Advances in Pure Mathematics, 2024, 14, 242-282
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Economy Function in the Mode of Sustainable Development

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How to cite this paper: Gonchar, N.S. (2024) Economy Function in the Mode of Sustainable Development. *Advances in Pure Mathematics*, **14**, 242-282.
https://doi.org/10.4236/apm.2024.144015

Received: March 8, 2024
Accepted: April 21, 2024
Published: April 24, 2024

Abstract

To implement the previously formulated principles of sustainable economic development, all solutions of the linear system of equations and inequalities, which are satisfied by the vector of real consumption, are completely described. It is established that the vector of real consumption with the minimum level of excess supply is determined by the solution of some quadratic programming problem. The necessary and sufficient conditions are established under which the economic system, described by the "input-output" production model, functions in the mode of sustainable development. A complete description of the equilibrium states for which markets are partially cleared in the economy model of production "input-output" is given, on the basis that all solutions of system of linear equations and inequalities are completely described. The existence of a family of taxation vectors in the "input-output" model of production, under which the economic system is able to function in the mode of sustainable development, is proved. Restrictions were found for the vector of taxation in the economic system, under which the economic system is able to function in the mode of sustainable development.

Keywords

Technological Mapping, Economic Balance, Clearing Markets, Vector of Taxation, Sustainable Economic Development

1. Introduction

The paper details the principles of economy sustainable development at the micro-economic level for the production model "input-output" and is a continuation of the papers [1] [2]. This model of production is the basis for the formulation of concepts: gross output, direct costs, gross domestic product, gross added value, gross accumulation, export vector, import vector. By sustainable economic development we understand the growth of the gross domestic product over a long period of time. During this period, the wages of employees increase, most companies operate in a profitable mode, and the exchange rate of the national currency is stable. In order to formulate this definition mathematically, such a concept as the consistency of the supply structure with the demand structure was introduced. At the firm level, this would mean studying the structure of demand for manufactured goods in such a way that the volume of goods produced is consistent with the structure of demand. The latter will ensure, according to proven theorems, the appropriate profit for firms.

Why is the concept of sustainable economic development important? Due to the cyclical nature of economic development, it is necessary to understand the internal causes of this phenomenon. The concept of sustainable economic development serves this purpose. Under conditions of uncertainty, firms produce goods by taking loans from banks or issue securities to finance the necessary costs of production. In order to repay loans, pay taxes on production, pay wages and ensure sufficient profit to restore fixed assets and expand production, it is necessary that the manufactured products be sold at prices at which the added value created is sufficient for this.

The model of sustainable development is formulated in such a way that for production technologies, described by technological mappings from the KTM class, there exists an equilibrium price vector for which firms are able to purchase production materials and services for the production of the final product, and consumers of the final product to satisfy their needs. In addition, it should be noted that there is a taxation system that ensures complete clearing of markets in a certain period of the economy's functioning. Therefore, the model of sustainable development is certain material and value balances, according to which the economic system is able to function continuously, creating a final product that is fully consumed in it and exported.

We call this mode of functioning of the economy the mode of sustainable development.

The purpose of the paper is to formulate the conditions under which the real economic system is able to function in the mode of sustainable development. Such conditions are limitations for the levels of taxation, under which the final product will be created in the economic system to satisfy the needs of consumers both in the sphere of the real economy and in the sphere of services. These are the restrictions on gross outputs, which are easily verified on the basis of Theorems established in Section 3.

An economic system may not be in a mode of sustainable development due to the fact that the supply of produced goods is in excess. This is due to the fact that during the reporting period, not all the products produced can be sold according to the concluded contracts. These can be food industry products, agricultural products, stockpiles of manufactured weapons. Therefore, the conditions of sustainable development, which provide for a complete clearing of the markets, will not be fully fulfilled. For this purpose, it is necessary to describe all states of economic equilibrium possible in the economic system, that is, to specify all equilibrium price vectors for which demand does not exceed supply. To describe all equilibrium states, one should be able to solve systems of nonlinear equations and inequalities. Mathematical methods for this did not exist before this paper and the papers of [1] [2] [3].

In this paper, we reduced this problem to the solution of two problems: the description of all solutions of a linear system of equations and inequalities and solving a linear homogeneous problem with respect to the equilibrium price vector for which partial clearing of the markets takes place.

It should be noted that the description of all solutions linear system of equations and inequalities in this work was obtained for the first time.

Each such equilibrium state is characterized by the level of excess supply. Its value ranges from zero to one. At zero level of excess supply, the markets are completely cleared. The greater the value of the level of excess supply, the closer the economic system is to the state of recession, see [4] [5] [6].

Among all the described equilibrium states, there is one with the smallest excess supply. As a rule, the economy system is in that state. To find it, you should solve the problem of quadratic programming, and based on this solution, construct the solution of the linear problem, finding the equilibrium state. Then calculate the level of excess supply. Having done this for each reporting year, its dynamics should be investigated. If it grows, then this is an indicator that the economic system may slip into a state of recession.

Check whether the existing taxation system satisfies the proved inequalities that must be satisfied by an economy operating in a mode of sustainable development.

If it satisfies the derived inequalities, it should be checked whether the components of the final consumption vector are strictly positive. The presence of negative components can be associated with a negative trade balance, with the withdrawal of the created final product into the shadow sector of the economy. In accordance with this, an economic policy can be developed to adjust its further development.

The proven Theorems show that in case of incomplete correspondence between the structure of demand and the structure of supply, there is a surplus of industrial goods, the price of which is not determined by demand and supply due to saturation of consumers with a certain group of goods. To characterize this phenomenon, an important concept of the vector of real consumption in the economic system is introduced. Based on this concept, the concept of the volume of unsold goods for a given period was introduced, which is defined as the relative value of unsold goods for a certain period.

Section 2 gives the definition of a polyhedral cone and conditions for a given vector to belong this cone. Theorem 1 describes all strictly positive solutions of the system of equations, provided that the right-hand side of the equations belongs to the polyhedral cone formed by the columns of the matrix of the left-hand side of the system of equations.

The consistency of the structure of supply with the structure of demand gives Definition 4. Definition 5 contains the consistency of the supply structure with the demand structure of a certain rank in the strict sense. Lemma 2 contains sufficient conditions for consistency of property vectors and demand vectors in the strict sense. Lemma 3 asserts the existence of a strictly positive solution of a certain system of equations. Theorem 2 formulates the necessary and sufficient conditions for the existence of a state of equilibrium under which the markets are completely cleared. Definition 6 contains the definition of consistency of the supply structure with the demand structure in a weak sense.

The definition of consistency of the supply structure with the demand structure in the weak sense of a certain rank is contained in Definition 7. Theorem 3 contains the necessary and sufficient conditions for the existence of a state of equilibrium, under which the markets are completely cleared under the condition that the supply structure is weakly consistent with the demand structure.

Theorems 4, 5 and their consequences provide sufficient conditions for the existence of a solution to the system of inequalities. Theorem 6 establishes sufficient conditions for the existence of an equilibrium vector of prices. Sufficient conditions for the existence of economic equilibrium are contained in Theorem 7, which are conditions for the aggregate supply vector. The proven Theorems are a description of equilibrium states in the economic system.

Theorem 8 gives necessary and sufficient conditions for the existence of solutions of the system of equations and inequalities, which are satisfied by the vector of real consumption.

The solutions of a certain system of linear equations and inequalities are completely described in Theorem 9.

The sufficient conditions for the existence of an equilibrium price vector are given in Theorem 10 provided that the vector of final consumption is a vector of real consumption generated by a certain non-negative vector.

Theorem 11 is another variant of Theorem 10.

The necessary and sufficient conditions for the existence of an equilibrium price vector corresponding to the vector of real consumption generated by some a certain non-negative vector are given in Theorem 12.

Theorem 13 is a statement that for not every vector proposed for consumption for a given demand structure there exists an equilibrium price vector. That is, there is no market mechanism for the sale of manufactured goods in the long term.

In Definition 8, it is stated that between every vector of real consumption, which is a solution of a system of equations and inequalities, and a vector of equilibrium prices there is a correspondence.

Theorem 14 is a justification of the above correspondences. In this case, there is an equilibrium price vector corresponding to market supply and demand.

The necessary and sufficient conditions for the existence of an equilibrium price vector are given in Theorem 15.

On the basis of proven Theorems, the concept of the volume of unsold products in a given period is introduced for the corresponding equilibrium state in the case of partial clearing of markets.

In Definition 9, it is stated that between the real consumption vector, generated by the quadratic programming problem, and the equilibrium price vector there exists a correspondence.

In Section 3, Theorem 16 gives the necessary and sufficient conditions for the functioning of the economic system in the mode of sustainable development described by the "input-output" production model.

In Theorem 17 for the "input-output" model, the existence of a taxation system under which the price vector formed in the economic system is an equilibrium price vector is established.

The best taxation system has been built, under which the economic system is able to function in the mode of sustainable development.

In Theorem 18, restrictions are found for the vector of taxation, under which a final product can be created in the economic system that ensures the functioning of the economic system in the mode of sustainable development.

In Theorem 19, a certain system of taxation is built, under which the economic system is able to function in the mode of sustainable development. In this case, the vector of gross output satisfies a certain system of equations, the solution of which always exists and which determines the vector of final consumption.

Section 4 is an illustration of the application of the results obtained in Sections 2 and 3 to the study of real economic systems described in value indicators. Theorem 20 is a reformulation of Theorem 18 for the case of real economic systems. It asserts the restrictions for the taxation systems in real economic systems under which the economic system is able to operate in a sustainable development mode. Theorem 21 is a partial case of Theorem 20.

2. Construction of Equilibrium States with Excess Supply

Here and further, $R_+^n \setminus \{0\}$ is a cone formed from the non-negative orthant R_+^n by discarding the null vector $\{0\} = \{0, \cdots, 0\}$. Next, the cone $R_+^n \setminus \{0\}$ is denoted by R_+^n.

We give a number of definitions useful for the future.

Definition 1 *Under the polyhedral non negative cone formed by the set of vectors* $\{a_i, i = \overline{1,t}\}$ *of the n-dimensional space* R^n *we understand the set of vectors of the form*

$$d = \sum_{i=1}^{t} \alpha_i a_i,$$

where $\alpha = \{\alpha_i\}_{i=1}^{t}$ runs through the set R_+^t.

Definition 2 *The dimension of the non negative polyhedral cone formed by a set of vectors* $\{a_i, i = \overline{1,t}\}$ *in the n-dimensional space* R^n *is the maximum* *number of linearly independent vectors from the set of vectors* $\{a_i, i=\overline{1,t}\}$.

Definition 3 *The vector b belongs to the interior of the non negative polyhedral r-dimensional cone* $r \le n$, *created by the set of vectors* $\{a_1,\cdots,a_t\}$ *in the n-dimensional vector space* R^n *if there exists a strictly positive vector* $\alpha = \{\alpha_i\}_{i=1}^t \in R_+^t$ *such that*

$$b = \sum_{s=1}^{t} a_s \alpha_s,$$

where $\alpha_s > 0, s=\overline{1,t}$.

We present the necessary and sufficient conditions under which a certain vector belongs to the interior of a polyhedral cone.

Lemma 1 *Let* $\{a_1,\cdots,a_m\}$, $1 \le m \le n$, *be a set of linearly independent vectors in* R_+^n. *The necessary and sufficient conditions for the vector b to belong to the interior of a non-negative cone* K_a^+ *formed by vectors* $\{a_i, i=\overline{1,m}\}$ *are conditions*

$$\langle f_i, b\rangle > 0, \quad i=\overline{1,m}, \quad \langle f_i, b\rangle = 0, \quad i=\overline{m+1,n}, \tag{1}$$

where $f_i, i=\overline{1,n}$, is a set of vectors biorthogonal to a set of linearly independent vectors $\bar{a}_i, i=\overline{1,n}$, and $\bar{a}_i = a_i, i=\overline{1,m}$.

For the proof of Lemma 1, see: [3], [7]. We will now describe the algorithm for constructing strictly positive solutions of the system of equations.

$$\psi = \sum_{i=1}^{l} C_i y_i, \quad y_i > 0, \quad i=\overline{1,l}, \tag{2}$$

relative to the vector $y=\{y_i\}_{i=1}^l$ or the same system of equations in coordinate form

$$\sum_{i=1}^{l} c_{ki} y_i = \psi_k, \quad k=\overline{1,n}, \tag{3}$$

for a vector $\psi=\{\psi_1,\cdots,\psi_n\}$ belonging to the interior of the polyhedral cone formed by the vectors $\left\{C_i=\{c_{ki}\}_{k=1}^n, i=\overline{1,l}\right\}$.

Theorem 1 *If a certain vector* ψ *belonging to the interior of the non-negative r-dimensional polyhedral cone formed by the vectors* $\left\{C_i=\{c_{ki}\}_{k=1}^n, i=\overline{1,l}\right\}$, *such that there exists a subset r of linearly independent vectors of the set of vectors* $\left\{C_i, i=\overline{1,l}\right\}$, *such that the vector* ψ *belongs to the interior of the cone formed by this subset of vectors, then there exist* $l-r+1$ *linearly independent nonnegative solutions* z_i *of the system of equations* (3) *such that the set of strictly positive solutions of the system of equations* (3) *is given by the formula*

$$y = \sum_{i=r}^{l} \gamma_i z_i, \tag{4}$$

where

$$z_i = \left\{\langle\psi, f_1\rangle - \langle C_i, f_1\rangle y_i^*, \cdots, \langle\psi, f_r\rangle - \langle C_i, f_r\rangle y_i^*, 0, \cdots, y_i^*, 0, \cdots, 0\right\}, \quad i=\overline{r+1,l},$$

$$z_r = \left\{\langle\psi, f_1\rangle, \cdots, \langle\psi, f_r\rangle, 0, \cdots, 0\right\},$$

$$y_i^* = \begin{cases} \min\limits_{s\in K_i} \dfrac{\langle \psi, f_s\rangle}{\langle C_i, f_s\rangle}, & K_i = \{s, \langle C_i, f_s\rangle > 0\}, \\ 1, & \langle C_i, f_s\rangle \le 0, \forall\, s = \overline{1,r}, \end{cases}$$

and the components of the vector $\{\gamma_i\}_{i=r}^{l}$ satisfy a set of inequalities

$$\sum_{i=r}^{l} \gamma_i = 1, \quad \gamma_i > 0, \quad i = \overline{r+1,l},$$

$$\sum_{i=r+1}^{l} \langle C_i, f_k\rangle y_i^* \gamma_i < \langle \psi, f_k\rangle, \quad k = \overline{1,r}. \tag{5}$$

For the proof of Theorem 1, see [3], [7].

Definition 4 *Let* $C_i = \{c_{si}\}_{s=1}^{n} \in R_+^n, i = \overline{1,l}$, *be a set of demand vectors and* $b_i = \{b_{si}\}_{s=1}^{n} \in R_+^n, i = \overline{1,l}$, *be a set of supply vectors. We say that the structure of supply is consistent with the structure of demand in the strict sense if the matrix B has the representation* $B = CB_1$, *where the matrix B consists of vectors* $b_i \in R_+^n, i = \overline{1,l}$, *as columns, and the matrix C consists of vectors* $C_i \in R_+^n, i = \overline{1,l}$, *as columns, and* B_1 *is a square non-negative non-decomposable matrix.*

Definition 5 *Let* $C_i = \{c_{si}\}_{s=1}^{n} \in R_+^n, i = \overline{1,l}$, *be a set of demand vectors and* $b_i = \{b_{si}\}_{s=1}^{n} \in R_+^n, i = \overline{1,l}$, *be a set of supply vectors. We say that the supply structure is consistent with the demand structure in the strict sense of the rank* $|I|$ *if there exists such a subset* $I \subseteq N$ *that the matrix* B^I *has the representation* $B^I = C^I B_1^I$, *where the matrix* B^I *consists of vectors* $b_i^I \in R_+^{|I|}, i = \overline{1,l}$, *in the form of columns, and the matrix* C^I *consists of vectors* $C_i^I \in R_+^n, i = \overline{1,l}$, *in the form of columns, and* $B_1^I = \left|b_{is}^{1,I}\right|_{i,s=1}^{l}$ *is a square non-negative non-decomposable matrix, where* $b_i^I = \{b_{ki}\}_{k\in I}$, $C_i^I = \{c_{ki}\}_{k\in I}$ *and, in addition, the inequalities hold*

$$\sum_{i=1}^{l} c_{ki} y_i^I < \sum_{i=1}^{l} b_{ki}, \quad k \in N \setminus I, \quad y_i^I = \sum_{s=1}^{l} b_{is}^{1,I}.$$

Lemma 2 *Suppose that the set of supply vectors* $b_i = \{b_{si}\}_{s=1}^{n} \in R_+^n, i = \overline{1,l}$, *belongs to the polyhedral cone formed by the set of vectors of demand* $\left\{C_i = \{c_{ki}\}_{k=1}^{n} \in R_+^n\ i = \overline{1,l}\right\}$. *Then for the matrix* $B = |b_{ki}|_{k=1,i=1}^{n,l}$ *formed by the columns of vectors* $b_i = \{b_{ki}\}_{k=1}^{n} \in R_+^n, i = \overline{1,l}$, *has a representation*

$$B = CB_1, \tag{6}$$

where the matrix $C = |c_{ki}|_{k=1,i=1}^{n,l}$ is formed by the columns of vectors $C_i = \{c_{ki}\}_{k=1}^{n} \in R_+^n, i = \overline{1,l}$, and the matrix $B_1 = \left|b_{mi}^1\right|_{m=1,i=1}^{l}$ is non-negative. If, in addition, the set of supply vectors $b_i \in R_+^n, i = \overline{1,l}$, belongs to the interior of the polyhedral cone formed by the demand vectors $\left\{C_i = \{c_{ki}\}_{k=1}^{n} \in R_+^n, i = \overline{1,l}\right\}$, matrix B_1 can be chosen strictly positive.

Proof. The proof of the first part of Lemma 2 follows from the second one. If each vector $b_i, i = \overline{1,l}$, belongs to the interior of the polyhedral cone formed by the vectors $C_i, i = \overline{1,l}$, then according to Theorem 1 there exists a strictly positive vector $y_i = \{y_{ki}\}_{k=1}^{l}$ such that

$$b_{ki} = \sum_{s=1}^{l} c_{ks} y_{si}, \quad k = \overline{1,n}.$$

Denote $y_{si} = b_{si}^1$, then we get

$$b_{ki} = \sum_{s=1}^{l} c_{ks} b_{si}^1, \quad k = \overline{1,n}, \quad i = \overline{1,l}.$$

This proves Lemma 2.

Lemma 3 *Let* $B_1 = \left\| b_{ki}^1 \right\|_{k,i=1}^{l}$ *be a square non-negative non-decomposable matrix. Then the problem*

$$\sum_{k=1}^{l} b_{ki}^1 d_k = \sum_{s=1}^{l} b_{is}^1 d_i, \quad i = \overline{1,l}, \tag{7}$$

has a strictly positive solution with respect to the vector $d = \{d_k\}_{k=1}^{l}$.

Proof. Due to the fact that the matrix B_1 is indecomposable, we have $\sum_{s=1}^{l} b_{is}^1 > 0, i = \overline{1,l}$. Let's consider the problem

$$\sum_{k=1}^{l} e_{ki} d_k^1 = d_i^1, \quad i = \overline{1,l}, \tag{8}$$

and prove that it has a strictly positive solution with respect to the vector $d^1 = \{d_k^1\}_{k=1}^{l}$, where we introduced the denotation

$$e_{ki} = \frac{b_{ki}^1}{\sum_{s=1}^{l} b_{ks}^1}.$$

To prove this, consider the problem

$$\frac{d_i^1 + \sum_{k=1}^{l} e_{ki} d_k^1}{2} = d_i^1, \quad i = \overline{1,l}, \tag{9}$$

in the set $P = \left\{ d^1 = \{d_k^1\}_{k=1}^{l}, d_k^1 \geq 0, k = \overline{1,l}, \sum_{k=1}^{l} d_k^1 = 1 \right\}$. Due to equalities $\sum_{i=1}^{l} e_{ki} = 1, \quad k = \overline{1,l}$, the map $H(d^1) = \{H_i(d^1)\}_{i=1}^{l}$, maps P into itself and is continuous on it, where

$$H_i(d^1) = \frac{d_i^1 + \sum_{k=1}^{l} e_{ki} d_k^1}{2}, \quad i = \overline{1,l}.$$

According to Brauer's Theorem [8], there exists a fixed point of the mapping $H(d^1)$, i.e

$$\frac{d_i^1 + \sum_{k=1}^{l} e_{ki} d_k^1}{2} = d_i^1, \quad i = \overline{1,l}.$$

The same fixed point satisfies the problem (8). Since the matrix $E = \left\| e_{ki} \right\|_{k,i=1}^{l}$ is non-negative and indecomposable, we have $E^{l-1} > 0$. From the fact that $Ed^1 = d^1$ implies $E^{l-1} d^1 = d^1$. This proves that the vector d^1 is strictly positive.

Lemma 3 is proved.

Theorem 2 *Let the supply structure agree with the demand structure in the strict sense with supply vectors* $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1,l}$ *, and demand vectors* $\left\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1,l}\right\}$ *, and let* $\sum_{s=1}^n c_{si} > 0, i = \overline{1,l}$ *,* $\sum_{i=1}^l c_{si} > 0, s = \overline{1,n}$ *. The necessary and sufficient conditions for the existence of a solution to the system of equations*

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p\rangle}{\langle C_i, p\rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1,n}, \tag{10}$$

relative to the vector p are that the vector $D = \{d_i\}_{i=1}^l$ belongs to the polyhedral cone formed by the vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1,n}$, where $B = CB_1$, $B_1 = \left|b_{ki}^1\right|_{k,i=1}^l$ is a non-negative non-decomposable matrix, vector $D = \{d_i\}_{i=1}^l$ is a strictly positive solution of the system of equations

$$\sum_{k=1}^l b_{ki}^1 d_k = y_i d_i, \quad y_i = \sum_{k=1}^l b_{ik}^1, \quad i = \overline{1,l}. \tag{11}$$

The proof of Theorem 2, see [7].

Definition 6 *Let* $C_i \in R_+^n, i = \overline{1,l}$ *, be a set of demand vectors and* $b_i \in R_+^n, i = \overline{1,l}$*, be a set supply vectors. We say that the structure of supply is consistent with the structure of demand in a weak sense, if the representation* $B = CB_1$ *is true for the matrix B, where the matrix B consists of vectors* $b_i \in R_+^n, i = \overline{1,l}$*, as columns, and the matrix C consists of vectors* $C_i \in R_+^n, i = \overline{1,l}$*, as columns, and* B_1 *is a square matrix satisfying the conditions*

$$\sum_{s=1}^l b_{is}^1 \geq 0, \quad i = \overline{1,l}, \quad B_1 = \left|b_{is}^1\right|_{i,s=1}^l. \tag{12}$$

Definition 7 *Let* $C_i \in R_+^n, i = \overline{1,l}$ *, be a set of demand vectors, and* $b_i \in R_+^n, i = \overline{1,l}$*, set of supply vectors. We say that the supply structure is consistent with the demand structure in the weak sense of the rank* $|I|$*, if there is such a subset* $I \subseteq N$ *that the representation* $B^I = C^I B_1$ *is true for the matrix* B^I*, where the matrix* B^I *consists of vectors* $b_i^I \in R_+^{|I|}, i = \overline{1,l}$*, in the form of columns, and the matrix* C^I *consists of vectors* $C_i^I \in R_+^n, i = \overline{1,l}$*, in the form of columns and* B_1^I *is a square matrix satisfying the conditions*

$$\sum_{s=1}^l b_{is}^{1,I} \geq 0, \quad i = \overline{1,l}, \quad B_1^I = \left|b_{is}^{1,I}\right|_{i,s=1}^l, \tag{13}$$

where $b_i^I = \{b_{ki}\}_{k\in I}$, $C_i^I = \{c_{ki}\}_{k\in I}$ and, in addition, the inequalities hold

$$\sum_{i=1}^l c_{ki} y_i^I < \sum_{i=1}^l b_{ki}, \quad k \in N \setminus I, \quad y_i^I = \sum_{s=1}^l b_{is}^{1,I}.$$

Theorem 3 *Let the supply structure agree with the demand structure in the weak sense with supply vectors* $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1,l}$ *, and demand vectors* $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n\ i = \overline{1,l}\}$*, and let* $\sum_{s=1}^n c_{si} > 0, i = \overline{1,l}$ *,* $\sum_{i=1}^l c_{si} > 0, s = \overline{1,n}$ *. The necessary and sufficient conditions for the existence of a solution to the system of equations*

$$\sum_{i=1}^{l} c_{ki} \frac{\langle b_i, p\rangle}{\langle C_i, p\rangle} = \sum_{i=1}^{l} b_{ki}, \quad k = \overline{1,n}, \tag{14}$$

are that the vector $D = \{d_i\}_{i=1}^{l}$ belongs to the polyhedral cone created by the vectors $C_k^T = \{c_{ki}\}_{i=1}^{l}, k = \overline{1,n}$, where $B = CB_1$, $B_1 = \left|b_{ki}^1\right|_{k,i=1}^{l}$ is a square matrix, vector $D = \{d_i\}_{i=1}^{l}$ is a strictly positive solution of the system of equations

$$\sum_{k=1}^{l} b_{ki}^1 d_k = y_i d_i, \quad y_i = \sum_{k=1}^{l} b_{ik}^1 \geq 0, \quad i = \overline{1,l}. \tag{15}$$

Proof. The proof of Theorem 3 is the same as Theorem 2.

Below we present the algorithms for constructing solutions of a linear system of inequalities.

Theorem 4 *Let* $A = \|a_{ki}\|_{k,i=1}^{n}$ *be a nonnegative non-decomposable matrix, and let the vector* $b = \{b_k\}_{k=1}^{n}$ *be strictly positive. In the set* $Y = \left\{ y = \{y_1, \cdots, y_n\}, y_i \geq 0, i = \overline{1,n}, \sum_{i=1}^{n} y_i = 1 \right\}$ *there exists a solution to the system of equations*

$$\frac{y_k + y_k \sum_{i=1}^{n} \frac{a_{ki}}{b_k} y_i + \varepsilon}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k \frac{a_{ki}}{b_k} y_i + n\varepsilon} = y_k, \quad k = \overline{1,n}, \tag{16}$$

which is strictly positive for every $\varepsilon > 0$.

Proof. Consider the mapping $\varphi^{\varepsilon}(y) = \{\varphi_k^{\varepsilon}(y)\}_{k=1}^{n}$, where

$$\varphi_k^{\varepsilon}(y) = \frac{y_k + y_k \sum_{i=1}^{n} \frac{a_{ki}}{b_k} y_i + \varepsilon}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k \frac{a_{ki}}{b_k} y_i + n\varepsilon}, \quad k = \overline{1,n}. \tag{17}$$

For any $\varepsilon > 0$, it is continuous on the set $Y = \left\{ y = \{y_1, \cdots, y_n\}, y_i \geq 0, i = \overline{1,n}, \sum_{i=1}^{n} y_i = 1 \right\}$ and maps it into itself. Based on Brauer's Theorem [8], there exists a fixed point of the map $\varphi^{\varepsilon}(y)$ in the set *Y*, that is,

$$\varphi^{\varepsilon}(y^{\varepsilon}) = y^{\varepsilon}. \tag{18}$$

Let us prove the strict positivity of $y^{\varepsilon} = \{y_k^{\varepsilon}\}_{k=1}^{n}$. The estimate

$$y_k^{\varepsilon} \geq \frac{\varepsilon}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k^{\varepsilon} \frac{a_{ki}}{b_k} y_i^{\varepsilon} + n\varepsilon} \geq \frac{\varepsilon}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} \frac{a_{ki}}{b_k} + n\varepsilon} > 0 \tag{19}$$

is valid. Theorem 4 is proved.

Theorem 5 *Let* $A = \|a_{ki}\|_{k,i=1}^{n}$ *be a nonnegative non-decomposable matrix, and the vector* $b = \{b_k\}_{k=1}^{n}$ *is strictly positive. In the set* $Y = \left\{ y = \{y_1, \cdots, y_n\}, y_i \geq 0, i = \overline{1,n}, \sum_{i=1}^{n} y_i = 1 \right\}$ *there exists a solution* $y_0 = \{y_k^0\}_{k=1}^{n}$

of the system of equations

$$\frac{y_k + y_k \sum_{i=1}^{n} \frac{a_{ki}}{b_k} y_i}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k \frac{a_{ki}}{b_k} y_i} = y_k, \quad k = \overline{1,n}. \tag{20}$$

For those k, for which $y_k^0 > 0$, the equalities

$$\frac{1 + \sum_{i=1}^{n} \frac{a_{ki}}{b_k} y_i^0}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k^0 \frac{a_{ki}}{b_k} y_i^0} = 1, \quad k = \overline{1,n}, \tag{21}$$

are true and for those k, for which $y_k^0 = 0$, the inequalities

$$\frac{1 + \sum_{i=1}^{n} \frac{a_{ki}}{b_k} y_i^0}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k^0 \frac{a_{ki}}{b_k} y_i^0} \le 1, \quad k = \overline{1,n}, \tag{22}$$

are valid.

Proof. Based on Theorem 4, for every $\varepsilon > 0$ there exists a solution $y^{\varepsilon} = \{y_k^{\varepsilon}\}_{k=1}^{n}$. of the system of equations (16), which is strictly positive. Due to the compactness of the set Y, there exists a subsequence $\varepsilon_m > 0$, such that $y^{\varepsilon_m} = \{y_k^{\varepsilon_m}\}_{k=1}^{n}$ goes to $y^0 = \{y_k^0\}_{k=1}^{n} \in Y$. Let the index k be such that $y_k^0 > 0$, then

$$\frac{y_k^0 + y_k^0 \sum_{i=1}^{n} \frac{a_{ki}}{b_k} y_i^0}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k^0 \frac{a_{ki}}{b_k} y_i^0} = y_k^0. \tag{23}$$

After reducing by $y_k^0 > 0$, we get the equality (21). Now let the index k be the one for which $y_k = 0$. Then the inequality follows from the system of equations (16) for every $\varepsilon > 0$.

$$\frac{y_k^{\varepsilon} + y_k^{\varepsilon} \sum_{i=1}^{n} \frac{a_{ki}}{b_k} y_i^{\varepsilon}}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k^{\varepsilon} \frac{a_{ki}}{b_k} y_i^{\varepsilon} + n\varepsilon} \le y_k^{\varepsilon}, \quad k = \overline{1,n}. \tag{24}$$

After reducing by $y_k^{\varepsilon} > 0$, we get the inequality.

Going to the limit in the inequality (25), we get the inequality

$$\frac{1 + \sum_{i=1}^{n} \frac{a_{ki}}{b_k} y_i^0}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} y_k^0 \frac{a_{ki}}{b_k} y_i^0} \le 1. \tag{26}$$

Theorem 5 is proved.

Consequence 1 *Let the quadratic form* $\sum_{k=1}^{n} \sum_{i=1}^{n} y_k \frac{a_{ki}}{b_k} y_i$ *be strictly positive on the set Y. Then there exists a solution* $z_0 = \{z_i^0\}_{i=1}^{n}$, *of the system of inequalities*

$$\sum_{i=1}^{n} a_{ki} z_i^0 \le b_k, \quad k = \overline{1,n}, \tag{27}$$

where $z_i^0 = \dfrac{y_i^0}{\sum_{k=1}^{n}\sum_{i=1}^{n} y_k^0 \dfrac{a_{ki}}{b_k} y_i^0}$, and $y_0 = \{y_i^0\}_{i=1}^n$ is a solution of the system of equations (20).

Consequence 2 *Let I be a set of indices* $k \in N_0 = [1,2,\cdots,n]$ *such that* $z_k^0 > 0$, *and* $J = N_0 \setminus I$. *The vector* $z_0^I = \{z_k^0\}_{k\in I}$ *satisfies the system of equations*

$$\sum_{i\in I} a_{ki} z_i^0 = b_k, \quad k \in I, \tag{28}$$

and inequalities

$$\sum_{i\in I} a_{ki} z_i^0 < b_k, \quad k \in J. \tag{29}$$

It is evident that the set *I* is non empty one.

Theorem 6 *Let I be a nonempty subset of the set* N_0, *and the minor* $\|a_{ij}\|_{i\in I, j\in I}$ *of the nonnegative matrix* $\|a_{ij}\|_{i,j=1}^n$ *is indecomposable and the vector b is strictly positive. The system of equations*

$$\sum_{i\in I} a_{ki} \frac{p_i b_i}{\sum_{s\in I} a_{si} p_s} = b_k, \quad k \in I, \tag{30}$$

and inequalities

$$\sum_{i\in I} a_{ki} \frac{p_i b_i}{\sum_{s\in I} a_{si} p_s} < b_k, \quad k \in J, \tag{31}$$

is always solvable in the set of strictly positive solutions with respect to the vector $p^I = \{p_i\}_{i\in I}$. There is one to one correspondence between the solutions of the system of equations and inequalities (30), (31) and the solutions of the system of equations and inequalities (28), (29) from Corollary 2.

Proof. Let *I* be a nonempty set. If $p_0^I = \{p_i^0\}_{i\in I}$ is a strictly positive solution of the system of equations and inequalities (30), (31), then introducing the denotation

$$z_i^0 = \frac{p_i^0 b_i}{\sum_{s\in I} a_{si} p_s^0}, i \in I,$$

we get a proof in one direction, i.e. $z_0^I = \{z_i^0\}_{i\in I}$ is a strictly positive solution of the system of equations and inequalities (28), (29) from Corollary 2. Let us first prove that there is no state of economic equilibrium under the condition that *I* is an empty set.

From the opposite. Let there exist a nonzero solution $p = \{p_i\}_{i=1}^n$ of the system of inequalities

$$\sum_{i=1}^{n} a_{ki} \frac{p_i b_i}{\sum_{s=1}^{n} a_{si} p_s} < b_k, \quad k = \overline{1,n}. \tag{32}$$

Introduce the denotation $y_i = \frac{p_i}{\sum_{s=1}^{n} a_{si} p_s}, i = \overline{1,n}$. Then the nonzero solution $p = \{p_i\}_{i=1}^{n}$ satisfies the system of equations

$$p_i = y_i \sum_{s=1}^{n} a_{si} p_s, \quad i = \overline{1,n}. \tag{33}$$

In the n-dimensional space R^n of vectors $p = \{p_i\}_{i=1}^{n}$ we introduce the norm $\|p\| = \sum_{s=1}^{n} b_s |p_s|$ and estimate the norm of the operator $[AY]^{\mathrm{T}}$, where $Y = \|\delta_{ij} y_i\|_{i,j=1}^{n}$. We have

$$\|AYp\| = \sum_{i=1}^{n} b_i y_i \left| \sum_{s=1}^{n} a_{si} p_s \right| \le \max_s \sum_{i=1}^{n} b_i y_i \frac{a_{si}}{b_s} \|p\|.$$

Hence $\left\|[AY]^{\mathrm{T}}\right\| \le \max_s \sum_{i=1}^{n} b_i y_i \frac{a_{si}}{b_s} < 1$. The last inequality holds due to the fulfillment of inequalities (32). This means that the system of equations (33) has only zero solution. Contradiction.

Therefore, let I be a nonempty set and there exists a strictly positive solution $z_0^I = \{z_i^0\}_{i \in I}$ of the system of equations and inequalities (28), (29) from Corollary 2. Let's put

$$z_i^0 = \frac{p_i b_i}{\sum_{s \in I} a_{si} p_s}, \quad i \in I. \tag{34}$$

With respect to the vector $p^I = \{p_i\}_{i \in I}$, we obtain a system of equations

$$p_i = y_i \sum_{s \in I} a_{si} p_s, \quad i \in I, \tag{35}$$

where the denotations $y_i = \frac{z_i^0}{b_i}, i \in I$, are introduced. Let us prove the existence of a strictly positive solution of the system of equations (35).

Consider the nonlinear mapping $\varphi(p) = \{\varphi_i(p)\}_{i \in I}$

$$\varphi_i(p) = \frac{p_i + y_i \sum_{s \in I} a_{si} p_s}{1 + \sum_{i \in I} y_i \sum_{s \in I} a_{si} p_s}, \quad i \in I, \tag{36}$$

on the set $P = \left\{ p = \{p_i\}_{i \in I}, p_i \ge 0, \sum_{i \in I} p_i = 1 \right\}$. The mapping $\varphi(p)$ is a continuous one of the set P into itself. According to Brauer's Theorem [8], there exists a fixed point $p_0^I = \{p_i^0\}_{i \in I}$ of this mapping in the set P, that is,

$$\frac{p_i^0 + y_i \sum_{s \in I} a_{si} p_s^0}{1 + \sum_{i \in I} y_i \sum_{s \in I} a_{si} p_s^0} = p_i^0, \quad i \in I. \tag{37}$$

This fixed point satisfies the system of equations

$$\lambda p_i^0 = y_i \sum_{s \in I} a_{si} p_s^0, \quad i \in I, \tag{38}$$

where $\lambda = \sum_{i \in I} y_i \sum_{s \in I} a_{si} p_s^0$. Let us show that $\lambda = 1$. Multiplying both the left and right parts of the equalities by b_i and summing over $i \in I$, we get

$$\lambda \sum_{i \in I} p_i^0 b_i = \sum_{s \in I} \left(\sum_{i \in I} a_{si} z_i^0 \right) p_s^0 = \sum_{s \in I} p_s^0 b_s. \tag{39}$$

Due to the fact that $b_i > 0, i \in I$, $p_0^I \neq 0$ we have $\sum_{s \in I} p_s^0 b_s > 0$. From the equality (39) we get that $\lambda = 1$.

The strict positivity of the vector $p_0^I = \left\{ p_i^0 \right\}_{i \in I}$ follows from the non-negativity of the matrix $\left\| a_{ij} \right\|_{i,j=1}^n$ and indecomposability of the minor $\left\| a_{ij} \right\|_{i \in I, j \in I}$. Indeed, denote by $A^I(y)$ the operator whose components are given by the formulas $\left[A^I(y) p \right]_i = y_i \sum_{s \in I} a_{si} p_s, i \in I$. Then the system of equations (38) in operator form can be written as follows $p_0^I = A^I(y) p_0^I$. Hence the vector p_0^I satisfies the system of equations $p_0^I = \left[A^I(y) \right]^{|I|-1} p_0^I$, where $|I|$ is the power of the set *I*. Due to the indecomposability of the minor $\left\| a_{ij} \right\|_{i \in I, j \in I}$, the matrix corresponding to the operator $\left[A^I(y) \right]^{|I|-1}$ is strictly positive. From here we get the strict positivity of p_0^I. Theorem 6 is proved.

Theorem 7 *Let* $A = \left\| a_{ij} \right\|_{i,j=1}^n$ *be a nonnegative matrix and I be a nonempty set. For any strictly positive vector* $\bar{b} \leq b$ $\bar{b}_i = b_i, i \in I$, $\bar{b}_i < b_i, i \in J$, $I \neq \varnothing$, $J = N_0 \setminus I$, *and such that it belongs to the interior of the cone formed by the column vectors* $a_i = \left\{ a_{ki} \right\}_{k=1}^n, i \in I$, *of the non-negative matrix* $\left\| a_{ij} \right\|_{i,j=1}^n$, *a minor* $\left\| a_{ij} \right\|_{i \in I, j \in I}$ *of which is indecomposable, there exists a state of economic equilibrium, that is, there is an equilibrium price vector* $p_0 = \left\{ p_i^0 \right\}_{i=1}^n$ *such that*

$$\sum_{i=1}^n a_{ki} \frac{p_i^0 b_i}{\sum_{s=1}^n a_{si} p_s^0} \leq b_k, \quad k = \overline{1,n}. \tag{40}$$

Proof. Under the conditions of Theorem 7

$$\bar{b} = \sum_{i \in I} a_{ki} z_i^0, \quad k = \overline{1,n}, \tag{41}$$

where $z_i^0 > 0, i \in I$. Therefore, there exists a solution of the system of equations and inequalities (28), (29) from Corollary 2, i.e.

$$\sum_{i \in I} a_{ki} z_i^0 = b_k, \quad k \in I. \tag{42}$$

$$\sum_{i \in I} a_{ki} z_i^0 < b_k, \quad k \in J. \tag{43}$$

Based on Theorem 6, there exists an equilibrium vector of prices. Theorem 7 is proved.

Theorem 7 provides a partial description of all equilibrium states of the economic system described by the "input-output" model. Note that the vector $b-\bar{b}$, if it is nonzero, does not find consumers in the market of goods and services. Vector $\bar{b}$, as before, will be called the vector of real consumption.

Theorem 8 *Let b be a strictly positive vector that does not belong to the positive cone formed by the column vectors of the non negative matrix A. The necessary and sufficient conditions of the solution existence of the system of equations and inequalities*

$$\sum_{j=1}^{n} a_{ij} z_j = b_i, \quad i \in I, \tag{44}$$

$$\sum_{j=1}^{n} a_{ij} z_j < b_i, \quad i \in J, \tag{45}$$

where I is a nonempty set, is a condition: there exists a non-negative vector $y = \{y_i\}_{i=1}^{n}$, $Ay \neq 0$ and such that

$$a\sum_{j=1}^{n} a_{ij} y_j = b_i, \quad i \in I,$$

where $a = \min_{1\le i\le n} \frac{b_i}{\sum_{j=1}^{n} a_{ij} y_j}$.

Proof. Necessity. Let there exist a solution of the system of equations (44) and inequalities (45) of Theorem 8. Then

$$a = \min_{1\le i\le n} \frac{b_i}{\sum_{j=1}^{n} a_{ij} z_j} = 1,$$

and, in addition,

$$\frac{b_i}{\sum_{j=1}^{n} a_{ij} z_j} = 1, \quad i \in I,$$

$$\frac{b_i}{\sum_{j=1}^{n} a_{ij} z_j} > 1, \quad i \in J.$$

It is obvious that $Az \neq 0$. The necessity is established.

Sufficiency. For any non-negative vector $y = \{y_i\}_{i=1}^{n}$, $Ay \neq 0$ let $a = \min_{1\le i\le n} \frac{b_i}{\sum_{j=1}^{n} a_{ij} y_j}$. Then $a < \infty$. We put

$$I = \left\{ i, \frac{b_i}{\sum_{j=1}^{n} a_{ij} y_j} = a \right\}.$$

Then I is a nonempty set and the inequalities hold

$$\sum_{j=1}^{n} a_{ij} z_j = b_i, \quad i \in I,$$

$$\sum_{j=1}^{n} a_{ij} z_j < b_i, \quad i \in J,$$

where $z = \{z_i\}_{j=1}^{n}$, $z_i = a y_i, i = \overline{1,n}$. Theorem 8 is proved.

We give a complete description of the solutions of the system of equations and inequalities (28), (29). We denote by $a_i = \{a_{ki}\}_{i=1}^{n}, i = \overline{1,n}$, the *i*-th column of the matrix *A*. Let us consider the numbers $d_i = \min_{1 \le i \le n} \frac{b_k}{a_{ki}}, i = \overline{1,n}$.

Theorem 9 *Let the strictly positive vector b not belong to the cone formed by the column vectors* $a_i = \{a_{ki}\}_{i=1}^{n}, i = \overline{1,n}$, *of the non-negative non-decomposable matrix A. Any solution of the system of equations (28) and inequalities (29) is given by the formula*

$$z = \{a(\alpha) \alpha_i d_i\}_{i=1}^{n},$$

where $\alpha = \{\alpha_i\}_{i=1}^{n} \in Q = \left\{ \alpha = \{\alpha_i\}_{i=1}^{n}, \alpha_i \ge 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}$,

$$a(\alpha) = \min_{1 \le k \le n} \frac{b_k}{\left[\sum_{i=1}^{n} \alpha_i d_i a_i \right]_k} \ge 1.$$

The function $a(\alpha)$ is bounded and convex down on the set *Q*.

Proof. Let $z_0 = \{z_i^0\}_{i=1}^{n}$ be a certain vector that is a solution of the system of equations (28) and inequalities (29). Let's denote

$$\alpha_i = \frac{\frac{z_i^0}{d_i}}{\sum_{j=1}^{n} \frac{z_j^0}{d_j}}, \quad i = \overline{1,n}.$$

Then

$$A z_0 = \sum_{i=1}^{n} a_i z_i^0 = \sum_{j=1}^{n} \frac{z_j^0}{d_j} \sum_{i=1}^{n} \alpha_i d_i a_i.$$

Because of

$$\min_{1 \le k \le n} \frac{b_k}{[A z_0]_k} = 1,$$

we get

$$\sum_{j=1}^{n} \frac{z_j^0}{d_j} = \min_{1 \le k \le n} \frac{b_k}{\left[\sum_{i=1}^{n} \alpha_i d_i a_i \right]_k}.$$

It is obvious that, conversely, every vector $\alpha = \{\alpha_i\}_{i=1}^{n} \in Q$ corresponds to solution of the system of equations (28) and inequalities (29), which is given by the formula

$$z = \{a(\alpha) \alpha_i d_i\}_{i=1}^{n},$$

where

$$a(\alpha)=\min_{1\le k\le n}\frac{b_k}{\left[\sum_{i=1}^{n}\alpha_i d_i a_i\right]_k}.$$

Let us establish that $a(\alpha)\ge 1$. It is obvious that $a_i d_i \le b$. Multiplying by $\delta_i \ge 0$ the left and right parts of the last inequality and summing over i and assuming that $\sum_{i=1}^{n}\delta_i > 0$ we will get

$$\frac{\sum_{i=1}^{n}\delta_i d_i a_i}{\sum_{i=1}^{n}\delta_i}\le b.$$

Denoting $\alpha_i=\frac{\delta_i}{\sum_{i=1}^{n}\delta_i}, i=\overline{1,n}$, we get what we need. It follows from the indecomposability of the matrix A that for every index $1\le i\le n$ there exists an index k such that

$$\sum_{j=1}^{n}a_{kj}z_j^0\ge a_{ki}z_i^0,$$

where $a_{ki}>0$. Hence

$$z_i^0\le\frac{b_k}{a_{ki}}\le\frac{\max_{1\le k\le n}b_k}{\min_{a_{ki}>0}a_{ki}}=c_0<\infty.$$

Due to the arbitrariness of the solution $z_0=\left\{z_i^0\right\}_{i=1}^{n}$ of the system of equations (28) and inequalities (29), we obtain

$$a(\alpha)\alpha_i d_i\le c_0.$$

Or

$$a(\alpha)\alpha_i\le\frac{c_0}{d_i}.$$

After summing over the index i, we get

$$a(\alpha)\le\sum_{i=1}^{n}\frac{c_0}{d_i}.$$

The boundedness of $a(\alpha)$ is established. Let's prove the convexity down. Because of

$$\gamma\frac{b_k}{\sum_{i=1}^{n}a_{ki}\alpha_i^1 d_i}+(1-\gamma)\frac{b_k}{\sum_{i=1}^{n}a_{ki}\alpha_i^2 d_i}\ge\frac{b_k}{\sum_{i=1}^{n}a_{ki}\left[\gamma\alpha_i^1+(1-\gamma)\alpha_i^2\right]d_i},$$

we have

$$\gamma\min_{1\le k\le n}\frac{b_k}{\sum_{i=1}^{n}a_{ki}\alpha_i^1 d_i}+(1-\gamma)\min_{1\le k\le n}\frac{b_k}{\sum_{i=1}^{n}a_{ki}\alpha_i^2 d_i}\ge\min_{1\le k\le n}\frac{b_k}{\sum_{i=1}^{n}a_{ki}\left[\gamma\alpha_i^1+(1-\gamma)\alpha_i^2\right]d_i}.$$

The latter means

$$\gamma a\left(\alpha^1\right)+\left(1-\gamma\right)a\left(\alpha^2\right)\geq a\left(\gamma\alpha^1+\left(1-\gamma\alpha^2\right)\right),$$

where $0\leq\gamma\leq 1$. Theorem 9 is proved.

Proposition 1 *In the set of solutions* $Z_0=\left\{z_0=\left\{z_i^0\right\}_{i=1}^n=\left\{a\left(\alpha\right)\alpha_i d_i\right\}_{i=1}^n,\alpha\in Q\right\}$ *of the system of equations (28) and inequalities (29) there exists a minimum of the function*

$$W\left(\alpha\right)=\sum_{k=1}^n\left[b_k-a\left(\alpha\right)\sum_{i=1}^n a_{ki}\alpha_i d_i\right]^2.$$

This minimum is global on the set of all solutions of the system of inequalities (27), i.e.

$$\min_{\alpha\in Q}W\left(\alpha\right)=\min_{z\in Z}\sum_{k=1}^n\left[b_k-\sum_{i=1}^n a_{ki}z_i\right]^2,$$

where Z is the set of all solutions of the system of inequalities (27).

Proof. The function $W\left(\alpha\right)$ is continuous on the closed bounded set Q, because so is the function $a\left(\alpha\right)$ due to its convexity and boundedness. According to the Weierstrass Theorem, there exists a minimum of the function $W\left(\alpha\right)$. For any solution $z=\left\{z_i\right\}_{i=1}^n\in Z$ let's denote

$$\alpha_i=\frac{\frac{z_i}{d_i}}{\sum_{j=1}^n\frac{z_j}{d_j}},\quad i=\overline{1,n}.$$

Then

$$Az=\sum_{i=1}^n a_i z_i=\sum_{j=1}^n\frac{z_j}{d_j}\sum_{i=1}^n\alpha_i d_i a_i\leq b.$$

From here

$$\sum_{j=1}^n\frac{z_j}{d_j}\leq\min_{1\leq k\leq n}\frac{b_k}{\left[\sum_{i=1}^n\alpha_i d_i a_i\right]_k}=a\left(\alpha\right).$$

Therefore

$$\sum_{k=1}^n\left[b_k-\sum_{i=1}^n a_{ki}z_i\right]^2\geq\sum_{k=1}^n\left[b_k-a\left(\alpha\right)\sum_{i=1}^n a_{ki}\alpha_i d_i\right]^2\geq\min_{\alpha\in Q}\sum_{k=1}^n\left[b_k-a\left(\alpha\right)\sum_{i=1}^n a_{ki}\alpha_i d_i\right]^2.$$

Taking the minimum of $z\in Z$, we have

$$\min_{z\in Z}\sum_{k=1}^n\left[b_k-\sum_{i=1}^n a_{ki}z_i\right]^2\geq\min_{\alpha\in Q}\sum_{k=1}^n\left[b_k-a\left(\alpha\right)\sum_{i=1}^n a_{ki}\alpha_i d_i\right]^2.$$

The inverse inequality is obvious due to the inclusion $Z\supset Z_0$, where Z_0 is the set of solutions of the system of equations (28) and inequalities (29). Proposition 1 is proved.

Note that the set of non-negative solutions Z of the system of equations and inequalities (27) is a closed bounded convex set. Indeed, the boundedness of the set of solutions follows from the indecomposability of the matrix A and the finiteness of the vector b. The convexity of the set of solutions is obvious. Then such a set is a convex combination of its extreme points. Such extreme points are some subset of the solutions constructed in Theorem 9, which do not have common equalities. Below we establish a number of theorems that will allow us to put into correspondence the vector of prices to every vector $z=\{z_i\}_{i=1}^n$, that satisfies the system of equations and inequalities

$$\sum_{j=1}^n a_{ij} z_j = b_i, \quad i \in I, \tag{46}$$

$$\sum_{j=1}^n a_{ij} z_j < b_i, \quad i \in J, \tag{47}$$

where I is a nonempty set.

Theorem 10 *Let* $A=\|a_{ij}\|_{ij=1}^n$ *be a strictly positive matrix, and* $z=\{z_i\}_{i=1}^n$ *be a nonnegative nonzero vector. Then there exists a solution to the system of equations*

$$\sum_{i=1}^n a_{ki} \frac{\bar{b}_i p_i}{\sum_{s=1}^n a_{si} p_s} = \bar{b}_k \quad k=\overline{1,n}, \tag{48}$$

in the set $P=\left\{p=\{p_i\}_{i=1}^n,\ p_i \geq 0,\ i=\overline{1,n},\ \sum_{i=1}^n p_i = 1\right\}$, where $\bar{b} = Az = \{\bar{b}_i\}_{i=1}^n$.

Proof. Consider the system of nonlinear equations

$$\frac{p_i + \frac{z_i}{\bar{b}_i}\sum_{s=1}^n a_{si} p_s}{1+\sum_{i=1}^n \frac{z_i}{\bar{b}_i}\sum_{s=1}^n a_{si} p_s} = p_i, \quad i=\overline{1,n},$$

in the set $P=\left\{p=\{p_i\}_{i=1}^n,\ p_i \geq 0,\ i=\overline{1,n},\ \sum_{i=1}^n p_i = 1\right\}$. The left part of this system of equations is a continuous mapping of the set P into itself. According to Brouwer's Theorem, there exists a fixed point $p_0=\{p_i^0\}_{i=1}^n$ of the mapping given by the left part of this system of equations. Or

$$z_i \sum_{s=1}^n a_{si} p_s^0 = \lambda \bar{b}_i p_i^0, \quad i=\overline{1,n},$$

where $\lambda = \sum_{i=1}^n \frac{z_i}{\bar{b}_i}\sum_{s=1}^n a_{si} p_s^0$. Summing over the index i, we will have the left and right parts of the above equalities

$$\sum_{i=1}^n \bar{b}_i p_i^0 = \lambda \sum_{i=1}^n \bar{b}_i p_i^0.$$

It is obvious that the vector p_0 is strictly positive, and $\sum_{i=1}^n \bar{b}_i p_i^0 > 0$, therefore $\lambda = 1$ and

$$z_i = \frac{\bar{b}_i p_i^0}{\sum_{s=1}^n a_{si} p_s^0}, \quad i=\overline{1,n}.$$

Using the fact that

$$\sum_{i=1}^{n} a_{ij} z_j = \bar{b}_i, \quad i = \overline{1,n},$$

and substituting z_i in the equality above, we get the required result. This proves Theorem 10.

It can be proved similarly

Theorem 11 *Let* $A = \|a_{ij}\|_{ij=1}^{n}$ *be a positive indecomposable matrix, and* $z = \{z_i\}_{i=1}^{n}$ *be a strictly positive vector. Then there is a solution to the system of equations*

$$\sum_{i=1}^{n} a_{ki} \frac{\bar{b}_i p_i}{\sum_{s=1}^{n} a_{si} p_s} = \bar{b}_k, \quad k = \overline{1,n},$$

in the set $P = \left\{ p = \{p_i\}_{i=1}^{n}, p_i \ge 0, i = \overline{1,n}, \sum_{i=1}^{n} p_i = 1 \right\}$, where $\bar{b} = Az = \{\bar{b}_i\}_{i=1}^{n}$.

Theorem 12 *Let* $A = \|a_{ij}\|_{ij=1}^{n}$ *be a nonnegative nonzero matrix. The necessary and sufficient conditions for the existence of a solution of the system of equations*

$$\sum_{i=1}^{n} a_{ki} \frac{\bar{b}_i p_i}{\sum_{s=1}^{n} a_{si} p_s} = \bar{b}_k, \quad k = \overline{1,n},$$

in the set P is the existence of a solution to the system of equations

$$z_i = \frac{\bar{b}_i p_i^0}{\sum_{s=1}^{n} a_{si} p_s^0}, \quad i = \overline{1,n},$$

in the set P for some non negative nonzero vector $z = \{z_i\}_{i=1}^{n}$ and $\bar{b} = Az = \{\bar{b}_i\}_{i=1}^{n}$.

Proof. The proof is obvious.

Theorem 13 *Let* $A = \|a_{ij}\|_{ij=1}^{n}$ *be a non-negative nonzero matrix and let* $z = \{z_i\}_{i=1}^{n}$ *be a non-negative vector, which is a solution to the system of in-equalities*

$$\sum_{i=1}^{n} a_{ij} z_j = b_i, \quad i \in I, \tag{49}$$

$$\sum_{i=1}^{n} a_{ij} z_j < b_i, \quad i \in J, \tag{50}$$

where I and J are nonempty sets. If b is a strictly positive vector, and $z_j > 0$ for some $j \in J$, then there is no solution to the system of equations

$$z_i = \frac{b_i p_i}{\sum_{s=1}^{n} a_{si} p_s}, \quad i = \overline{1,n}, \tag{51}$$

in the set $P = \left\{ p = \{p_i\}_{i=1}^{n}, p_i \ge 0, i = \overline{1,n}, \sum_{i=1}^{n} p_i = 1 \right\}$.

Proof. Argument from the opposite. Suppose that there is a nonzero nonnegative vector $p_0 = \{p_i^0\}_{i=1}^n$ which is a solution of the system of equations of Theorem 13. Then $\sum_{s=1}^n a_{si} p_s^0 \neq 0$ and

$$z_i \sum_{s=1}^n a_{si} p_s^0 = b_i p_i^0, \quad i = \overline{1,n}.$$

Hence we have $p_j^0 > 0$ for $j \in J$ specified in Theorem 13. Summing up the left and right parts by *i*, we get

$$\sum_{s=1}^n \left(\sum_{i=1}^n a_{si} z_j \right) p_s^0 = \sum_{i=1}^n b_i p_i^0.$$

But

$$\sum_{s=1}^n \left(\sum_{i=1}^n a_{si} z_j \right) p_s^0 < \sum_{i=1}^n b_i p_i^0.$$

Contradiction. This proves the theorem 13.

Definition 8 *Let A be a nonnegative matrix, and the vector b is strictly positive, which does not belong to the cone formed by the column vectors of the matrix A. We say that the vector* $z = \{z_i\}_{i=1}^n$, *which is a solution of the system of inequalities*

$$\sum_{j=1}^n a_{ij} z_j = b_i, \quad i \in I, \tag{52}$$

$$\sum_{j=1}^n a_{ij} z_j < b_i, \quad i \in J, \tag{53}$$

where I is a nonempty set, corresponds to the equilibrium price vector if there exists a solution to the system of equations

$$\sum_{i=1}^n a_{ki} \frac{\bar{b}_i p_i}{\sum_{s=1}^n a_{si} p_s} = \bar{b}_k, \quad k = \overline{1,n}, \tag{54}$$

in the set $P = \left\{ p = \{p_i\}_{i=1}^n, p_i \geq 0, i = \overline{1,n}, \sum_{i=1}^n p_i = 1 \right\}$, where $\bar{b} = Az = \{\bar{b}_i\}_{i=1}^n$.

Consequence 3 *There is no equilibrium state corresponding to the vector* $z = \{z_i\}_{i=1}^n$ *which is a solution of the system of equations (51) of Theorem 13 with respect to the vector* $p = \{p_i\}_{i=1}^n$.

Proof. Indeed, if it were not so, then the solution of the system of equations (51) of Theorem 13 with respect to the vector $p = \{p_i\}_{i=1}^n$ should exist. But it is not so.

Theorem 14 is the basis for the determining of the equilibrium price vector in the case of partial clearing of markets.

Theorem 14 *Let* $\|a_{ij}\|_{i,j=1}^n$ *be a nonnegative matrix,* $b = \{b_i\}_{i=1}^n$ *be a strictly positive vector, and let the matrix* $\|a_{ij}\|_{i,j \in I}$ *be an indecomposable one for a certain non empty set I. Then the equilibrium price vector* $p = \{p_i\}_{i=1}^n$, *being a solution of the system of equations and inequalities*

$$\sum_{j=1}^{n} a_{ij} \frac{b_j p_j}{\sum_{s=1}^{n} a_{sj} p_s} = b_i, \quad i \in I,$$

$$\sum_{j=1}^{n} a_{ij} \frac{b_j p_j}{\sum_{s=1}^{n} a_{sj} p_s} < b_i, \quad i \in J, \tag{55}$$

in the set $P = \left\{ p = \{p_i\}_{i=1}^{n}, p_i \geq 0, i = \overline{1,n}, \sum_{i=1}^{n} p_i = 1 \right\}$ is a solution of the system of equations

$$z_i = \frac{\bar{b}_i p_i}{\sum_{s=1}^{n} a_{si} p_s}, \quad i = \overline{1,n}, \tag{56}$$

where $\bar{b} = \{\bar{b}_i\}_{i=1}^{n}$, $\bar{b} = Az$, the vector $z = \{z_i\}_{i=1}^{n}$ is determined as follows $z_i = z_i^0, i \in I$, $z_i = 0, i \in J$. The vector $z_0^I = \{z_i^0\}_{i \in I}$ satisfies the system of equations and inequalities

$$\sum_{j \in I} a_{ij} z_j^0 = b_i, \quad i \in I, \tag{57}$$

$$\sum_{j \in I} a_{ij} z_j^0 < b_i, \quad i \in J. \tag{58}$$

Proof. Let there exist a solution of the system of equations and inequalities (55) with respect to the vector $p = \{p_i\}_{i=1}^{n}$ in the set *P*. Let us show that $p_i = 0, i \in J$. We lead the proof from the opposite. Let at least one component $p_k, k \in J$, of the vector *p* be strictly positive. Then, multiplying by $p_i, i = \overline{1,n}$, the *i*-th equation or inequality and summing the left and right parts, respectively, we get the inequality $\sum_{j=1}^{n} b_j p_j < \sum_{i=1}^{n} b_i p_i$. Due to the strict positivity of the vector *b*, this inequality is impossible, because $\sum_{j=1}^{n} b_j p_j > 0$. Therefore, our assumption is not correct, and therefore $p_i = 0, i \in J$. The remaining component $p_i, i \in I$, of the vector *p* is a solution of the system of equations and inequalities

$$\sum_{j \in I} a_{ij} \frac{b_j p_j}{\sum_{s \in I} a_{sj} p_s} = b_i, \quad i \in I, \tag{59}$$

$$\sum_{j \in I} a_{ij} \frac{b_j p_j}{\sum_{s \in I} a_{sj} p_s} < b_i, \quad i \in J. \tag{60}$$

Let us introduce the denotation

$$z_j^0 = \frac{b_j p_j}{\sum_{s \in I} a_{sj} p_s}, \quad j \in I. \tag{61}$$

It is evident that the equalities and inequalities

$$\sum_{j \in I} a_{ij} z_j^0 = b_i. \quad i \in I,$$

$$\sum_{j\in I} a_{ij} z_j^0 < b_i, \quad i \in J,$$

are valid. If we introduce a vector $z = \{z_i\}_{i=1}^n$ where $z_i = z_i^0, i \in I$, $z_i = 0, i \in J$, then we obtain a system of equations and inequalities

$$\sum_{j=1}^n a_{ij} z_j = b_i, \quad i \in I,$$

$$\sum_{j=1}^n a_{ij} z_j < b_i, \quad i \in J.$$

The price vector $p = \{p_i\}_{i=1}^n$ is a solution of the system of equations (56), due to the fact that $\bar{b} = Az$, and therefore is also a solution of the system equations

$$\sum_{j=1}^n a_{ij} \frac{\bar{b}_j p_j}{\sum_{s=1}^n a_{sj} p_s} = \bar{b}_i, \quad i = \overline{1,n}. \tag{62}$$

Theorem 14 is proved.

Theorem 15 *Let the matrix* $\|a_{ij}\|_{i,j=1}^n$ *be nonnegative, the vector* $b = \{b_i\}_{i=1}^n$ *be strictly positive, and let the matrix* $\|a_{ij}\|_{i,j\in I}$ *be indecomposable for some nonempty set I. The necessary and sufficient condition of existence of the equilibrium price vector* $p = \{p_i\}_{i=1}^n$, *which is a solution of the system of equations and inequalities*

$$\sum_{j=1}^n a_{ij} \frac{b_j p_j}{\sum_{s=1}^n a_{sj} p_s} = b_i, \quad i \in I,$$

$$\sum_{j=1}^n a_{ij} \frac{b_j p_j}{\sum_{s=1}^n a_{sj} p_s} < b_i, \quad i \in J, \tag{63}$$

in the set $P = \left\{ p = \{p_i\}_{i=1}^n, p_i \geq 0, i = \overline{1,n}, \sum_{i=1}^n p_i = 1 \right\}$, such that $p_i > 0, i \in I$ is the existence of a strictly positive solution of the system of equations and inequalities

$$\sum_{j\in I} a_{ij} z_j = b_i, \quad i \in I,$$

$$\sum_{j\in I} a_{ij} z_j < b_i, \quad i \in J. \tag{64}$$

Proof. Necessity. Let there exist a solution of the system of equations and inequalities (63) with respect to the vector $p = \{p_i\}_{i=1}^n$ in the set P and such that $p_i > 0, i \in I$. Whereas in the previous Theorem 14 we establish that $p_i = 0, i \in J$. The remaining component $p_i, i \in I$, of the vector p is the solution of the system of equations and inequalities

$$\sum_{j\in I} a_{ij} \frac{b_j p_j}{\sum_{s\in I} a_{sj} p_s} = b_i, \quad i \in I, \tag{65}$$

$$\sum_{j\in I} a_{ij} \frac{b_j p_j}{\sum_{s\in I} a_{sj} p_s} < b_i, \quad i \in J. \tag{66}$$

Let's introduce the denotation

$$z_i = \frac{b_i p_i}{\sum_{s\in I} a_{si} p_s}, \quad i \in I.$$

Then the equalities and inequalities

$$\sum_{j\in I} a_{ij} z_j = b_i \quad i \in I,$$

$$\sum_{j\in I} a_{ij} z_j < b_i \quad i \in J.$$

are valid. It is obvious that $z_i > 0, i \in I$. The necessity is established.

Sufficiency. If there exists a strictly positive solution of the system of equations and inequalities (64), then the conditions of Theorem 6 are true, that is, there exists a strictly positive solution of the system of equations

$$\sum_{i\in I} a_{ki} \frac{p_i b_i}{\sum_{s\in I} a_{si} p_s} = b_k, \quad k \in I, \tag{67}$$

and inequalities

$$\sum_{i\in I} a_{ki} \frac{p_i b_i}{\sum_{s\in I} a_{si} p_s} < b_k, \quad k \in J. \tag{68}$$

Let's construct the equilibrium vector of prices $p = \{p_i\}_{i=1}^n$ by setting $p_i = 0, i \in J$, and choosing the components $p_i, i \in I$, to be equal to the corresponding components of solution of the system of equations and inequalities (67), (68). The price vector constructed in this way is the solution of the system of equations and inequalities (63). Theorem 15 is proved.

For any vector $z_0 \in Z$, let

$$\bar{b} = \sum_{i=1}^n z_i^0 a_i. \tag{69}$$

If $I = \{i, \bar{b}_i = b_i\}$ is a nonempty set, then the vector $\bar{b}$ will be called the vector of real consumption. In accordance with Theorems 10 - 12, it corresponds to the equilibrium price vector p_0, which is the solution of the system of equations (62).

For a part of the goods, the indices of which are included in the set $J = N_0 \setminus I$, the equilibrium price is $p_i^0 = 0, i \in J$. The latter means that this part of the goods does not find consumers on the market of goods of the economic system. But certain funds were spent on their production, which are called the cost of these goods. Let's introduce the generalized equilibrium vector of prices by putting $p_u = \{p_i^u\}_{i=1}^n$ $p_i^u = p_i^0, i \in I$, $p_i^u = p_i^c, i \in J$, where p_i^c is the cost price of the produced goods that do not find consumers on the market of goods of the economic system. Each such equilibrium state will be matched with the level of excess supply

$$R = \frac{\langle b - \bar{b}, p_u \rangle}{\langle b, p_u \rangle}, \tag{70}$$

where $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, $x = \{x_i\}_{i=1}^{n}$, $y = \{y_i\}_{i=1}^{n}$.

Finding solutions of the system of equations and inequalities (28), (29) with the smallest excess supply will require finding all possible solutions of such a system of equations and inequalities and finding among them the minimum excess supply, which can turn out to be an infeasible problem for large dimensions of the matrix *A*. Based on Theorem 9 and Proposition 1 below, the solution of this problem is proposed as a quadratic programming problem.

Definition 9 *Let A be an indecomposable nonnegative matrix, and let b be a strictly positive vector that does not belong to the cone formed by the column vectors of the matrix A. The solution* z_0 *of the quadratic programming problem*

$$\min_{z\in Z} \sum_{i=1}^{n} \left[b_i - \sum_{k=1}^{n} a_{ik} z_k \right]^2, \tag{71}$$

where $Z = \left\{ z = \{z_i\}_{i=1}^{n}, z_i \geq 0, i = \overline{1,n}, \sum_{i=1}^{n} a_{ki} z_i \leq b_k, k = \overline{1,n} \right\}$ corresponds to the real consumption vector $\bar{b} = A z_0 \leq b$. Assume that for the non-empty set $I = \{i, \bar{b}_i = b_i\}$ there exists an equilibrium price vector $p_0 = \{p_i^0\}_{i=1}^{n}$ which is a solution of the system equations

$$\sum_{i=1}^{n} a_{ki} \frac{p_i^0 \bar{b}_i}{\sum_{s=1}^{n} a_{si} p_s^0} = \bar{b}_k, \quad k = \overline{1,n}. \tag{72}$$

Then the value

$$R = \frac{\langle b - \bar{b}, p_0 \rangle}{\langle b, p_0 \rangle} \tag{73}$$

will be called the generalized excess supply corresponding to the generalized equilibrium vector of prices p_0, where $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, $x = \{x_i\}_{i=1}^{n}$, $y = \{y_i\}_{i=1}^{n}$.

According to the formula (73), the level of excess supply for the generalized equilibrium vector is the smallest. This is the state of economic equilibrium below which the economic system cannot fall.

3. Economic Systems Capable of Operating in Sustainable Mode

A description of sustainable economic development at the macroeconomic level is proposed in [9] [10] [11] [12]. In this section, we formulate the principles of sustainable development at the microeconomic level. Each business project, which starts the production of a certain group of goods, plans to receive added value. In the production process, there are direct costs for production materials and labor costs. If a business project is such that it has sales markets with positive added value, then we say that this business project has contracts with suppliers of materials and raw materials and a certain number of workers who sell their labor power. The totality of concluded contracts will be characterized by technological mapping. Consider a technological mapping that describes the production of the *i*-th type of product by spending a vector of goods $\{a_{ki}\}_{k=1}^n, i=\overline{1,n}$, on one unit of the *i*-th manufactured product. Such a technological mapping will be characterized by the "input-output" matrix $\|a_{ij}\|_{k,i=1}^n$. Our goal is to formulate principles that will ensure sustainable development of the economic system. Suppose that the economic system produces $x_i, i=\overline{1,n}$, units of the i-th product. Then the gross input vector is equal to $X_i=\{x_i a_{ki}\}_{k=1}^n$ and the gross output vector is equal to $Y_i=\{x_i \delta_{ki}\}_{k=1}^n$. Each *i*-th consumer in the economic system is characterized by two vectors $b_i=\{b_{ki}\}_{k=1}^n, i=\overline{1,l}$ and $C_i=\{c_{ki}\}_{k=1}^n, i=\overline{1,l}$. The vectors $b_i=\{b_{ki}\}_{k=1}^n, i=\overline{1,l}$, we call the property vectors. Every *i*-th consumer wants to exchange the vector $b_i=\{b_{ki}\}_{k=1}^n$ for the vector $C_i=\{c_{ki}\}_{k=1}^n$, which we call the demand vector. We assume that in the production process all produced goods are distributed according to the rule

$$\sum_{i=1}^n (Y_i - X_i) = \sum_{i=1}^l b_i. \tag{74}$$

Is there such a market mechanism that would provide such distribution of the product in society for a certain vector of prices?

Definition 10 *The distribution of the product in society will be called economically expedient if, in the process of production and distribution, in accordance with the concluded agreement, the ith consumer owns a set of goods* $b_i=\{b_{ki}\}_{k=1}^n, i=\overline{1,l}$, *so that the vector* $b=\sum_{i=1}^l b_i$ *belongs to the interior of the cone formed by the vectors* $C_i=\{c_{ki}\}_{k=1}^n, i=\overline{1,l}$.

Let's introduce two matrices $C=|c_{ki}|_{k=1,i=1}^{n,l}$ and $B=|b_{ki}|_{k=1,i=1}^{n,l}$.

Definition 11 *If the representation* $B=CB_1$ *is valid for the matrix B and such that there is a solution to the problem*

$$\sum_{k=1}^l b_{ki}^1 d_k = \sum_{s=1}^l b_{is}^1 d_i \tag{75}$$

with respect to the vector $d=\{d_k\}_{k=1}^l$, which belongs to the cone formed by the vectors $C_i^T=\{c_{ki}\}_{k=1}^n$, then we will call the distribution of the product in society rational.

In this work, we adhere to the concept of describing economic systems developed in [3]. The essence of this description is that the supply of firms is primary, and the choice of consumers is secondary. But at the same time, it is important that the structure of the supply corresponds to the structure of the demand. The axioms of this description are presented in [3], where random fields of consumer choice and decision-making by firms are constructed based on these axioms.

We describe firms by technological mappings $y_i=F_i(x_i), x_i \in X_i$ from the CTM class (compact technological mapping) [1], [2] [3], and the demand of the *i* consumer by the product vector $C_i=\{c_{ki}\}_{k=1}^n$, which he wants to consume in a certain period of the economy functioning. Suppose that m firms function in the economic system, which are described by technological mappings $y_i = F_i(x_i)$, $x_i \in X_i$, $i = \overline{1,m}$, from the CTM class. If firms have chosen production processes $x_i \in X_i$, $y_i \in F(x_i)$, $i = \overline{1,m}$, then the produced final product in the economic system will be equal to $\sum_{i=1}^{m}(y_i - x_i)$, and the i-th consumer will receive a vector of goods $b_i, i = \overline{1,l}$, in accordance with the signed contracts. The condition of economic equilibrium is the fulfillment of inequalities

$$\sum_{i=1}^{l} c_{ki} \frac{\langle b_i, p\rangle}{\langle C_i, p\rangle} \le \sum_{i=1}^{l} b_{ki}, \quad k = \overline{1,n}. \tag{76}$$

The main idea that we lay down is the following: firms must produce such a quantity of goods, the sale of which will ensure their functioning in the next production cycle, and for this they must have sufficient income. In addition, the necessary balances for the state must be ensured: state spending on defense, ensuring the freedoms and rights of citizens, public administration, updating fixed assets, funding education, etc.

Is such a distribution of products the result of market exchange based on the existence of an equilibrium vector of prices. Among all the possible distributions of output produced by firms, what matters is when firms will be profitable so that they can undertake the next production cycle. Under the firm we also understand any owner of labor force that he sells on the market of the economic system for the appropriate salary, which he spends on its restoration and satisfaction of other needs. In this case, it is convenient to consider all those engaged in the production of products and services as firms that produce labor and sell it to other firms. We assume that the economic system has l consumers, m firms, $l > m$ and produces n type of goods. In order for the economic system to function, it is necessary to ensure the protection of the rights and freedoms of citizens, public administration, protection from external threats and etc. For this, the income of firms should be taxed. Let us consider the taxation vector $\pi = \{\pi_i\}_{i=1}^{m}$, $0 \le \pi_i < 1$, $i = \overline{1,m}$, whose economic meaning is as follows:
$\pi_i \langle p, y_i - x_i\rangle$ is the part of the income that is withdrawn to finance education, defense, public administration and other institutions for the safe functioning of the state. Here $\langle p, y_i - x_i\rangle = \sum_{k=1}^{n} p_k (y_{ki} - x_{ki})$, where $x_i = \{x_{ki}\}_{k=1}^{n}$, $y_i = \{y_{ki}\}_{k=1}^{n}$, are the input and output vectors and $p = \{p_k\}_{k=1}^{n}$ is an equilibrium price vector.

The general formulation of the problem is as follows: firms implemented production processes (x_i, y_i), $x_i \in X_i$, $y_i \in F_i(x_i)$, $i = \overline{1,m}$ as a result of which the final product $\sum_{i=1}^{m}(y_i - x_i)$ is produced in the economic system, which must be distributed so that the process of production and distribution of products is continuous. A condition for this is a system of equalities

$$\sum_{i=1}^{m} x_{ki} \frac{(1-\pi_i)\langle y_i, p\rangle}{\langle x_i, p\rangle} + \sum_{i=m+1}^{l} \frac{C_{ki} D_i(p)}{\langle C_i, p\rangle} = \sum_{i=1}^{m} y_{ki}, \quad k = \overline{1,n}, \tag{77}$$

$$\sum_{i=m+1}^{l} y_i^0 C_{ki} = \sum_{i=1}^{m} \pi_i y_{ki}, \quad k = \overline{1,n}. \tag{78}$$

Equalities (78) are material balances to ensure public procurement, defense orders, construction of educational institutions, renewal of fixed assets, etc. If we put $D_i(p) = y_i^0 \langle C_i, p\rangle$ and take into account (78), then we get

$$\sum_{i=1}^{m} x_{ki} \frac{(1-\pi_i)\langle y_i, p\rangle}{\langle x_i, p\rangle} = \sum_{i=1}^{m} (1-\pi_i) y_{ki}, \quad k = \overline{1,n}, \tag{79}$$

where we denoted

$$\langle y_i, p\rangle = \sum_{k=1}^{n} y_{ki} p_k, \quad \langle x_i, p\rangle = \sum_{k=1}^{n} x_{ki} p_k,$$

and $\pi = \{\pi_i\}_{i=1}^{m}$ is the taxation vector.

In order for the process of functioning of the economic system to be continuous, it is necessary that there should be an equilibrium vector of prices p_0, so that equalities (77) and inequalities $\langle y_i - x_i, p_0\rangle > 0, i = \overline{1,m}$, were valid.

The ability of the economic system to function continuously will be called the ability to function in the mode of sustainable development.

Below we consider the "input-output" production model. Suppose that x_i^0 units of the i-th product are produced in the economic system, $i = \overline{1,n}$. In this case, the input vector is equal to $X_i = \{x_i^0 a_{ki}\}_{k=1}^{n}$. The output vector is equal to $Y_i = \{x_i^0 \delta_{ki}\}_{k=1}^{n}$. Then the problem (79) can be written in the form

$$\sum_{i=1}^{n} a_{ki} \frac{(1-\pi_i) x_i^0 p_i}{\sum_{s=1}^{n} a_{si} p_s} = (1-\pi_k) x_k^0, \quad k = \overline{1,n}. \tag{80}$$

Denote $(1-\pi_i) x_i^0 = x_i, i = \overline{1,n}$, then the problem (80) and the conditions of profitability are rewritten in the form

$$\sum_{i=1}^{n} a_{ki} \frac{x_i p_i}{\sum_{s=1}^{n} a_{si} p_s} = x_k, \quad k = \overline{1,n}, \tag{81}$$

$$p_i - \sum_{s=1}^{n} a_{si} p_s > 0, \quad i = \overline{1,n}. \tag{82}$$

Theorem 16 *Let* $A = \|a_{ki}\|_{k,i=1}^{n}$ *be a non negative productive indecomposable matrix. The necessary and sufficient conditions for the functioning of the economic system in the mode of sustainable development are that the vector* $x = \{x_i\}_{i=1}^{n}$ *belongs to the interior of the cone formed by the column vectors of the matrix* $A(E-A)^{-1}$.

Proof. Necessity. Assume that there exists a vector of equilibrium prices p_0 that satisfies the system of equations (81) and inequalities (82). Substituting p_i^0 from equalities

$$p_i^0 - \sum_{s=1}^{n} a_{si} p_s^0 = \delta_i^0, \quad \delta_i^0 > 0, \quad i = \overline{1,n}. \tag{83}$$

into the system of equations (81) we get

$$x_k = \sum_{i=1}^{n} a_{ki} \frac{x_i\left(\sum_{s=1}^{n} a_{si} p_s^0 + \delta_i^0\right)}{\sum_{s=1}^{n} a_{si} p_s} = \sum_{i=1}^{n} a_{ki} x_i + \sum_{i=1}^{n} a_{ki} \frac{x_i \delta_i^0}{\sum_{s=1}^{n} a_{si} p_s^0}, \quad k = \overline{1,n}. \tag{84}$$

If we introduce the vector $\alpha = \{\alpha_k\}_{k=1}^{n}$, where $\alpha_k = \frac{x_k \delta_k^0}{\sum_{s=1}^{n} a_{sk} p_s^0} > 0, k = \overline{1,n}$, then we get from of equalities (84) the equality $x = A(E-A)^{-1}\alpha$. The latter proves the necessity.

Sufficiency. From the very beginning, we assume that A^{-1} exists. For a diagonal matrix

$$X = \left\| \delta_{ij} x_j \right\|_{i,j=1}^{n} \tag{85}$$

the representation $X = AB_1$ is true, where $B_1 = A^{-1}X = \left\| a_{ki}^{-1} x_i \right\|_{k,i=1}^{n}$. From the assumptions of Theorem 16 we have

$$b_k^1 = \sum_{i=1}^{n} a_{ki}^{-1} x_i = \left[(E-A)^{-1} \alpha \right]_k > 0, k = \overline{1,n}. \tag{86}$$

We will prove that the system of equations

$$\sum_{k=1}^{n} a_{ki}^{-1} x_i d_k = b_i^1 d_i, \quad i = \overline{1,n}, \tag{87}$$

has a strictly positive solution belonging to the cone formed by the vectors $\{a_{ki}\}_{k=1}^{n}, i = \overline{1,n}$. From (86) we get

$$x_k = \sum_{i=1}^{n} a_{ki} b_i^1. \tag{88}$$

The problem (87) is equivalent to the problem

$$d_k = \sum_{i=1}^{n} a_{ik} \frac{b_i^1}{x_i} d_i = \sum_{i=1}^{n} a_{ik} \frac{b_i^1}{\sum_{k=1}^{n} a_{ik} b_k^1} d_i, \quad i = \overline{1,n}. \tag{89}$$

Let's introduce the denotation

$$u_{ik} = a_{ik} \frac{b_i^1}{\sum_{k=1}^{n} a_{ik} b_k^1}, \quad i,k = \overline{1,n}, \tag{90}$$

and the matrix $U = \left| u_{ik} \right|_{i,k=1}^{n}$. Consider a nonlinear system of equations

$$d_k = \frac{d_k + \sum_{i=1}^{n} u_{ik} d_i}{1 + \sum_{k=1}^{n} \sum_{i=1}^{n} u_{ik} d_i}, \quad k = \overline{1,n}, \tag{91}$$

on the set $D = \left\{ d = \{d_i\}_{i=1}^{n}, d_i \geq 0, i = \overline{1,n}, \sum_{i=1}^{n} d_i = 1 \right\}$.

Thanks to Schauder's theorem [8], there exists a solution of the system of equations (91) in the set D. The system of equations (91) can be written as

$$\sum_{i=1}^{n} u_{ik} d_i = \lambda d_k, \tag{92}$$

where $\lambda = \sum_{k=1}^{n} \sum_{i=1}^{n} u_{ik} d_i$.

We will prove that $\lambda > 0$ and the solution $d = \{d_i\}_{i=1}^{n}$ of the system of equations is strictly positive due to the indecomposability of the matrix A. Indeed, the vector $d = \{d_i\}_{i=1}^{n}$ satisfies the system of equations (92), which can be written in operator form

$$U^{\mathrm{T}} d = \lambda d, \tag{93}$$

or

$$\left[U^{\mathrm{T}}\right]^{n-1} d = \lambda^{n-1} d. \tag{94}$$

Due to the fact that the vector d belongs to the set D, and the matrix U is non-negative and indecomposable, the vector $\left[U^{\mathrm{T}}\right]^{n-1} d$ is strictly positive. It follows that $\lambda > 0$ and the vector d is strictly positive. Let's prove that $\lambda = 1$. The problem (92) is equivalent to the problem

$$\lambda \sum_{k=1}^{n} a_{ki}^{-1} x_i d_k = b_i^1 d_i, \quad i = \overline{1,n}. \tag{95}$$

The summation by index i of the left and right parts of equalities (95) gives $\lambda \sum_{k=1}^{n} b_k^1 d_k = \sum_{i=1}^{n} b_i^1 d_i$. The latter proves necessary. Therefore, there is a solution to the problem (89). From the Theorem 3 and equalities (89), if we put

$$p_i = \frac{b_i^1 d_i}{\sum_{s=1}^{n} a_{is} b_s^1} \tag{96}$$

and introduce the price vector $p = \{p_i\}_{i=1}^{n}$, then $d_k = \sum_{s=1}^{n} a_{sk} p_s$, $k = \overline{1,n}$.

Or, in relation to the equilibrium price vector, we obtain a system of equations

$$p_i = \frac{b_i^1}{\sum_{s=1}^{n} a_{is} b_s^1} \sum_{s=1}^{n} a_{si} p_s, \quad i = \overline{1,n}. \tag{97}$$

Due to the fact that the representation $x = A(E-A)^{-1}\alpha$ is valid for the vector x, where α is a strictly positive vector, then from (88) we get $b^1 = (E-A)^{-1}\alpha$, where $b^1 = \{b_k^1\}_{k=1}^{n}$. We have from the last one

$$\frac{b_i^1}{\sum_{s=1}^{n} a_{is} b_s^1} = 1 + \frac{\alpha_i}{\sum_{k=1}^{\infty} \sum_{s=1}^{n} a_{is}^k \alpha_s} > 1, \quad i = \overline{1,n}, \tag{98}$$

where we have denoted by a_{is}^k the matrix elements of the matrix A^k. Therefore, the constructed solution of the problem (89) is such that the inequalities

$$p_i - \sum_{s=1}^{n} a_{si} p_s > 0, \quad i = \overline{1,n}, \tag{99}$$

hold, if the conditions of Theorem 16 are valid.

Assume that the matrix A is degenerate. The proof of necessity is the same as in the previous case. To prove sufficiency, consider a non-degenerate matrix $A + \varepsilon E$, where $\varepsilon > 0$ and small enough such that $(A + \varepsilon E)^{-1}$ exists for all $\varepsilon > 0$. This is possible due to the fact that $\varepsilon = 0$ is a root of the equation $\det(A + \varepsilon E) = 0$ and this equation has a finite number of roots. We assume that $x(\varepsilon) = (A + \varepsilon E)(E - A - \varepsilon E)^{-1} \alpha$, where the vector α is strictly positive. As before, repeating all the above arguments, we come to the fact that the vector $d(\varepsilon) = \{d_k(\varepsilon)\}_{k=1}^{n}$ satisfies the system of equations

$$d_k(\varepsilon) = \sum_{i=1}^{n} (a_{ik} + \delta_{ik}\varepsilon) \frac{b_i^1(\varepsilon)}{\sum_{k=1}^{n} (a_{ik} + \delta_{ik}\varepsilon) b_k^1(\varepsilon)} d_i(\varepsilon), \quad i = \overline{1,n}, \tag{100}$$

and it is strictly positive.

Due to the fact that $b^1(\varepsilon) = \{b_i^1(\varepsilon)\}_{i=1}^{n} = (E - A - \varepsilon E)^{-1} \alpha$, $(A + \varepsilon E) b^1(\varepsilon) = (A + \varepsilon E)(E - A - \varepsilon E)^{-1} \alpha$ we get that there is a limit

$$\lim_{\varepsilon \to 0} b^1(\varepsilon) = (E - A)^{-1} \alpha = b^1 = \{b_k^1\}_{k=1}^{n}.$$

Because $d(\varepsilon)$ is bounded, there exists a subsequence ε_n that goes to zero, so that there is a limit

$$\lim_{\varepsilon_n \to 0} d(\varepsilon_n) = d = \{d_k\}_{k=1}^{n},$$

which satisfies the system of equations (89).

Following the above arguments, we come to the fact that the introduced vector $p = \{p_i\}_{i=1}^{n}$ satisfies the system of equations (97). Let us prove that the vector x satisfies the system of equations (81). Really, the equality

$$\frac{x_i p_i}{\sum_{s=1}^{n} a_{si} p_s} = \frac{b_i^1 x_i}{\sum_{s=1}^{n} a_{is} b_s^1} = b_i^1 \tag{101}$$

is valid.

Multiplying the left and right parts by a_{ki} and summing over i, we get

$$\sum_{i=1}^{n} a_{ki} \frac{x_i p_i}{\sum_{s=1}^{n} a_{si} p_s} = \sum_{i=1}^{n} a_{ki} b_i^1 = x_k, \quad k = \overline{1,n}. \tag{102}$$

The remaining statements of Theorem 16 are obtained as before. The theorem 16 is proved.

The following Theorems are central in this section.

Theorem 17 *Let* $A = \|a_{ki}\|_{k,i=1}$ *be an indecomposable and productive matrix for the "input-output" production model. Suppose that the strictly positive gross output vector* $x = \{x_i\}_{i=1}^{n}$ *satisfies the system of equations*

$$x_k - \sum_{i=1}^{n} a_{ki} x_i = c_k + e_k - i_k, \quad k = \overline{1,n}, \tag{103}$$

and the price vector $p=\{p_i\}_{i=1}^n$ satisfies the system of equations

$$p_i-\sum_{s=1}^n a_{si}p_s=\delta_i, \quad i=\overline{1,n}, \tag{104}$$

where $c=\{c_i\}_{i=1}^n$, $e=\{e_i\}_{i=1}^n$, $i=\{i_k\}_{k=1}^n$, $\delta=\{\delta_i\}_{i=1}^n$, $c_k\geq 0$, $e_k\geq 0$, $i_k\geq 0$, $\delta_k>0$, $k=\overline{1,n}$, are vectors of final consumption, export, import and added values, respectively. Then there exists a vector of taxation $\pi=\{\pi_i\}_{i=1}^n$ such that the vector $p=\{p_i\}_{i=1}^n$ satisfies also the system of equations

$$\sum_{i=1}^n a_{ki}\frac{(1-\pi_i)x_ip_i}{\sum_{s=1}^n a_{si}p_s}=(1-\pi_k)x_k, \quad k=\overline{1,n}. \tag{105}$$

Proof. Since the price vector $p=\{p_i\}_{i=1}^n$ is a strictly positive one, as a solution to the set of equations (104), let's denote $X_i=x_ip_i$, $\Delta_i=\delta_ix_i$, $\bar{a}_{ki}=\frac{p_ka_{ki}}{p_i}$. In these denotations, the system of equations (105) is written in the form

$$\sum_{i=1}^n \bar{a}_{ki}\frac{(1-\pi_i)X_i}{\sum_{s=1}^n \bar{a}_{si}}=(1-\pi_k)X_k, \quad k=\overline{1,n}. \tag{106}$$

Let us put

$$V_i=\frac{X_i(1-\pi_i)}{\sum_{s=1}^n \bar{a}_{si}}, \quad i=\overline{1,n}.$$

Then the vector $V=\{V_i\}_{i=1}^n$ satisfies the system of equations

$$\sum_{i=1}^n \bar{a}_{ki}V_i=\sum_{s=1}^n \bar{a}_{sk}V_k, \quad k=\overline{1,n}. \tag{107}$$

On the basis of Lemma 3, there exists a strictly positive solution of this system of equations, which is determined with accuracy up to a constant. Let us denote $V_0=\{V_i^0\}_{i=1}^n$ the solution of this system of equations, the sum of the components of which is equal to one. Then $V=\{V_i\}_{i=1}^n=c_0V_0=\{c_0V_i^0\}_{i=1}^n$. From here

$$1-\pi_i=\frac{c_0V_i^0\sum_{s=1}^n \bar{a}_{sk}}{X_i}=c_0\frac{V_i^0}{X_i}\left(1-\frac{\Delta_i}{X_i}\right), \quad i=\overline{1,n},$$

or

$$\pi_i=1-c_0\frac{V_i^0}{X_i}\left(1-\frac{\Delta_i}{X_i}\right), \quad i=\overline{1,n}. \tag{108}$$

The constant $c_0>0$ can be chosen such that the inequalities

$$1-c_0\frac{V_i^0}{X_i}\left(1-\frac{\Delta_i}{X_i}\right)>0, \quad i=\overline{1,n},$$

are satisfied. Theorem 17 is proved.

Consequence 4 *The best system of taxation* $\pi=\{\pi_i\}_{i=1}^n$ *under the condition* *that the final product will be created in the economic system is one that satisfies the equality*

$$\frac{1-\pi_i}{\frac{V_i^0}{X_i}\left(1-\frac{\Delta_i}{X_i}\right)}=\frac{1}{\max\limits_{1\le i\le n}\frac{V_i^0}{X_i}\left(1-\frac{\Delta_i}{X_i}\right)},\quad i=\overline{1,n}. \tag{109}$$

Proof. We choose the constant c_0 in the formula (108) so that the value of $\pi_i, i=\overline{1,n}$, is the smallest. For this, we should put

$$c_0=\frac{1}{\max\limits_{1\le i\le n}\frac{V_i^0}{X_i}\left(1-\frac{\Delta_i}{X_i}\right)}. \tag{110}$$

Then equalities (109) will hold.

In the following Theorem 18, we find out for the taxation system $\pi=\{\pi_i\}_{i=1}^n$, the conditions under which in the economic system a final product can be created and the economy can function in the mode of sustainable development.

Theorem 18 *Let the matrix* $A=\|a_{ki}\|_{k,i=1}^n$ *be non-decomposable and productive for the "input-output" economy model. Then for the equilibrium price vector* $p=\{p_i\}_{i=1}^n$, *which is a solution of the system of equations (104), there always exists a solution of the system of equations (105) with respect to the vector* $x=\{x_i\}_{i=1}^n$, *which satisfies the system of equations (103) with a strictly positive right-hand side of this system of equations, provided that the taxation system* $\pi=\{\pi_i\}_{i=1}^n$, *satisfies the conditions*

$$0<\pi_i\le 1-b\left(1-\frac{\Delta_i}{X_i}\right),\quad i=\overline{1,n}, \tag{111}$$

for a certain *b*, satisfying inequalities

$$\max_{1\le i\le n}(1-\pi_i)<b<\frac{1}{\max\limits_{1\le i\le n}\left(1-\frac{\Delta_i}{X_i}\right)},$$

where as before $\Delta_k=\delta_k x_k, X_k=p_k x_k, k=\overline{1,n}$

Proof. Here and further we use the denotations of Theorem 17. Let the economic system be described by the "input-output" model, where the output vector satisfies the system of equations (105), and the equilibrium price vector is determined by the system of equations (104).

Let's put $X_k=x_k p_k$, $\bar{A}=|\bar{a}_{ki}|_{k,i=1}^n$, $\bar{a}_{ki}=\frac{p_k a_{ki}}{p_i}$, $C_k=c_k p_k$, $E_k=e_k p_k$, $I_k=i_k p_k$, $\Delta_k=\delta_k p_k$. In these denotations, the system of equations (105) can be written in the form

$$\sum_{i=1}^n \bar{a}_{ki}\frac{(1-\pi_i)X_i}{\sum_{s=1}^n \bar{a}_{si}}=(1-\pi_k)X_k,\quad k=\overline{1,n}. \tag{112}$$

and the system of equations (104) is in the form

$$X_i-\sum_{s=1}^n \bar{a}_{si}X_i=\Delta_i,\quad i=\overline{1,n}. \tag{113}$$

Introducing the denotation

$$\frac{(1-\pi_i)X_i}{\sum_{s=1}^{n}\bar{a}_{si}} = X_i^0, \quad i=\overline{1,n}, \tag{114}$$

the system of equations (112) can be rewritten in the form

$$\sum_{i=1}^{n}\bar{a}_{ki}X_i^0 = \sum_{s=1}^{n}\bar{a}_{sk}X_k^0, \quad k=\overline{1,n}. \tag{115}$$

On the basis of Lemma 3, there is a strictly positive solution of this system of equations, which is determined with accuracy up to a constant. Let $X_0 = \{X_i^0\}_{i=1}^n$ be a strictly positive solution of the system of equations (115) whose component sum is equal to one. Then any other is equal to $c_0X_0 = \{c_0X_i^0\}_{i=1}^n$, where $c_0 > 0$. For the components of the vector $X = \{X_i\}_{i=1}^n$ we get the formulas

$$X_i = c_0\frac{\sum_{s=1}^{n}\bar{a}_{si}}{1-\pi_i}X_i^0 = c_0\frac{1-\frac{\Delta_i}{X_i}}{1-\pi_i}X_i^0 = c_0\frac{1-\frac{\delta_i}{p_i}}{1-\pi_i}X_i^0, \quad i=\overline{1,n}. \tag{116}$$

To prove that the obtained solution is a vector of gross output for some vector of final consumption, it is necessary to establish that the inequalities

$$X_k - \sum_{i=1}^{n}\bar{a}_{ki}X_i > 0, \quad k=\overline{1,n}, \tag{117}$$

are valid. Because the inequalities (111) hold, the inequalities

$$1 \le \frac{1-\pi_i}{b\left(1-\frac{\Delta_i}{X_i}\right)} = \frac{1-\pi_i}{b\sum_{s=1}^{n}\bar{a}_{si}} \tag{118}$$

are true, therefore

$$X_k - \sum_{i=1}^{n}\bar{a}_{ki}X_i \ge X_k - \frac{1}{b}\sum_{i=1}^{n}\bar{a}_{ki}\frac{(1-\pi_i)X_i}{\sum_{s=1}^{n}\bar{a}_{si}} = \left(1-\frac{1-\pi_k}{b}\right)X_k > 0, \quad k=\overline{1,n}. \tag{119}$$

So,

$$X_k - \sum_{i=1}^{n}\bar{a}_{ki}X_i = Y_k, \quad k=\overline{1,n}, \tag{120}$$

where the vector $Y = \{Y_k\}_{k=1}^n$ has strictly positive components. Introducing the denotations $Y = C + E - I$, $C = \{C_i\}_{i=1}^n$, $E = \{E_i\}_{i=1}^n$, $I = \{I_i\}_{i=1}^n$, we get that the vector $X = \{X_i\}_{i=1}^n$ is the gross output vector for the tax system under consideration. The proof of Theorem 18 ends with a remark about the problem (105) and (112) are equivalent due to the strict positivity of the price equilibrium vector.

The following Theorem 19 considers the case of a taxation system under which the economic system can still function in the mode of sustainable development. This case is interesting because the added value created in the relevant industry is equal to the value of the final product created in the same industry in a state of economic equilibrium.

Theorem 19 *Let the matrix* $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^{n}$ *be indecomposable, whose matrix elements satisfy the system of inequalities*

$$\sum_{s=1}^{n} \bar{a}_{si} < 1, \quad i = \overline{1,n}. \tag{121}$$

Then for taxation systems $\pi = \{\pi_i\}_{i=1}^{n}$, where $\pi_i = \frac{\delta_i}{p_i} = \frac{\Delta_i}{X_i}, i = \overline{1,n}$, the economic system is able to function in the mode of sustainable development. The gross output vector $X = \{X_i\}_{i=1}^{n}$ satisfies the system of equations

$$\sum_{i=1}^{n} \bar{a}_{ki} X_i = \sum_{s=1}^{n} \bar{a}_{sk} X_k, \quad k = \overline{1,n}, \tag{122}$$

the solution of which exists and is strictly positive. The vector of final consumption in value indicators is given by the formula $Y = \{Y_i\}_{i=1}^{n}$, $Y_i = d\left(1 - \sum_{s=1}^{n} \bar{a}_{si}\right) X_i, i = \overline{1,n}$, $d > 0$ and is arbitrary, where $X_0 = \{X_i^0\}_{i=1}^{n}$ is one of the possible solutions of the system of equations (122), the sum of whose components is equal to one. The gross added value Δ_i^0 of the *i*-th industry is given by the formula $\Delta_i^0 = d\left(1 - \sum_{s=1}^{n} \bar{a}_{si}\right) X_i^0, i = \overline{1,n}$.

Proof. Due to the fact that $(1 - \pi_i) = \sum_{s=1}^{n} \bar{a}_{si}$, the system of equalities (112) turns into a system of equations (122) with respect to the gross output vector $X = \{X_i\}_{i=1}^{n}$ in value indicators. Based on Lemma 3, there exists a strictly positive solution of the system of equations (122). Let us denote one of the possible solutions $X_0 = \{X_i^0\}_{i=1}^{n}$, the sum of whose components is equal to one. We will rewrite the system of equations (122) in the form

$$dX_k^0 - \sum_{i=1}^{n} \bar{a}_{ki} dX_i^0 = \left(1 - \sum_{s=1}^{n} \bar{a}_{sk}\right) dX_k^0, \quad k = \overline{1,n}. \tag{123}$$

The gross output vector $dX_0 = \{dX_i^0\}_{i=1}^{n}$ is uniquely determined from the system of equations (123). Therefore, the final consumption vector is given by the formula $Y = \{Y_i\}_{i=1}^{n}$, $Y_i = d\left(1 - \sum_{s=1}^{n} \bar{a}_{si}\right) X_i^0, i = \overline{1,n}$, $d > 0$. The created gross added value is given by the formulas $\Delta_i^0 = d\left(1 - \sum_{s=1}^{n} \bar{a}_{si}\right) X_i^0, i = \overline{1,n}$. The theorem 19 is proved.

In the future, we will approach the models of real economic systems with models of economic systems capable of functioning in the mode of sustainable development. One of such algorithms is presented in the following corollary.

Consequence 5 *Let* $Y = \{Y_i\}_{i=1}^{n}$ *be the vector of the final product created in the real economic system. Then, in the mean square, the closest to the vector of the created final product in the real economic system is the vector of final* *consumption* $Y_0=\{Y_i^0\}_{i=1}^n$, *in the economic system capable of functioning in the mode of sustainable development under the consideration of Theorem 19 taxation system, where*

$$Y_i^0=d_0\left(1-\sum_{s=1}^n\bar{a}_{si}\right)X_i^0,\quad i=\overline{1,n},\quad d_0=\frac{\sum_{i=1}^n Y_i\left(1-\sum_{s=1}^n\bar{a}_{si}\right)X_i^0}{\sum_{i=1}^n\left[\left(1-\sum_{s=1}^n\bar{a}_{si}\right)X_i^0\right]^2}.$$

The set of gross added values has the form

$$\Delta_i^0=d_0\left(1-\sum_{s=1}^n\bar{a}_{si}\right)X_i^0,\ i=\overline{1,n}. \tag{124}$$

4. Research of Real Economic Systems

Information about the country's economic system is given in value indicators, the structure of which is given by a table that can be characterized by the matrix of direct costs $\bar{A}=\|\bar{a}_{ki}\|_{k,i=1}^n$, the vector of gross outputs $X=\{X_i\}_{i=1}^n$, the vector of gross added values $\Delta=\{\Delta_k\}_{k=1}^n$, the final consumption vector $C=\{C_i\}_{i=1}^n$, the export vector $E=\{E_i\}_{i=1}^n$, the import vector $I=\{I_i\}_{i=1}^n$, between which there are relations

$$X_k-\sum_{i=1}^n\bar{a}_{ki}X_i=C_k+E_k-I_k,\quad k=\overline{1,n}. \tag{125}$$

Gross added value consists of taxes on production, which we denote by $T_1=\{T_i^1\}_{i=1}^n$ and wages and profits and mixed income, which we denote by $Z_1=\{Z_i^1\}_{i=1}^n$. Let's put $\pi_0=\{\pi_i^0\}_{i=1}^n$, where the denotation $\pi_i^0=\frac{T_i^1}{\Delta_i}$ is introduced. The vector $\pi_0=\{\pi_i^0\}_{i=1}^n$ is called the vector of taxation in the real economic system. Then the conditions of sustainable development can be written in the form

$$\sum_{i=1}^n\bar{a}_{ki}\frac{\left(1-\pi_i^0\right)X_i}{\sum_{s=1}^n\bar{a}_{si}}=\left(1-\pi_k^0\right)X_k,\quad k=\overline{1,n}, \tag{126}$$

$$1-\sum_{s=1}^n\bar{a}_{si}>0,\quad i=\overline{1,n}. \tag{127}$$

But these conditions may not be fulfilled for real economic systems.

To study real economic systems, the statistical indicators of which are given in value indicators, it is convenient to introduce the concept of a relative equilibrium vector of prices. If the price vector $p=\{p_i\}_{i=1}^n$, implemented in the economic system is not in equilibrium, then by introducing the relative price vector $\hat{p}=\{\hat{p}_i\}_{i=1}^n$, where $\hat{p}_i=\frac{p_i^0}{p_i}$, and $p_0=\{p_i^0\}_{i=1}^n$ is the equilibrium price vector , the condition of economic equilibrium in value indicators with respect to the vector of relative equilibrium prices $\hat{p}=\{\hat{p}_i\}_{i=1}^n$ can be written in the form

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{\left(1-\pi_i^0\right) \hat{p}_i X_i}{\sum_{s=1}^{n} \hat{p}_s \bar{a}_{si}} \leq \left(1-\pi_k^0\right) X_k, \quad k=\overline{1,n}, \tag{128}$$

In the future, we will study the system of inequalities (128) and find out the conditions for the existence of a relative equilibrium vector of prices $\hat{p}_i = \frac{p_i^0}{p_i}$, we will partially describe the set of equilibrium states and find out the quality of each equilibrium state , based on proven Theorems 5, 6, 7 and Corollaries 1, 2 with Theorems 5, 6. First, let's find out when the conditions of sustainable development are fulfilled, that is, when there is equality in a system of inequalities (128). Let $\hat{p} = \{\hat{p}_i\}_{i=1}^n$ be a strictly positive relative equilibrium price vector. We will find out the conditions when the conditions of sustainable development will be fulfilled

$$\sum_{i=1}^{n} \bar{a}_{ki} \frac{\left(1-\pi_i^0\right) \hat{p}_i X_i}{\sum_{s=1}^{n} \hat{p}_s \bar{a}_{si}} = \left(1-\pi_k^0\right) X_k, \quad k=\overline{1,n}, \tag{129}$$

$$\hat{p}_i - \sum_{s=1}^{n} \hat{p}_s \bar{a}_{si} = \hat{\delta}_i,, \quad i=\overline{1,n}. \tag{130}$$

where $\hat{\delta}_i > 0, i=\overline{1,n}$.

An analogue of Theorem 18 in the case under consideration there is

Theorem 20 *Let the matrix* $\hat{A} = \|\hat{a}_{ki}\|_{k,i=1}^n$ *be indecomposable and productive in the "input-output" economy model, given in value indicators. Then for the relative equilibrium strictly positive price vector* $\hat{p} = \{\hat{p}_i\}_{i=1}^n$, *which satisfies the system of equations (130) there is always a solution to the system of equations (129) with respect to the vector* $X = \{X_i\}_{i=1}^n$, *for which the final product will be created, i.e. the inequalities*

$$X_k - \sum_{i=1}^{n} \bar{a}_{ki} X_i = Y_k > 0, \quad k=\overline{1,n},$$

are valid, provided that the taxation system $\pi_0 = \{\pi_i^0\}_{i=1}^n$ satisfies the conditions

$$0 < \pi_i^0 \leq 1 - b\left(1 - \frac{\hat{\Delta}_i}{\hat{X}_i}\right), \quad i=\overline{1,n}, \tag{131}$$

for a certain b, satisfying inequalities

$$\max_{1 \leq i \leq n}\left(1-\pi_i^0\right) < b < \frac{1}{\max_{1 \leq i \leq n}\left(1 - \frac{\hat{\Delta}_i}{\hat{X}_i}\right)},$$

where $\hat{a}_{ki} = \frac{\hat{p}_k \bar{a}_{ki}}{\hat{p}_i}$, $\hat{\Delta}_k = \hat{\delta}_k X_k, \hat{X}_k = \hat{p}_k X_k, k=\overline{1,n}$. Under these conditions, the economic system is able to function in the mode of sustainable development.

The most interesting for applications is the case when the relative equilibrium vector of prices $\hat{p} = \{\hat{p}_i\}_{i=1}^n$, is such that $\hat{p}_i = 1, i = \overline{1,n}$. Then $\Delta_i = X_i\left(1-\sum_{s=1}^n \bar{a}_{si}\right), i = \overline{1,n}$ We reformulate the Theorem 20 for this case.

Theorem 21 *Let the matrix* $\bar{A} = \|\bar{a}_{ki}\|_{k,i=1}^n$ *be indecomposable and productive in the "input-output" economy model, given in value indicators. Then there exists always a solution to the system of equations (129) with respect to the vector* $X = \{X_i\}_{i=1}^n$, *for which the final product will be created, i.e. the inequalities*

$$X_k - \sum_{i=1}^n \bar{a}_{ki} X_i = Y_k > 0, \quad k = \overline{1,n},$$

are valid, provided that the taxation system $\pi_0 = \{\pi_i^0\}_{i=1}^n$ satisfies the conditions

$$0 < \pi_i^0 \le 1 - b\left(1 - \frac{\Delta_i}{X_i}\right), \quad i = \overline{1,n}, \tag{132}$$

for a certain b, satisfying inequalities

$$\max_{1\le i\le n}\left(1-\pi_i^0\right) < b < \frac{1}{\max_{1\le i\le n}\left(1-\frac{\Delta_i}{X_i}\right)}.$$

If, in addition $Y_k = C_k + E_k - I_k, k = \overline{1,n}$, then, under these conditions, the economic system, described in value indicators, is able to function in the mode of sustainable development.

If the inequalities (128) hold, then there exists a nonempty subset $I \subseteq N$, where $N = \{1, 2, \cdots, n\}$, such that for indices $k \in I$ inequalities (128) are transformed into equalities. Under the condition $I = N$, we are talking about a complete clearing of the markets. If $I \subset N$, we are talking about partial clearing of the markets. We call the vector $\bar{\psi} = \{\bar{\psi}_k\}_{k=1}^n$ the real consumption vector, where

$$\bar{\psi}_k = \sum_{i\in I} \bar{a}_{ki} \frac{\left(1-\pi_i^0\right)\hat{p}_i X_i^0}{\sum_{s\in I} \hat{p}_s \bar{a}_{si}}, \quad k = \overline{1,n}, \tag{133}$$

In the case of partial market clearing, the real consumption vector $\bar{\psi}$ does not coincide with the supply vector $\psi = \{\psi_k\}_{k=1}^n$, where $\psi_k = \left(1-\pi_k^0\right)\hat{X}_k^0$. The vector of real consumption can be presented in the form

$$\bar{\psi}_k = \sum_{i\in I} \bar{a}_{ki} y_i, \quad k = \overline{1,n}, \tag{134}$$

where the non-zero vector $y = \{y_i\}_{i\in I}$ is the solution of the system of equations and inequalities

$$\sum_{i\in I} \bar{a}_{ki} y_i = \psi_k, \quad k \in I, \tag{135}$$

$$\sum_{i\in I} \bar{a}_{ki} y_i < \psi_k, \quad k \in N \setminus I, \tag{136}$$

and the equilibrium vector of prices $\hat{p} = \{\hat{p}_i\}_{i\in I}$ satisfies the system of equations

$$y_i = \frac{\left(1-\pi_i^0\right)\hat{p}_i X_i^0}{\sum_{s\in I} \hat{p}_s \bar{a}_{si}}, \quad i \in I. \tag{137}$$

According to Theorem 6 and Corollary 2, there exists one to one correspondence between the solutions of the system of equations and inequalities (135), (136) and the system of equations (137) with respect to the equilibrium price vector $\hat{p} = \{\hat{p}_i\}_{i \in I}$. Let us consider the relative generalized equilibrium vector of prices $p_u = \{p_i^u\}_{i=1}^n$, $p_i^u = \hat{p}_i, i \in I$, $p_i^u = 1, i \in J$, where the vector $\hat{p}^I = \{\hat{p}_i\}_{i \in I}$ satisfies the system of equations (137).

The number of goods, the cost of which is $\langle \psi - \bar{\psi}, p_u \rangle$, does not find a consumer on the market of goods of the economic system. We will characterize this phenomenon as the level of excess supply

$$R = \frac{\langle \psi - \bar{\psi}, p_u \rangle}{\langle \psi, p_u \rangle}. \tag{138}$$

If there is no state of economic equilibrium corresponding to the vector $y = \{y_i\}_{i=1}^n$, which satisfies the system of inequalities and equations

$$\sum_{i=1}^n \bar{a}_{ki} y_i = \psi_k, \quad k \in I, \tag{139}$$

$$\sum_{i=1}^n \bar{a}_{ki} y_i < \psi_k, \quad k \in N \setminus I, \tag{140}$$

with a non-empty set *J*, as it happened in Theorem 13, then in this case the generalized relative equilibrium price vector $\hat{p} = \{\hat{p}_k\}_{k=1}^n$, is a solution of the system of equations

$$y_i = \frac{\bar{\psi}_i \hat{p}_i}{\sum_{s=1}^n \bar{a}_{si} \hat{p}_s}, \quad i = \overline{1,n}. \tag{141}$$

where the vector of real consumption $\bar{\psi}$ is given by the formula $\bar{\psi} = \{\bar{\psi}_k\}_{k=1}^n$, $\psi_k = \sum_{i=1}^n \bar{a}_{ki} y_i, k = \overline{1,n}$ and the vector $y = \{y_i\}_{i=1}^n$ is a solution of the set of equalities (139) and inequalities (140). In this case, the level of excess supply for such an equilibrium state will be described by the formula

$$R = \frac{\langle \psi - \bar{\psi}, \hat{p} \rangle}{\langle \psi, \hat{p} \rangle}. \tag{142}$$

5. Conclusion

In the work, a method of researching real economic systems for the possibility of their functioning in the mode of sustainable development is built. Section 2 develops algorithms for constructing equilibrium states for the case of partial market clearing. For this purpose, the vector of real consumption, which satisfies the system of linear equations and inequalities, is considered. A complete description of the solutions of such a system of linear inequalities and equations is given. On this basis, a complete description of the equilibrium states under which partial clearing of the markets takes place in the production model "input-output" is given. These states are characterized by the level of excess supply in a certain period of the economy function. In order to find equilibrium states with the lowest level of excess supply, it is proved that the vector of real consumption is the solution of a certain problem of quadratic programming. Section 3 establishes the necessary and sufficient conditions for the operation of the economic system in the mode of sustainable development. It is shown that there is a family of taxation vectors under which the economic system described by the production model "input-output" functions in the mode of sustainable development. A restriction was found for taxation systems in real economic systems, under which the final product is created in the economic system, and the economic system functions in the mode of sustainable development.

Acknowledgements

This work is partially supported by the Fundamental Research Program of the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine "Building and researching financial market models using the methods of nonlinear statistical physics and the physics of nonlinear phenomena N 0123U100362".

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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