onto the admissible set Z_{ad}. Similar to the discussion in Becker *et al.* [25], (2.6) is equivalent to

$$Z = R_{Z_{ad}}\left(-\frac{1}{\gamma}p\right).$$

Let $B(q,v) = (\nabla q, \nabla v)$. Then the bilinear form of the continuous first-order system reads:

$$\begin{cases}B(q,v) = (f + z, v), & \forall v\in H_0^1(\Omega),\\ B(r,v) = (q - q_d, v), & \forall v\in H_0^1(\Omega),\\ (\gamma z + r, w - z)\ge 0, & \forall v\in Z_{ad}.\end{cases} \tag{2.7}$$

From classical Scott-Dupont theory [26] we first introduce a local estimate.

Lemma 2.1 For any $w\in H^s(\Omega)$, there is a piecewise polynomial $w_\pi\in\mathbb{P}_1(\mathcal{T}_h)$ satisfying

$$|w - w_\pi|_{m,K} \le C h_K^{s-m}|w|_{s,E},\quad \forall E\in\mathcal{T}_h, \tag{2.8}$$

where $0\le m\le s\le 2$

For any function $w\in H_0^1(\Omega)$, using degrees of freedom (2.3) and (2.4) we can define the interpolation $I_h w\in W_h$. We denote by φ_i the ith degrees of freedom given by (2.3) and (2.4), where $i = 1,\cdots,N^E$ and N^E is the number of degrees of freedom (2.3) and (2.4). Then we have

$$\varphi_i(I_h w) = \varphi_i(w),\quad i = 1,\cdots,N^E$$

From [23] the following lemma provides the interpolation error estimation of the nonconforming virtual element

Lemma 2.2 For any $w \in H_0^1(\Omega) \cap H^s(\Omega)$ and $1 \le s \le 2$, we have

$$\|w - I_h w\|_E + h_E |w - I_h w|_{1,E} \le C h_E^s |w|_{s,E}, \forall E \in \mathcal{T}_h. \tag{2.9}$$

3. Virtual Element Approximation

3.1. Virtual Element Discrete Scheme for State Equation

The bilinear form of the state equation is as follows

$$B(q,w) = (f + z, w) =: \mathcal{L}(w), \quad \forall w \in H_0^1(\Omega).$$

The corresponding virtual element discrete scheme of (1.2) can be defined by

$$B_h(q_h(z), w_h) = \mathcal{L}_h(w_h), \quad \forall w_h \in W_h,$$

where

$$\begin{cases} B_h(q_h(z), w_h) := \sum_{E \in \mathcal{T}_h} B_h^E(q_h(z), w_h) \\ \quad = \sum_{E \in \mathcal{T}_h} \left(B^E\left(\Pi_1^\nabla q_h(z), \Pi_1^\nabla w_h\right) + S^E\left(q_h(z) - \Pi_1^\nabla q_h(z), w_h - \Pi_1^\nabla w_h\right)\right), \\ \mathcal{L}_h(w_h) := \sum_{E \in \mathcal{T}_h} \left(f + z, \Pi_1^0 w_h\right)_{0,E}. \end{cases}$$

There are many choices for S^E, and following [12] we take the simple choice

$$\begin{aligned} &S^E\left(q_h(z) - \Pi_1^\nabla q_h(z), v_h - \Pi_1^\nabla w_h\right) \\ &= \sum_{y=1}^{N^E} \mathrm{dof}_y\left(q_h(z) - \Pi_1^\nabla q_h(z)\right) \mathrm{dof}_y\left(w_h - \Pi_1^\nabla w_h\right) \end{aligned}$$

such that it satisfies the following property:

$$\alpha_* B^E(w_h, w_h) \le S^E(w_h, w_h) \le \alpha^* B^E(w_h, w_h), \ \forall w_h \in W_h^E \ \text{ with } \Pi_1^0 w_h = 0.$$

Here N_E is the number of degrees of freedom on the element E and denotes the $\mathrm{dof}_y\left(r_h(z) - \Pi_1^0 q_h(z)\right)$ value of the *yth* local degree of freedom defining $q_h(z) - \Pi_1^\nabla q_h(z)$ in W_h^E.

We define consistency and stability as follows:

Definition 3.1 see ([23])

- Consistency: For all $p \in \mathbb{P}_1(E)$ and for all $w_h \in W_h^E$

$$B_h^E(r, w_h) = \int_E \nabla w \cdot \Pi_0^0 \nabla w_h \mathrm{d}x. \tag{3.1}$$

- Stability: There exist positive constants α^* and α_* independent of h and the mesh element E such that

$$\alpha_* B^E(w_h, w_h) \le B_h^E(w_h, w_h) \le \alpha^* B^E(w_h, w_h), \quad \forall w_h \in W_h^E. \tag{3.2}$$

Lemma 3.1 See ([23]) The Discrete bilinear form B_h^E satisfies the polynomial consistency property and the stability property. Then, we obtain B_h is coercive.

The virtual element approximation of control problem (1.1)-(1.2) is to find

$(q_h, z_h) \in W_h \times Z_{ad}$ such that

$$\min_{z_h \in Z_{ad}} J\left(\Pi_1^0 q_h, z_h\right) := \frac{1}{2} \sum_{E \in \mathcal{T}_h} \int_E \left(\Pi_1^0 q_h - q_d\right)^2 \mathrm{d}x + \frac{\gamma}{2} \int_\Omega z_h^2 \mathrm{d}x \tag{3.3}$$

subject to

$$B_h(q_h, w_h) = \sum_{E \in \mathcal{T}_h} \left(f + z_h, \Pi_1^0 w_h\right)_{0,E}, \quad \forall w_h \in W_h. \tag{3.4}$$

In [27], it also showed that (3.3) and (3.4) has a unique solution (q_h, z_h) and that (q_h, z_h) is the solution of (3.3) and (3.4) if and only if there is a co-state $r_h \in W_h$ such that (q_h, r_h, z_h) satisfies the following discrete first-order optimality conditions:

$$\begin{cases} B_h(q_h, v_h) = \sum_{E \in \mathcal{T}_h} \left(f + z_h, \Pi_1^0 v_h\right)_{0,E}, & \forall v_h \in W_h, \\ B_h(r_h, v_h) = \sum_{E \in \mathcal{T}_h} \left(q_h - q_d, \Pi_1^0 v_h\right)_{0,E}, & \forall v_h \in W_h, \\ \sum_{E \in \mathcal{T}_h} \left(\gamma z_h + \Pi_1^0 r_h, w_h - z_h\right)_{0,E} \geq 0, & \forall w_h \in Z_{ad}. \end{cases} \tag{3.5}$$

3.2. A Priori Error Estimate

Lemma 3.2 see ([28])) There exists a positive constant C such that, for all $E \in \mathcal{T}_h$ and all smooth enough functions w defined on E, it holds:

$$\begin{aligned} \left\|v - \Pi_1^0 v\right\|_{p,E} &\leq C h_E^{t-p} |v|_{p,E}, \ p, t \in \mathbb{N}, \ p \leq t \leq 2, \\ \left\|v - \Pi_1^\nabla v\right\|_{p,E} &\leq C h_E^{t-p} |w|_{t,E}, \ p, r \in \mathbb{N}, \ p \leq t \leq 2, \ t \geq 1. \end{aligned} \tag{3.6}$$

To derive a priori error estimate we need to introduce the following auxiliary problems:

$$\begin{cases} B_h(q_h(z), v_h) = \sum_{E \in \mathcal{T}_h} \left(f + z, \Pi_1^0 v_h\right)_{0,E}, & \forall v_h \in W_h, \\ B_h(r_h(q), v_h) = \sum_{E \in \mathcal{T}_h} \left(q - q_d, \Pi_1^0 v_h\right)_{0,E}, & \forall v_h \in W_h, \\ B_h(r_h(z), v_h) = \sum_{E \in \mathcal{T}_h} \left(q_h(z) - q_d, \Pi_1^0 v_h\right)_{0,E}, & \forall v_h \in W_h. \end{cases} \tag{3.7}$$

We make the following data assumption:

Assumption 3.1 (Data assumption) We assume the solution (z, q, r) of the optimal control problem and (f, q_d) satisfy:

$$f, z, q_d \in H^1(\mathcal{T}_h), \quad q, r \in H^2(\mathcal{T}_h).$$

Then we have the following estimates.

Note that $W_h \not\subset H_0^1(\Omega)$, from integration by parts, we get

$$\sum_{E \in \mathcal{T}_h} B^E(q, v_h) = \sum_{E \in \mathcal{T}_h} \int_k -\Delta q v_h \mathrm{d}x + \sum_{E \in \mathcal{T}_h} \int_{\partial E} \frac{\partial q}{\partial n_k} v_h \mathrm{d}s, \ \forall q \in H^2(\Omega), \forall v_h \in W_h$$

The consistency error is

$$\mathcal{N}_h\left(y,w_h\right)=\sum_{E\in\mathcal{T}_h}\int_{\partial E}\frac{\partial y}{\partial \boldsymbol{n}_k}w_h\mathrm{d}s=\sum_{e\in\mathcal{S}_h}\int_e\frac{\partial y}{\partial \boldsymbol{n}_e}\left[w_h\right]\mathrm{d}s,\ \forall w_h\in V_h \tag{3.8}$$

Similar to the theoretical analysis in Xiao *et al.* [23], we have

$$\left|\mathcal{N}_h\left(q,v_h\right)\right|\le Ch|q|_2\left|v_h\right|_{1,h},\forall v_h\in W_h. \tag{3.9}$$

Theorem 3.1 Suppose that (q,r) is the solution of (2.7), and $\left(q_h(z),r_h(q)\right)$ is the solution of auxiliary problem (3.7), under Assumpiton 2.1 and assumption 3.1 we have

$$\left\|q-q_h(z)\right\|_{1,h}+\left\|r-r_h(z)\right\|_{1,h}\le Ch$$

Proof:

$$q-q_h(z)=q-I_hq+I_hq-q_h(z)=\xi_h+\theta_h, \tag{3.10}$$

where $\xi_h=q-I_hq$ and $\theta_h=I_hq-q_h(z)$

From the coercivity of $B_h(\cdot,\cdot)$

$$\begin{aligned}\alpha_*\left|\theta_h\right|_{1,h}^2&\le B_h\left(\theta_h,\theta_h\right)=B_h\left(I_hq,\theta_h\right)-B_h\left(q_h(z),\theta_h\right)\\&=\sum_{K\in\mathcal{T}_h}B_h^K\left(I_hq,\theta_h\right)-\sum_{E\in\mathcal{T}_h}\left(f+z,\Pi_1^0\theta_h\right)_{0,E}\\&=\sum_{K\in\mathcal{T}_h}\left(B_h^K\left(I_hq-q_\pi,\theta_h\right)+B_h^K\left(q_\pi,\theta_h\right)\right)-\sum_{E\in\mathcal{T}_h}\left(f+z,\Pi_1^0\theta_h\right)_{0,E}\\&=\sum_{K\in\mathcal{T}_h}\left(B_h^K\left(I_hq-q_\pi,\theta_h\right)+B^K\left(q_\pi,\theta_h\right)+B_h^K\left(q_\pi,\theta_h\right)-B^K\left(q_\pi,\theta_h\right)\right)\\&\quad-\sum_{E\in\mathcal{T}_h}\left(f+z,\Pi_1^0\theta_h\right)_{0,E}\\&=\sum_{K\in\mathcal{T}_h}\left(B_h^K\left(I_hq-q_\pi,\theta_h\right)+B^K\left(q_\pi-q,\theta_h\right)\right)+B_h\left(q_\pi,\theta_h\right)\\&\quad-B\left(q_\pi,\theta_h\right)+B\left(q,\theta_h\right)-\sum_{E\in\mathcal{T}_h}\left(f+z,\Pi_1^0\theta_h\right)_{0,E}\\&=\sum_{K\in\mathcal{T}_h}\left(B_h^K\left(I_hq-q_\pi,\theta_h\right)+B^K\left(q_\pi-q,\theta_h\right)\right)\\&\quad-\left(B\left(q_\pi,\theta_h\right)-B_h\left(q_\pi,\theta_h\right)+\sum_{E\in\mathcal{T}_h}\left(f+z,\Pi_1^0\theta_h\right)_{0,E}-\left(f+z,\theta_h\right)_{0,\Omega}\right)\\&\quad+B\left(q,\theta_h\right)-\left(f+z,\theta_h\right)_{0,\Omega}.\end{aligned} \tag{3.11}$$

Using formula (2.8), interpolation estimates (2.9) and stability property of $a_h(\cdot,\cdot)$, we get

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\left(B_h^K\left(I_hq-q_\pi,\theta_h\right)+B^K\left(q_\pi-q,\theta_h\right)\right)\\&\le C\sum_{K\in\mathcal{T}_h}\left(\left\|\nabla\left(I_hq-q_\pi\right)\right\|_K\left\|\nabla\theta_h\right\|_K+\left\|\nabla\left(q_\pi-q\right)\right\|_K\left\|\nabla\theta_h\right\|_K\right)\\&\le C\sum_{K\in\mathcal{T}_h}\left(h_K|q|_{2,K}\left\|\nabla\theta_h\right\|_K+h_K|q|_{2,K}\left\|\nabla\theta_h\right\|_K\right)\\&\le Ch|q|_2\left|\theta_h\right|_{1,h}.\end{aligned} \tag{3.12}$$

By the consistency and the stability of y_π, we obtain

$$
\begin{aligned}
\left|B(q_\pi,\theta_h)-B_h(q_\pi,\theta_h)\right| &= \left|\sum_{K\in\tau_h}\left(\int_K \nabla q_\pi\cdot\nabla\theta_h \mathrm{d}x-\int_K \prod_0^0 \nabla q_\pi\cdot\prod_0^0\nabla\theta_h \mathrm{d}x\right)\right| \\
&= \left|\sum_{K\in\tau_h}\int_K\left(I-\prod_0^0\right)\nabla\theta_h\cdot\nabla q_\pi \mathrm{d}x\right| \\
&= \left|\sum_{K\in\mathcal{T}_h}\int_K\left(I-\prod_0^0\right)\nabla\theta_h\cdot\left(I-\prod_0^0\right)(\nabla q_\pi)\mathrm{d}x\right| \\
&\le Ch|\theta_h|_{1,h}\|q\|_2 .
\end{aligned} \tag{3.13}
$$

Then, by the minkowski inequality and the property of the L^2 projection, we get

$$
\begin{aligned}
\left|\sum_{K\in\mathcal{T}_h}\left(f+z,\Pi_1^0\theta_h\right)-(f+z,\theta)\right| &\le \sum_{E\in\mathcal{T}_h}\|f+z\|\left\|\Pi_1^0\theta_h-\theta_h\right\| \\
&\le \left(\|f\|+\|z\|\right)\left\|\Pi_1^0\theta_h-\theta_h\right\| \\
&\le Ch|\theta_h|_{1,h}\left(\|f\|+\|z\|\right)
\end{aligned} \tag{3.14}
$$

By integration by parts, we write

$$
B(q,\theta_h)-(f+z,\theta_h)=\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial q}{\partial \boldsymbol{n}_K}\theta_h \mathrm{d}s=\sum_{e\in\mathcal{E}_h}\int_e\frac{\partial q}{\partial \boldsymbol{n}_e}[\theta_h]\mathrm{d}s.
$$

Furthermore, from (3.9) we have

$$
B(q,\theta_h)-(f+z,\theta_h)\le Ch|z|_2|\theta_h|_{1,h}. \tag{3.15}
$$

By substituting (3.12), (3.13), (3.14) and (3.15) into (3.11), we have

$$
\alpha_*|\theta_h|_{1,h}^2\le Ch\left(|z|_2+|f|_1\right)|\theta_h|_{1,h},
$$

$$
|\theta_h|_{1,h}\le Ch\left(|z|_2+|f|_1\right). \tag{3.16}
$$

From Lemma 2.2 (2.9) we get

$$
|q-I_hq|_{1,h}\le Ch|z|_2. \tag{3.17}
$$

By combining (3.16) and (3.17), we have

$$
\|q-q_h(z)\|_{1,h}\le Ch \tag{3.18}
$$

By repeating the proof procedure of (3.18), we can get

$$
\|r-r_h(q)\|_{1,h}\le Ch
$$

□

Similar to the discussion in Theorem 3.1 and theoretical analysis in Andrea *et al.* [18] we have the L^2 error estimates between solution of (2.7) and the solution of auxiliary problem (3.7)

Lemma 3.3 see ([18]) Suppose that (q,r) is the solution of (2.7), and $(q_h(z),r_h(q))$ is the solution of auxiliary problem (3.7), under Assumpiton 2.1 and assumption 3.1 we have

$$
\|q-q_h(z)\|+\|r-r_h(q)\|\le Ch^2
$$

Theorem 3.2 (A priori error estimate) Let (q,r,z) and (q_h,r_h,z_h) are the solutions of (2.7) and (3.5) respectively. Under the Assumptions 2.1 and 3.1 we derive

$$\|z-z_h\|+\|q-q_h\|+\|r-r_h\|\le Ch^2$$

and

$$\|q-q_h\|_{1,h}+\|r-r_h\|_{1,h}\le Ch.$$

Proof: We decompose the errors $q-q_h$ and $r-r_h$ into

$$q-q_h=q-q_h(z)+q_h(z)-q_h$$

and

$$r-r_h=r-r_h(z)+r_h(z)-r_h.$$

From the discrete first-order optimality system of the optimal control problem (3.5) and auxiliary problems (3.7) we get

$$B_h\left(q_h(u)-q_h,\theta_h\right)=\sum_{E\in\mathcal{T}_h}\left(z-z_h,\Pi_1^0\theta_h\right)_{0,E}.$$

Let $\theta_h=q_h(u)-q_h$, and from (3.2) we have

$$\begin{aligned}C\alpha_*\|q_h(z)-q_h\|_{1,h}^2&\le B_h\left(q_h(z)-q_h,q_h(u)-q_h\right)\\&=\sum_{E\in\mathcal{T}_h}\left(z-z_h,\Pi_1^0\left(q_h(u)-q_h\right)\right)_{0,E}\\&\le\sum_{E\in\mathcal{T}_h}\|z-z_h\|_{0,E}\cdot\|q_h(z)-q_h\|_{0,E}\\&\le\|z-z_h\|\cdot\|q_h(z)-q_h\|_{1,h}.\end{aligned}\tag{3.19}$$

then

$$\|q_h(z)-q_h\|_{1,h}\le C\|z-z_h\|.$$

By the Lemma 3.2 and Theorem 3.1, we have

$$\|q-q_h(z)\|\le Ch^2,\quad \|q-q_h(z)\|_{1,h}\le Ch.$$

Combining these inequalities, we get

$$\|q-q_h\|\le C\left(h^2+\|z-z_h\|\right)$$

and

$$\|q-q_h\|_{1,h}\le C\left(h+\|z-z_h\|\right).$$

By the Lemma 3.2 and Theorem 3.1, we have

$$\|r-r_h(q)\|\le Ch^2,\quad \|r-r_h(z)\|_{1,h}\le Ch.$$

Let $v_h=r_h(q)-r_h(z)$. From (3.2) we derive

$$\begin{aligned}C\alpha_*\|r_h(q)-r_h(z)\|_{1,h}^2&\le B_h\left(r_h(q)-r_h(z),r_h(q)-r_h(z)\right)\\&=\sum_{E\in\mathcal{T}_h}\left(q-q_h(z),\Pi_1^0\left(r_h(q)-r_h(z)\right)\right)_{0,E}\\&\le\sum_{E\in\mathcal{T}_h}\|q-q_h(z)\|_{0,E}\cdot\|r_h(q)-r_h(z)\|\\&\le\|q-q_h(z)\|\cdot\|r_h(q)-r_h(z)\|_{1,h}.\end{aligned}\tag{3.20}$$

Further we have

$$\left\|r_h(q)-r_h(z)\right\|_{1,h} \le C\left\|q-q_h(z)\right\| \le Ch^2.$$

By triangle inequality we derive

$$\left\|r-r_h(z)\right\|+h\left\|r-p_h(z)\right\|_{1,h} \le Ch^2. \tag{3.21}$$

similar to (3.20) we obtain

$$\left\|r_h(z)-r_h\right\|_{1,h} \le C\left\|q_h(z)-q_h\right\| \le C\left\|z-z_h\right\|.$$

Through the triangle inequality, we can infer

$$\left\|r-r_h\right\| \le C\left(h^2+\left\|z-z_h\right\|\right)$$

and

$$\left\|r-r_h\right\|_{1,h} \le C\left(h+\left\|z-z_h\right\|\right).$$

Because both the estimation of the state and the adjoint state depend on the estimation of the control variables, now we need estimate $\left\|z-z_h\right\|$ Define

$$\hat{J}_h'(z)(w-z) := \sum_{E\in\mathcal{T}_h}\int_E\left(\gamma z+\Pi_1^0 r_h(z)\right)(w-z)\mathrm{d}x.$$

Then, we can prove that

$$\hat{J}_h'(w)(w-z)-\hat{J}_h'(z)(w-z) \ge \gamma\left\|v-u\right\|^2, \tag{3.22}$$

$$\begin{aligned}
&\hat{J}_h'(w)(w-z)-\hat{J}_h'(z)(w-z)\\
&=\sum_{E\in\mathcal{T}_h}\int_E\left(\gamma w+\Pi_1^0 r_h(w)-\gamma z-\Pi_1^0 r_h(z)\right)(w-z)\mathrm{d}x\\
&=\gamma\sum_{E\in\mathcal{T}_h}\int_E(w-z)^2\,\mathrm{d}x+\sum_{E\in\mathcal{T}_h}\int_E\left(\Pi_1^0 r_h(w)-\Pi_1^0 r_h(z)\right)(w-z)\mathrm{d}x\\
&=\gamma\int_\Omega(w-z)^2\,\mathrm{d}x+\sum_{E\in\mathcal{T}_h}\int_E\left(\Pi_1^0 r_h(w)-\Pi_1^0 r_h(z)\right)(w-z)\mathrm{d}x.
\end{aligned}$$

Using (3.7) and the property of the projection of L^2 we can deduce

$$\begin{aligned}
&\sum_{E\in\mathcal{T}_h}\int_E\left(\Pi_1^0 r_h(w)-\Pi_1^0 r_h(z)\right)(w-z)\mathrm{d}x\\
&=B_h\left(q_h(w)-q_h(z),r_h(w)-r_h(z)\right)\\
&=B_h\left(r_h(w),q_h(w)-q_h(z)\right)-B_h\left(r_h(z),q_h(w)-q_h(z)\right)\\
&=\sum_{E\in\mathcal{T}_h}\left(q_h(w)-q_d,\Pi_1^0\left(q_h(w)-q_h(z)\right)\right)_{0,E}\\
&\quad-\sum_{E\in\mathcal{T}_h}\left(q_h(z)-q_d,\Pi_1^0\left(q_h(w)-q_h(z)\right)\right)_{0,E}\\
&=\sum_{E\in\mathcal{T}_h}\left(\Pi_1^0\left(q_h(w)-q_h(z)\right),\Pi_1^0\left(q_h(w)-q_h(z)\right)\right)\ge 0.
\end{aligned}$$

Then from (3.22) we have

$$\begin{aligned}
&\gamma\left\|z-z_h\right\|^2\\
&\le\sum_{E\in\mathcal{T}_h}\int_E\left(\gamma z+\Pi_1^0 r_h(z)\right)(z-z_h)\mathrm{d}x-\sum_{E\in\mathcal{T}_h}\int_E\left(\gamma z_h+\Pi_1^0 r_h(z_h)\right)(z-z_h)\mathrm{d}x\\
&=\sum_{E\in\mathcal{T}_h}\int_E\left(\gamma u+\Pi_1^0 r_h(z)-\gamma z_h-\Pi_1^0 r_h(z_h)\right)(z-z_h)\mathrm{d}x
\end{aligned}$$

$$= (\gamma z + r, z - z_h) + \sum_{E \in \mathcal{T}_h} \left(\gamma z_h + \Pi_1^0 r_h(z_h), z_h - z\right)_{0,E} + \sum_{E \in \mathcal{T}_h} \left(\Pi_1^0 r_h(z) - r, z - z_h\right)_{0,E}$$

$$\leq 0 + 0 + \sum_{E \in \mathcal{T}_h} \left(\Pi_1^0 r_h(z) - r, z - z_h\right)_{0,E}.$$

This shows

$$\|z - z_h\| \leq C \left(\sum_{E \in \mathcal{T}_h} \left\| \Pi_1^0 r_h(z) - r \right\|_{0,E}^2 \right)^{\frac{1}{2}}.$$

Note that

$$\begin{aligned} \left\| \Pi_1^0 r_h(z) - r \right\|_{0,E} &\leq \left\| \Pi_1^0 r_h(z) - \Pi_1^0 r \right\|_{0,E} + \left\| \Pi_1^0 r - r \right\|_{0,E} \\ &\leq \left\| r_h(z) - r \right\|_{0,E} + \left\| \Pi_1^0 r - r \right\|_{0,E}. \end{aligned}$$

Then by Lemma 3.1 and (3.21) we have

$$\|z - z_h\| \leq Ch^2$$

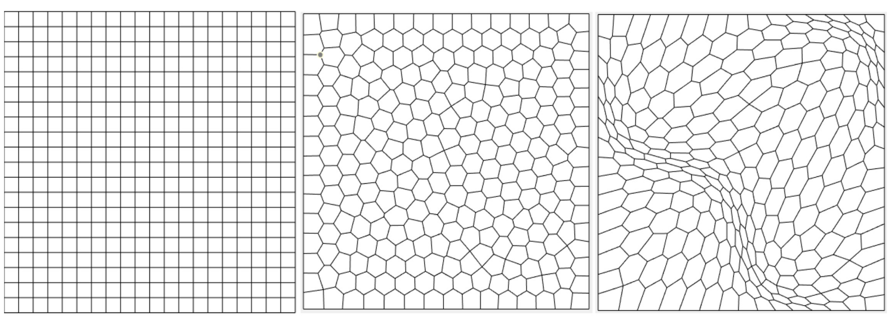
Inserting the $\|z - z_h\|$ into the $\|q - q_h\|$ and $\|r - r_h\|$, we obtain the result.

□

Remark 1 (comparison with conforming VEM). The global virtual element space defined differently for the VEM and NCVEM, For the VEM simply take $W = H_0^1(\Omega)$. For NCVEM, we introduce the nonconforming broken Sobolev space $H^1(\mathcal{T}_h)$ by imposing certain weak inter-element continuity requirements. In contrast to conforming VEM, since W_h is not a subset of $H_0^1(\Omega)$ in general, the substitution of discrete function W_h in the weak formulation leads to a nonconformity error such as (3.15).

4. Numerical Experiments

In this section, we present three different sequences of meshes to validate the performance of our error analysis presented in this paper. Through decomposing the domain into multiple squares, we obtain the first sequence of meshes (labeled square). The second meshes (labeled Lloyd) is given through Voronoi mesh generator [29]. The third sequence (labeled distorted) is to divide the Lloyd meshes into multiple distorted Lloyd meshes. These three sequences of meshes are respectively shown in **Figures 1(a)-(c)**.



(a) Square (b) Lloyd (c) Distorted

Figure 1. Three meshes.

We will confirm the priori error on the three grids by showing the relative errors in L^2 and H^1 norm between q,r,z and the solution q_h,r_h,z_h given by the NCVEM. We use Q_0,R_0,Z_0 to denote the relative errors in the L^2 norm between q,r,z and q_h,r_h,z_h. Similarly, we respectively use Q_1,R_1 to denote the relative errors in the H^1 norm between q,r and q_h,r_h.

Example: The optimal control problem (1.1)-(1.2) is restricted to the unit square $\Omega=[0,1]\times[0,1]$. Let $z_a=-1$, $z_b=0$, $\gamma=1$. We chose the following exact solution

$$q(x,y)=2\sin(\pi x)\sin(\pi y),$$
$$r(x,y)=100\left(\left(x^2-x\right)\left(x^2-y\right)\right),$$
$$z(x,y)=\max\left(-1,\min(-r,0)\right).$$

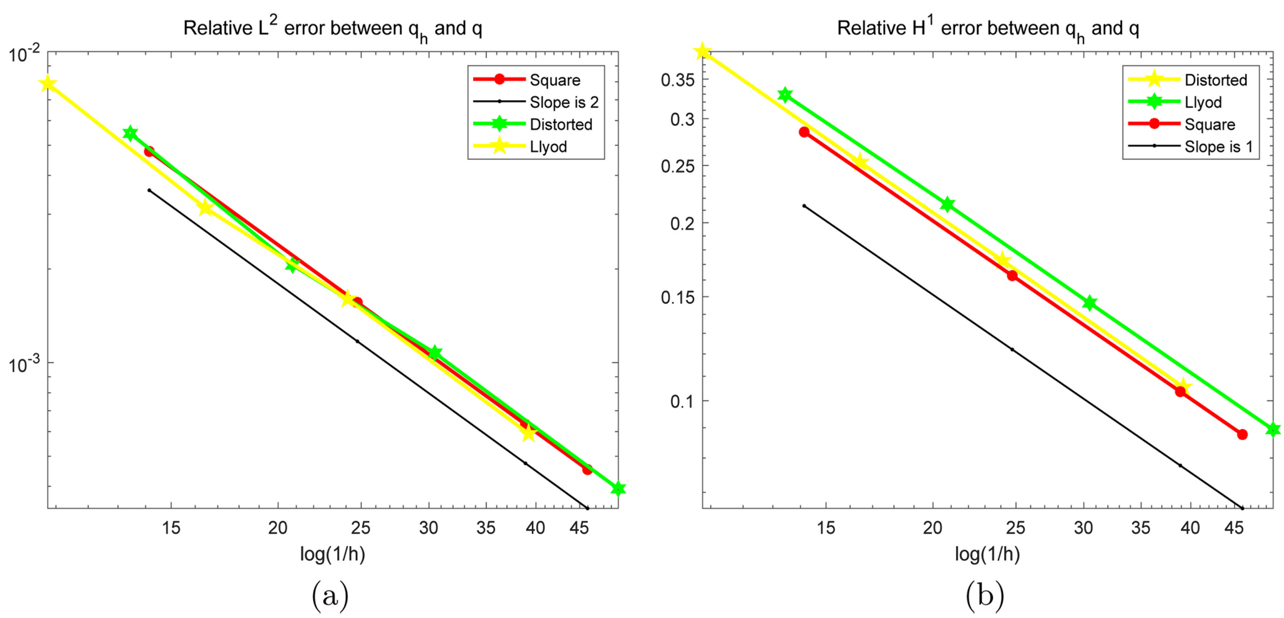
f and y_d can be determined from the exact solutions y,p,u.

In Table 1, there different meshes data of size parameter (mesh diameter), number of elements and vertices are shown.

In Figures 2-4, we present the convergence rate curves of the state, adjoint

Table 1. Mesh data for three grid meshes. h represents the mesh size parameter, and $\mathcal{E}$ and $\mathcal{V}$ represent the number of elements and vertices of the mesh.

	Square			Lloyd			Distorted		
n	h	$\mathcal{E}$	$\mathcal{V}$	h	$\mathcal{E}$	$\mathcal{V}$	h	$\mathcal{E}$	$\mathcal{V}$
1	0.0707	400	441	0.0744	300	602	0.0929	300	602
2	0.0404	1225	1296	0.0481	700	1402	0.0609	700	1402
3	0.0257	3025	3136	0.0328	1500	3001	0.0415	1500	3001
4	0.0218	4225	4356	0.0200	4000	8001	0.0255	4000	8001



(a) (b)

Figure 2. Relative errors of state variables in L^2 and H^1 norm.

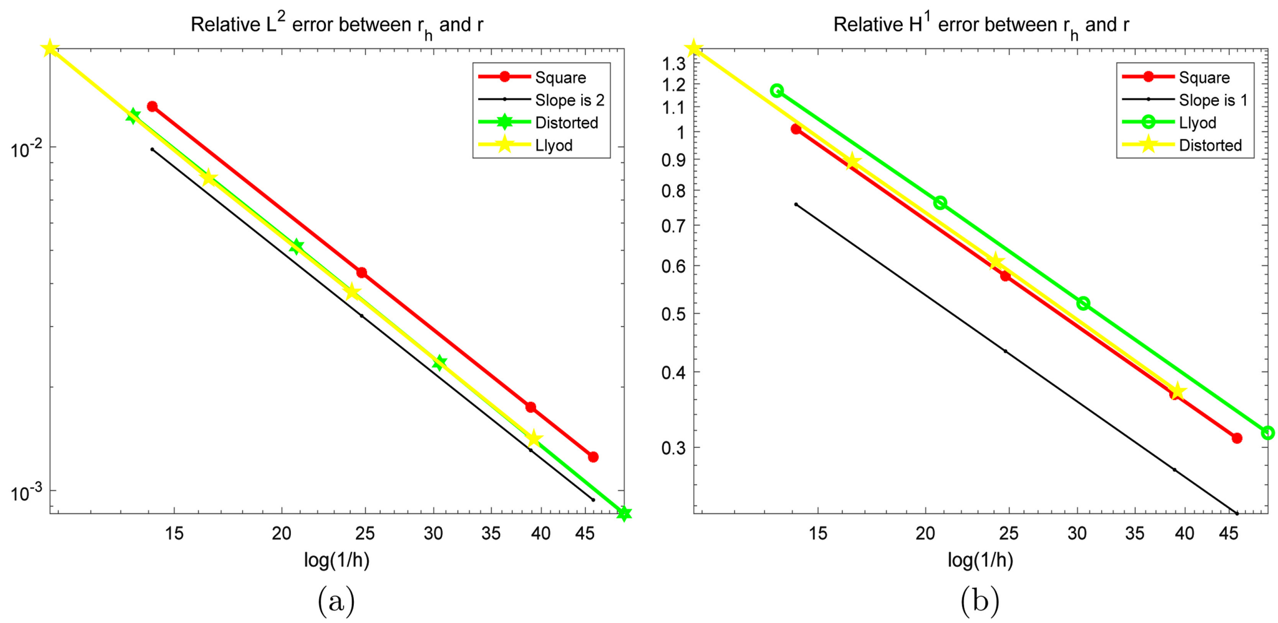


Figure 3. Relative errors of adjoint state variables in L^2 and H^1 norm.

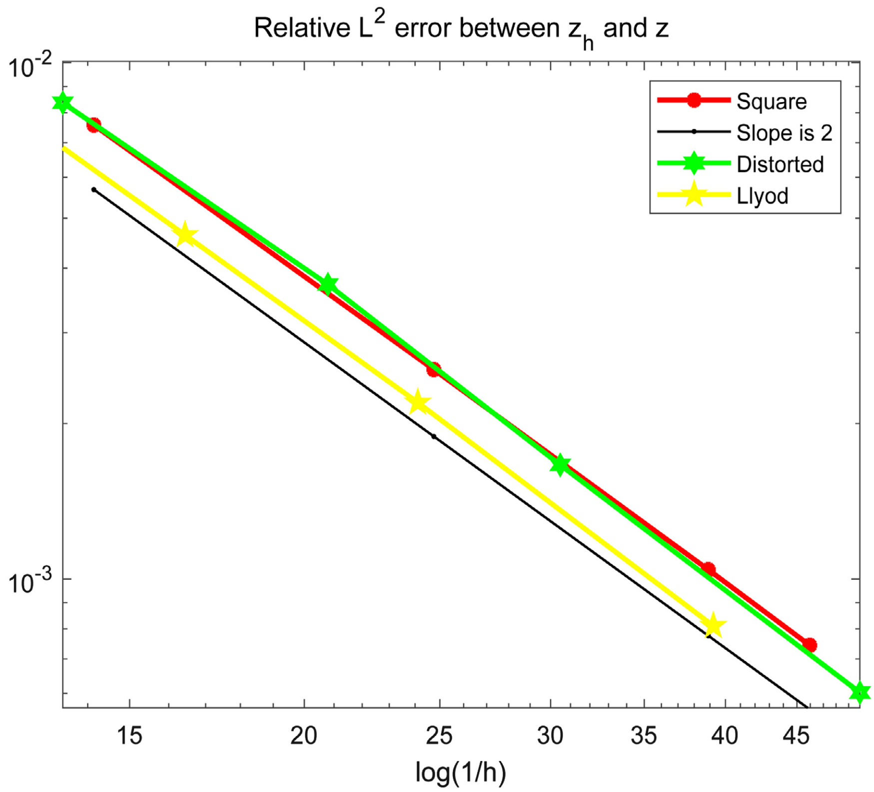


Figure 4. Relative errors of control variables in L^2 and H^1 norm.

state and control variables in L^2 and H^1 norm in **Tables 2-7**, the numerical results about the relative errors and convergence are shown on three different meshes. In **Figures 5-7**, we present the figure of the solution of three variables in three meshes.

Table 2. Relative errors and convergence rates of state, adjoint state, control variables in L^2 norm on Square mesh of Example 4.1.

h	$\mathcal{Z}_0$	Rate	$\mathcal{Q}_0$	Rate	$\mathcal{R}_0$	Rate
0.0707	7.600e−3		4.800e−3		1.310e−2	
0.0404	2.500e−3	1.949	1.600e−3	1.979	4.30e−3	1.989
0.0257	1.000e−3	1.976	6.325e−4	1.992	1.700e−3	1.996
0.0218	7.436e−4	2.020	4.529e−4	1.999	1.300e−3	1.998

Table 3. Relative errors and convergence rates of state, adjoint state variables in H^1 norm on Square mesh of Example 4.1.

h	$\mathcal{Q}_1$	Rate	$\mathcal{R}_1$	Rate
0.0707	2.848e−1		1.013e−0	
0.0404	1.628e−1	0.996	5.768e−1	1.0014
0.0257	1.1036e−1	0.997	3.670e−1	1.003
0.0218	8.77e−2	0.999	3.106e−1	1.003

Table 4. Relative errors and convergence rates of state, adjoint state, control variables in L^2 norm on Lloyd mesh of Example 4.1.

h	$\mathcal{Z}_0$	Rate	$\mathcal{Q}_0$	Rate	$\mathcal{R}_0$	Rate
0.0744	8.400e−3		5.500e−3		1.230e−2	
0.0481	3.700e−3	1.857	2.100e−3	2.352	5.100e−3	2.061
0.0328	1.700e−3	2.015	1.100e−3	1.705	2.400e−3	2.036
0.0200	6.024e−4	2.058	3.927e−4	2.037	8.565e−4	2.047

Table 5. Relative errors and convergence rates of state, adjoint state variables in H^1 norm on Lloyd mesh of Example 4.1.

h	$\mathcal{Q}_1$	Rate	$\mathcal{R}_1$	Rate
0.0744	3.289e−1		1.169e−0	
0.0481	2.174e−1	0.977	7.623e−1	0.980
0.0328	1.463e−1	1.003	5.194e−1	1.003
0.0200	8.940e−2	0.999	3.169e−1	1.001

Table 6. Relative errors and convergence rates of state, adjoint state, control variables in L^2 norm on Distorted mesh of Example 4.1.

h	$\mathcal{Z}_0$	Rate	$\mathcal{Q}_0$	Rate	$\mathcal{R}_0$	Rate
0.0929	1.050e−2		7.900e−3		1.930e−2	
0.0609	4.600e−3	1.936	3.100e−3	2.171	8.100e−3	2.051
0.0415	2.200e−3	1.949	1.600e−3	1.761	3.800e−3	1.993
0.0255	8.119e−4	2.048	5.913e−4	2.047	1.400e−3	2.021

Table 7. Relative errors and convergence rates of state, adjoint state variables in H^1 norm on Distorted mesh of Example 4.1.

h	$\mathcal{Q}_1$	Rate	$\mathcal{R}_1$	Rate
0.0929	3.894e−1		1.37e−0	
0.0609	2.531e−1	1.017	8.922e−1	1.017
0.0415	1.727e−1	0.981	6.093e−1	0.995
0.0255	1.055e−1	1.013	3.712e−1	1.019

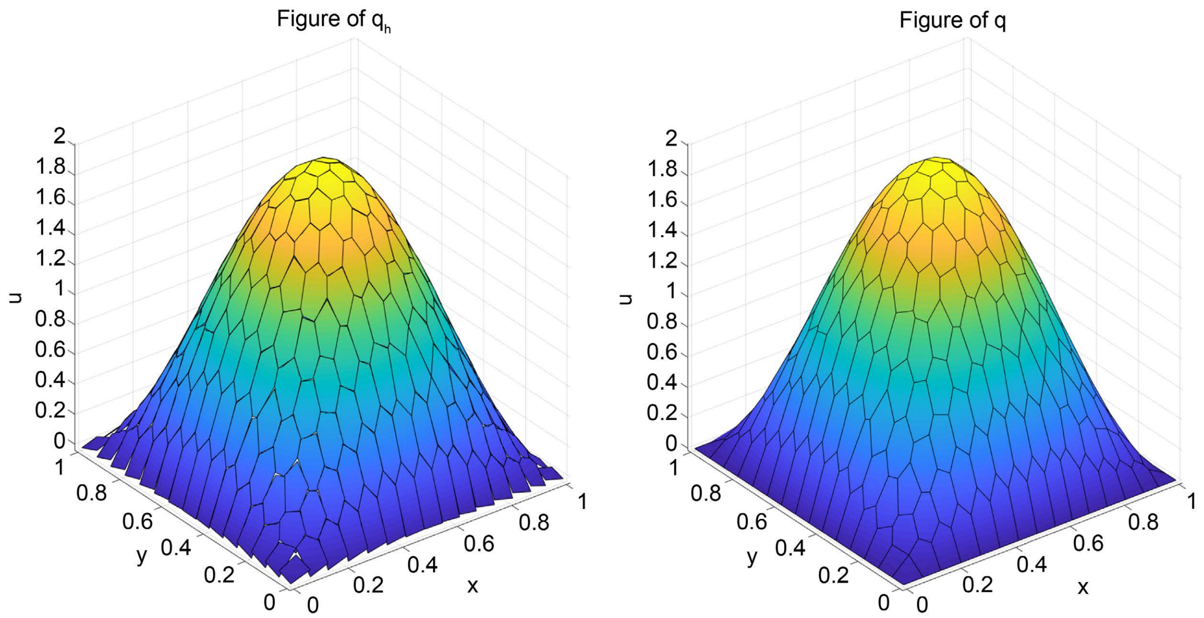


Figure 5. The solution of state variables in Lloyd meshes.

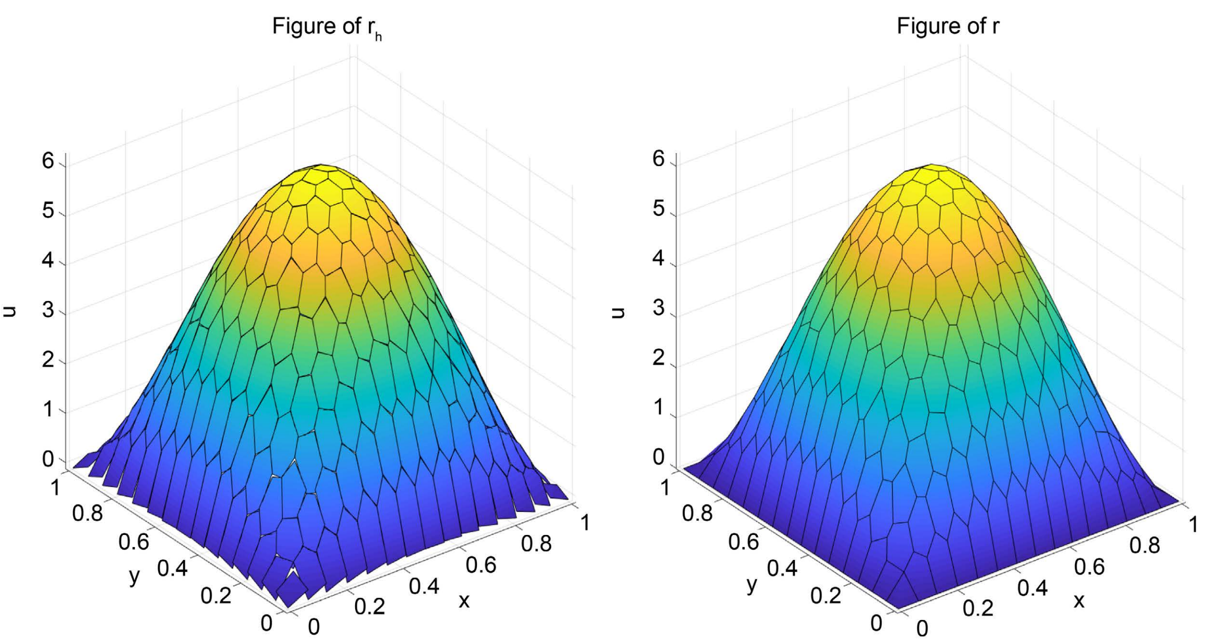


Figure 6. The solution of adjoint state variables in Lloyd meshes.

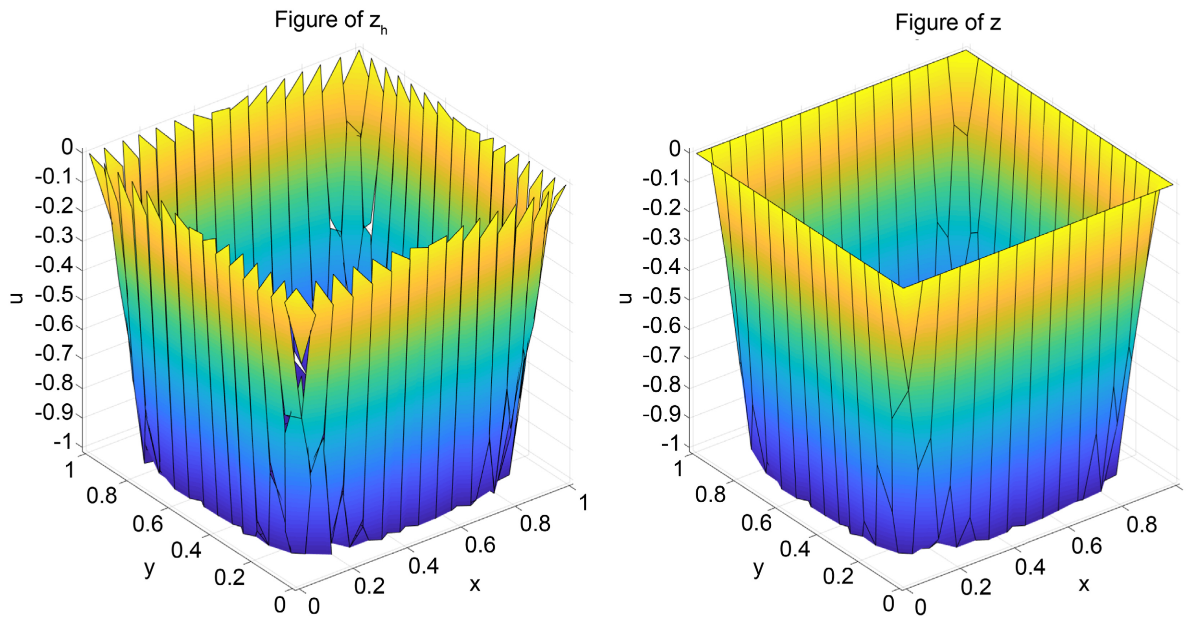


Figure 7. The solution of control variables in Lloyd meshes.

5. Conclusions

In this paper, NCVEM is applied to approximate elliptic optimal control problems with pointwise control constraints. The priori error estimates are derived, numerical examples verifies the theoretical results.

In our future work, we will increase the complexity of the elliptic problem, generalize the problem to linear indefinite elliptic problems. And because of the flexibility of the VEM, we will derive a posteriori error estimate for the problem, and derive an adaptive grid algorithm to guide mesh refinement.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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