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About the Strange Tree Paradox and Possible Inconsistency of Set Theory

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Abstract

The existence of "strange trees" is proven and their paradoxical nature is discussed, due to which set theory is suspected of being contradictory. All proofs rely on informal set-theoretic reasoning, but without using elements that were prohibited in axiomatic set theories in order to overcome the difficulties encountered by Cantor's naive set theory. Therefore, in fact, the article deals with the possible inconsistency of existing axiomatic set theories, in particular, the ZFC theory. Strange trees appear when uncountable cardinals appear.

Keywords

Set Theory, Inconsistency, Tree, Strange Tree, Through Way, Almost Through Way, Isomorphism, Almost Isomorphism

1. Introduction

Set theory was created by the outstanding German mathematician Georg Cantor in the last third of the 19th century. Cantor's ideas were met at first with great caution, then by many (but not all) with admiration. One of the greatest mathematicians of all time, David Hilbert, wrote: "I believe that it [Cantor's set theory] represents the highest manifestation of human genius, as well as one of the highest achievements of human spiritual activity". And even when the paradoxes of set theory began to shake mathematics and another outstanding mathematician, Henri Poincaré, subjected Cantor's ideas and achievements to harsh criticism, Hilbert insistently asserted: "No one will expel us from the paradise that Cantor created for us." At present, set theory is the foundation on which the entire edifice of modern mathematics is built.

Paradoxes in set theory were discovered at the end of the 19th century, when it seemed that everything essential in a new building had already been built. In

1899, Cantor described the paradox of the set of all sets. According to this paradox, it cannot be argued that the set of subsets of the set of all sets is greater (in the language of set theory, has greater cardinality) than the set of all sets, which contradicted the theorem that Cantor proved in 1891.

Along with the paradox of the set of all sets, several other paradoxes (mathematical and semantic) were discovered and described, the most famous and often cited of which is Russell's paradox about the set of all sets that are not elements of themselves. This set, as is easy to see, is and is not at the same time an element of itself, which means a contradiction. And, as the mathematician Hadamard put it, an earthquake in mathematics began, real battles in which the creator of set theory could no longer take part due to health reasons. We will call that situation the first crisis in mathematics, "the crisis of unexpected paradoxes".

The solution, according to most mathematicians, was found by Zermelo, who created the first axiomatic set theory in 1908. After the improvement made by Frenkel, this theory came into general use as the Zermelo-Frenkel theory, abbreviated as the ZF theory [1] [2]. Zermelo set the goal of preserving the existing means of mathematics without leading to paradoxes as much as possible, and he achieved his goal. The ZF theory retains the possibility of using impredicative definitions (*i.e.*, those when the definition of a set includes an object that is an element of this set), which, as it has been established, is impossible to do without in mathematics. Thus, Weil especially emphasized the impredicative nature of some definitions of calculus. The restrictions were introduced only on the means of constructing new sets to make it impossible to obtain too large sets like the set of all sets or the set of all sets that are not elements of itself, leading to paradoxes. These sets simply do not exist in ZF. Note that in all obtained paradoxes non-predicative definitions of sets are given.

Then the axiom of arbitrary choice was added to the ZF axioms (the author of this axiom was also Zermelo). The ZF theory with the axiom of arbitrary choice is called the ZFC theory [2]. Subsequently, other axiomatic set theories appeared [3], but the ZF and ZFC theories (let us note for clarity that these theories, as it was established, have the same strength: if one is consistent, then the other is consistent) retained their leading position and are now considered as standard. This is probably due to the simplicity, clarity of construction and the possibilities of the theories. These theories are usually considered as two variants of the same theory—ZFC is a strengthened version of ZF.

The ZFC theory is simple and elegant. The number of its axioms is small, and there are only two initial mathematical concepts: "set" and "belonging". It contains only one mathematical symbol, denoting the two-place predicate of membership. At the same time, it turns out that everything that has been and is being done in mathematics (with the exception of that part of it that deals with some specific problems of set theory itself) can be done in the ZFC theory. That is why it has received universal recognition. Of course, in ordinary research mathematicians do not explicitly use the ZFC axiomatics, but for everyone who is well acquainted with it, it is obvious that translating the obtained results into the ZFC language is only a matter of technique and time.

Meanwhile, the emergence of axiomatic set theories was not at all a solution to the problem of paradoxes (*i.e.*, the problem of consistency). The old, well-known paradoxes have disappeared. But new ones could appear. And set theory was not something exceptional in this regard. In principle, contradictions could arise in calculus and even in arithmetic (where there are no non-predicative definitions). But, of course, for set theory the question of consistency was especially acute. And then Hilbert set a grandiose task for mathematics (this task was called the Hilbert program)—to prove directly by finite means (constructive proofs formalized in arithmetic), as such generally accepted mathematical means with which all mathematicians would agree, the consistency of all significant existing axiomatic theories. Previously, evidence of consistency was always relative. It was proven, for example, that if it is known that the geometry of Euclid is consistent, then the Lobachevsky-Bolyai geometry is also consistent. And it was supposed to start with Peano arithmetic, as the simplest significant axiomatic theory, from the inconsistency of which the inconsistency of all other theories would follow. At the heart of Hilbert's program was his confidence in the scientific knowability of the world. Hilbert was confident that everything true in mathematics could be proven. "Wir müssen wissen—wir werden wissen."

Hilbert's program meant the creation of a new branch of mathematics—"metamathematics", which deals with the study of the logical foundations of mathematics itself. It was a grandiose program in its conception. But its goal turned out to be unattainable. This became clear when Gödel's famous first theorem was published in 1931, which established the existence of true arithmetic statements that cannot be proven by arithmetic. Gödel's second theorem followed from the first theorem (for a clear and complete presentation of Gödel's theorems and the first crisis in mathematics, see, for example, [4]). By virtue of Gödel's second theorem, the consistency of a consistent axiomatic system cannot be proven by means formalized in the system itself (since such a proof immediately implies the inconsistency of this system). And the second crisis in mathematics came, "the crisis caused by Gödel's theorems".

It is interesting to note that even without Gödel's theorems, a weak point can be discerned in Hilbert's program. Let us assume that we have proven the consistency of the system by means formalized in the system itself. What does this give us? Only faith in consistency based on faith in intuition. Because if a system is contradictory, then by its means it is possible to prove any statement that can be formalized in it, including its consistency. But, of course, Gödel's evidence brought the highest and final clarity to this issue.

In the field of view of metamathematics first of all, there are now two axiomatic systems: classical Peano arithmetic and ZFC set theory (or any other theory equivalent in strength). At the same time, the general opinion is that Peano arithmetic is certainly consistent, and set theory is almost certainly, but the reason for the inconsistency can only be connected with the incompleteness of the rules associated with the ban on constructing new sets (and it is silently suggested that in this situation the rules could be corrected and the inconsistency would disappear). It is this point of view that, according to the author's opinion, was expressed by Kolmogorov and Dragalin in their book [5]: "At present, the consistency of the theories Ar and Ar2 [the ordinary arithmetic theory and the second-order arithmetic theory] can be considered reliably established. The consistency of a theory like ZF is much more problematic."

Mathematics has a great fear of contradiction. After all, according to the laws of logic, everything follows from contradiction. And not only in classical, but also in intuitionistic mathematics (where every proof of the existence of an object must determine the method of its construction).

The axioms of ZFC are transparent and completely consistent with our intuition, but the difference between ZFC and arithmetic is that in ZFC results were obtained that are very counterintuitive. And the "strange tree paradox", described later in the article, is just one of them. The "doubling ball paradox" is well known. It is shown that a ball in three-dimensional space can be divided into a finite number of sets and, using the movement of parts in space, like rigid bodies, another ball of two larger diameter can be assembled. This is not the case in arithmetic.

Thus, the way out of the second crisis was in the refusal of mathematicians to prove its consistency while maintaining confidence that consistency takes place. In a humorous form, Andre Weil expressed this statement this way: "God exists because mathematics is consistent, and the devil exists because we will never prove it". At the same time, we have to admit that in mathematics there is ignoramus et ignorabimus.

The confidence of mathematicians in the absence of contradictions in Peano arithmetic, based on faith in intuition, is universal, but not one hundred percent. In this regard, I would like to draw the reader's attention to article [6]. The article is devoted to a new crisis in mathematics, "the crisis of complexity", which occurred after 1970. Proofs are becoming longer and more complex, and checking their correctness is becoming more and more difficult. I will call it the third crisis. The third crisis, in relation to the problem of consistency, gives the situation new features associated with the rejection of unconditional faith in the consistency of arithmetic.

Let there be a provable arithmetic statement, written in the language of arithmetic, the shortest proof of which requires hundreds of millions of text pages (so its proof will be of unimaginable length). But the statement itself has a fairly short notation. The author considers a good candidate for the role of such a statement to be a statement about the inconsistency of arithmetic (written in the language of arithmetic itself) and provides some group-theoretic considerations for this statement. (Although he stipulates that he is simply expressing an assumption.) We find ourselves in a situation where theoretically there is proof, but in fact we will never have it. And in any case, whether or not there is proof of the inconsistency of arithmetic, we are at the mercy of ignoramus et ignorabimus, in different guises.

Thus, Davis's article (implicitly) offers such a solution of the problem of consistency-inconsistency of arithmetic. Inconsistency is possible, but its proof (if it exists) will be astronomically long. Therefore, we can safely ignore the question of the inconsistency of arithmetic.

Davis's article says nothing about set theory.

Three crises in mathematics were described above. It is easy to see that these crises are closely related to each other in their content. Therefore, it is perhaps more correct to speak of three stages of one crisis, which has its roots in the philosophical problem of truth. The initial view of mathematicians, that a priori there is a clear division in mathematics into true and false, that everything true is provable, and everything provable is true, turned out to be untenable and mathematicians are painfully trying to find a replacement for it.

This article introduces strange trees, proves their existence if there are uncountable cardinals, and discusses the paradoxical nature of the situation that has arisen, which we call the situation of strange tree paradox. Theorems 1 - 3 and two intuitive considerations highlight the paradoxical nature of the existence of strange trees and raise the question of the possible inconsistency of set theory described by existing axiomatic theories. In the continuation of this work, an approach to proving the inconsistency of set theory will be formulated and with its help new strong results will be obtained.

The questions that this article is devoted to were formulated in preliminary form in works [7] [8]. Note also that in [9] [10] issues related to possible semantic incompleteness and inconsistency of Peano arithmetic were considered.

2. Definitions

Let T be a partially ordered set with order relation "≤" which can be represented as a *tree with levels* numbered by ordinals from 0 to $\propto$ inclusive. Note that we consider an ordinal as the set of all smaller ordinals (as it is customary in axiomatic set theories). Each *vertex of a tree* is at some level, and all levels are non-empty with possible exception of the upper level $\propto$. For the sake of brevity, we will identify the levels with their numbers. The notation v^k will usually mean that a vertex v^k is at the level k. The following conditions are met: 1˚ there exists only one element $v_R \in T$ for which $v_R \le v$ for all $v \in T$, v_R is at the level 0, and there are no other vertices at the level 0; 2˚ for each v^m for all $k < m$ there is a unique v^k for which $v^k \le v^m$; 3˚ if there is a sequence ($v^k, k < l$), l being a limit ordinal, such that $v^i \le v^j$ for $i < j$ (such sequences will be called *continuing sequences*), then there is a unique limit v^l such that $v^k \le v^l$ for all $k < l$.

The vertex v_R will be called the *root* of a tree, and the ordered pairs of vertices $\langle u, v \rangle$, when v immediately follows u, are called *edges* (or *transitions* from the vertex u to a *child vertex* v). For generality of considerations, the empty set will also be considered a tree (*empty tree*).

The ordinal $\propto$ will be called the *height* of a tree T and denoted by $\propto = height(T)$. The level $\propto$ of the tree T will be called the *upper level*. This level will be empty in the case (and only in this case) when $\propto$ is a limit ordinal and there are no continuing sequences ($v^k, k < \propto$) in a tree. But vertices exist always at all lesser levels.

A tree will be called *finite* if $\propto < \omega$ and the set of transitions from any vertex to its child ones is finite. If $\propto \geq \omega$, then we will talk about a *transfinite* tree.

By a *path* in a tree T we mean the continuing sequence of vertices $(v^k, v^{k+1}, \cdots)$. A path is a well-ordered set of vertices, and a path from a vertex u to vertex v exists if and only if $u \leq v$ (and if it exists, it is unique). Each path has the starting vertex, but may not have the last one. There is a path from the root vertex to any vertex of a tree T. If the vertex v is at level k, then the path from the root to v (inclusive) is the set of vertices isomorphic to the ordinal $k+1$. A path can also be characterized by the sequence of tree edges.

Let a vertex not be at the upper level. If a given vertex is not followed by any vertex, then we will call it *final*, otherwise—*non-final*. All vertices of the upper level will be considered non-final, unless otherwise is stated. In the latter case the division of vertices at the upper level into final and non-final is determined by additional agreements. For example, if a tree $T^{\propto}$ of height $\propto$ is the lower part of a tree T^{β} of height β, then it often makes sense to introduce the division of vertices into final and non-final ones for the upper level $\propto$ of the tree $T^{\propto}$ induced by the tree T^{β}. If the height of a tree is expressed by a limit ordinal, then there may be a case when there are no vertices at the upper level (but at all lower levels they are).

Let us introduce one more (optional, but often convenient) condition 4˚: final vertices cannot appear at levels with non-limit numbers. It is easy to see that the use of this rule does not impose any restrictions on the generality of the results obtained. Later we will usually assume that condition 4˚ is satisfied.

For greater clarity, we will imagine a tree in the form of a graphic picture on the *plane of trees* where the root vertex is at the very bottom and diverging paths go up from it. Since paths in a tree can break off at levels lower than the upper one, the tree can be represented as a bush where the root of the bush is the root of the tree. In Figure 1 the enlarged view of a tree of the height 3ω with selected levels 0, ω, 2ω, 3ω and several paths is shown. The final vertices at levels less than the upper one are marked in black.

Any non-final vertex u of a tree defines (generates) some *subtree* (it will be the set of vertices v such that $u \leq v$) for which this vertex is the root. Such subtrees will be called *trees generated by the vertices of a given tree*.

If a path starts at the root vertex of a tree T and goes through all levels with a possible exception of the upper level $\propto$ (when $\propto$ is a limit ordinal), then such a path will be called a *through* path. A finite tree always has a through path. A non-final vertex on a non-upper level will be called a through vertex if there is a through path passing through it. A path will be called *strongly through* if it ends

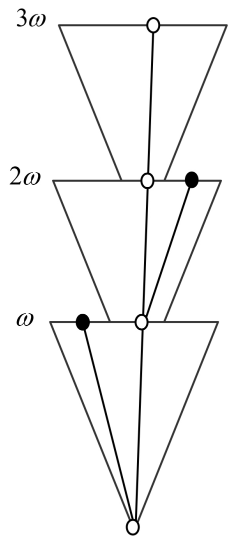

Figure 1. A tree of height 3ω.

(actually or potentially) at a non-final upper-level vertex. Accordingly, in this case we will speak of strongly through vertices.

A tree, all vertices of which at non-upper levels are through (strongly through), will be called through (strongly through).

Let us introduce (fundamental for this paper) the concept of the *almost through vertex*. This concept is introduced for the case when the height of a tree is a limit ordinal. When $\propto$ is a limit ordinal, a non-final vertex, located at the level $i < \propto$, will be called almost through vertex if for any $i < j < \propto$ there exists a path that starts at it and ends at some vertex of level *j*. Each through vertex is obviously an almost through vertex. The root vertex is an almost through vertex. If a tree contains a through path, then the root vertex is a through vertex.

A tree, all non-final vertices of which at non-upper levels are almost through vertices, will be called *almost through* tree. A vertex will be called *actually almost through* vertex if it is an almost through vertex, but not a through one. A tree with actually almost through non-final vertices will be called *strange*. To put it in another way: a strange tree is an almost through tree with no vertices at the upper level. In a strange tree, $\propto$ is a limit ordinal. A vertex of a tree *T* will be called *strange* if the tree generated by it is strange.

The *numbering of transitions* (for brevity, just the *numbering*) can be entered in a tree. Then transitions from each non-final vertex *v* to child ones (edges connecting non-final vertices with child ones) receive numbers that are elements of some set *M* (for each vertex *v*, the set of transitions can be its own, and hence in the general case $M = M(v)$). Without loss of generality, we will assume further that the set of ordinals less than some cardinal is taken: $M(v) = m(v)$ where $m(v)$ is a cardinal. By virtue of the definition of a tree, each of its vertices is uniquely determined by specifying the numbers of transitions in the way that goes from the root vertex to it. So, by specifying transitions, the encoding of the tree vertices can be performed. Such trees will be called *trees with numbered* *transitions* (for brevity, *numbered trees*).

The encoding of a tree vertices using the sequences of edge numbers leading to them will be called the *numerical encoding*.

Being numerically encoded, each vertex of a tree is represented by a sequence of ordinals characterizing transitions from non-final vertices preceding it to child ones. The case, when $m(v)$ is the same for all v at the level k, is representative and can be usually chosen in the proofs of lemmas. If the vertex of a tree is at the level l, then in the numerical encoding this vertex is represented by the sequence $(x_0, x_1, \cdots) = (x_k, k < l)$ where $x_k < m^k$ (and it is assumed that all such x_k are used). Each x_k is interpreted as the number of the edge from the vertex $(x_i, i < k)$ of the level k to the vertex $(x_i, i < k+1)$ of the level $k+1$. At the level zero, there is the empty sequence.

The numerical encoding may be separated from a tree T and considered as a new tree which is obviously isomorphic to T (the concept of tree isomorphism will be introduced below). Such trees (when a vertex at a level k is the sequence of ordinals $(x_i, i < k)$ that obeys the requirements formulated above) will be called trees in *numerical form* (or in m^k-*numerical form* if we want to emphasize that at the level k there are m^k transitions from each non-final vertex to child ones). In such trees, the vertex order relation is the relation of the continuation of sequences of ordinals.

By *tree isomorphism* we mean a one-to-one correspondence that preserves the order relation between vertices of different levels. If the nonfinality correspondence is also satisfied for vertices of the upper level, then we will speak of a *strong isomorphism*. An ordinary isomorphism with vertices of one sort on the upper level will also be called strong.

Each tree is isomorphic to a tree in numerical form, and when it is convenient, we will assume that a tree in question is given in numerical form.

It will be shown that for $height(T) < \omega_1$ strange trees do not exist, while for $height(T) = \omega_1$ strange trees exist, and an example of a strange tree will be given.

In strange trees at the upper level $\propto$ there are no vertices although there is a path from each non-final vertex of the level $k < \propto$ to the vertex of any level $l : k < l < \propto$, and this looks like an intuitive paradox (see further Section 5) which can generate a real contradiction.

We define $cut(T, k)$ as a tree of the height k obtained by cutting the tree T by level k: $cut(T, k) = \{v \in T : lev(v) \le k\}$ where $lev(v)$ is the level at which the vertex v is located. We will say that the operation of cutting the tree T by level k is performed. This function is defined at $k \le height(T)$. We will assume that by default in the tree $T^k = cut(T, k)$, where k is less than the height of T, those and only those vertices are non-final at the upper level k that were non-final in the tree T.

For to isomorphisms, the use of the cut off function means that the part of an isomorphism is taken. Thus $cut(ism, k)$, where *ism* is an isomorphism of a tree

T_a to tree T_b, means the isomorphism of the tree $cut(T_a, k)$ to the tree $cut(T_b, k)$ taken as a part of the isomorphism *ism*.

A tree T is said to be *homogeneous* if, for any two non-final vertices located at the same level less than the upper one, the trees generated by them are isomorphic. A tree T of height $\propto$, when $\propto$ is a limit ordinal, will be called *almost homogeneous* if for all $k < \propto$ the tree $cut(T, k)$ is homogeneous. If $\propto$ is a non-limit ordinal, then the concept of almost homogeneity is not introduced. If we mean strong isomorphism, then we will speak of *strong homogeneity*.

We say that trees T_a, T_b of the same height $\propto$, where $\propto$ is a limit ordinal, are *almost isomorphic* if for all $k < \propto$, the trees $cut(T_a, k)$ and $cut(T_b, k)$ are isomorphic. If $\propto$ is a non-limit ordinal, then the concept of almost isomorphism is not introduced. The concept of almost isomorphism will play an important role in the paper.

Let us further agree to assume that usually in the notation of a tree the superscript indicates the height of a tree. And as a rule, we will assume by default $T^k = cut(T, k)$. For clarity, note that the operation $cut(T, k)$ is unique.

Remark 1. It is easy to see that if the trees T_a, T_b of height $\propto$ are isomorphic, then for $k < \propto$ the trees T_a^k, T_b^k will always be strongly isomorphic. And in this case, speaking of isomorphism, we will by default mean strong isomorphism. This fact should be kept in mind in the future, as it may not be specially stipulated.

Let us introduce an ordering relation for trees. We will assume that

$$T_a \le T_b \text{ means that } T_a = cut(T_b, height(T_a)). \tag{1}$$

It is easy to see that if $T^l = cut(T^m, l)$ and $T^k = cut(T^l, k)$, then $T^k = cut(T^m, k)$. Hence,

$$\text{if } T^k \le T^l \text{ and } T^l \le T^m \text{, then } T^k \le T^m. \tag{2}$$

By virtue of (2), the introduced relation (1) is a partial order relation for which the item 2˚ of tree definition is fulfilled.

Note that further, when we talk about trees as vertices of a multilevel object, it is always assumed that the order relations between them satisfy condition (1).

As it was said, the sequence of vertices $(v^k, k = 0,1,\cdots)$ in a tree T will be called *continuing* if for all k, l ($k < l$) $v^k \le v^l$. We will talk about a continuing sequence of trees $(T^k, k = 0,1,\cdots)$ if for all k, l ($k < l$) $T^k \le T^l$.

We will call a sequence $(ism(T_a^k, T_b^k), k < \propto)$ the continuing sequence of isomorphisms if for all $k < l$ the isomorphism $ism(T_a^l, T_b^l)$ of trees T_a^l, T_b^l is the continuation of the isomorphism $ism(T_a^k, T_b^k)$ of trees T_a^k, T_b^k.

Let a continuing sequence of trees $(T^k, k < \propto)$ be given. Let us introduce the concept of path in it. A sequence of vertices $(v_0, v_1, \cdots)$ will be called a path in a continuing sequence of trees if any of its proper initial segment is a path in some tree T^k. It is easy to see that in this case this segment will be a path for all T^l where $l > k$. A path will be called a *through path* if it passes through all levels of a continuing sequence.

Let there be a continuing sequence of trees $\left(T^k, k<\propto\right)$ where ∝ is a limit ordinal. A tree *T* of height ∝ will be called the *limit* of this sequence if $T^k = cut\left(T, k\right)$ for all $k<\propto$. For non-limit ∝, the concept of the limit of a sequence of trees is not introduced.

For clarity, we can represent the continuing sequence of trees as the process of constructing the tree that is the limit of this sequence. The concept of continuing sequence of trees will play the key role in many further considerations.

A construction in which at each level less than ∝, where ∝ is a limit ordinal, there are vertices (level ∝ is excluded from consideration), and items 1˚ - 3˚ of the beginning of the section are satisfied (without item 3˚ for level ∝), will be called *almost a tree*. From any tree, the corresponding almost a tree is obtained by discarding vertices of level ∝ (if there were any). And a tree from almost a tree is obtained by adding vertices that complete the through paths. The word "almost" can be omitted if it is clear from the text that we say about almost trees.

Given a continuing sequence $\left(T^k, k<\propto\right)$ where $T^k = cut\left(T, k\right)$ for almost a tree T for all $k<\propto$, T will also be considered as the limit of the sequence $\left(T^k, k<\propto\right)$. Thus, the limit of the sequence of trees exists both as a tree and almost a tree.

In case of almost trees of height ∝, we will use the notation $T^{\propto-0}$. We will assume that the writing of the form $T^{\propto-0}$ means that ∝ is a limit ordinal. The continuing sequence $\left(T^k, k<\propto\right)$ uniquely defines the almost tree $T^{\propto-0}$ and can be identified with it.

Extending the notion of the function *cut*, we will write $T^{\propto-0} = cut\left(T^{\propto}, \propto-0\right)$.

We will assume that all trees (and almost trees) considered below have a height less than ω_1 or equal to it.

3. Some Preliminary Results

Note that after formulating lemmas, we will give proofs (or indications for the proofs) only where it is really necessary, without being obvious enough or relatively easily obtained from what was proved earlier.

Lemma 1. The continuing sequence of trees $\left(T^k, k<\propto\right)$ (∝ is a limit ordinal) has a limit. In the class of almost trees, the limit is unique: it is the union of vertices included in $\left(T^k, k<\propto\right)$, preserving the order relations that they had in the sequence. In the class of trees, the limit is not unambiguous, but all limits differ only in vertices at the upper level which terminate through paths in the continuing sequence $\left(T^k, k<\propto\right)$. Thus, if an unambiguous way of choosing final vertices of through paths is indicated, then the limit becomes unambiguous.

In what follows, our special interest will be associated with trees and almost trees, whose vertices are themselves trees with the order relation (1) introduced above.

Lemma 2. Let there be a continuing sequence of trees $\left(T^k, k<\propto\right)$, where the vertices of the trees T^k are trees themselves. Suppose that for each continuing sequence of vertices $\left(v_{i_k}^k, k<\propto\right)$ ($v_{i_k}^k$ are trees) there is a way to choose uniquely the limit in the class of trees. This determines the unique choice of the limit in the class of trees for the sequence $\left(T^k, k < \propto\right)$.

Lemma 3. A tree T is almost homogeneous if and only if for any two non-final vertices on the same level, the trees generated by them are almost isomorphic.

Lemma 4. Let $k < l < m$ and $v^k \le v^m$, $v^l \le v^m$. Then $v^k \le v^l$.

Indeed, let $\overline{v}^k \le v^l$. Then $\overline{v}^k \le v^m$. Then $\overline{v}^k = v^k$ and so $v^k \le v^l$ (see 2°).

Remark 2. It is often more convenient to use the notion of almost a tree. On the one hand, the limit of a continuing sequence in the class of almost trees is unique, and on the other one, each through path in a tree $T^{\propto}$ generates (by dropping the terminating vertex) a through path in the corresponding almost a tree $T^{\propto - 0}$. And all our interest revolves around the problem of the existence of through paths in different trees. Note also that if we take trees in the numerical form (each vertex is the sequence of ordinals), then the vertices of the upper level will be uniquely determined by the vertices of lower levels, and the limit of the sequence of trees will be unambiguous.

Lemma 5. If a homogeneous (almost homogeneous) tree T_a is isomorphic (almost isomorphic) to a tree T_b, then the tree T_b is homogeneous (almost homogeneous).

Let the division of vertices into final and non-final take place for the upper level $\propto$ as well (see Section 2). A tree $T^{\propto}$ we call *strongly through* if for each non-final vertex v^k with $k < \propto$ there is a path from v^k to some non-final vertex $v^{\propto}$. Note that if $T^{\propto}$ is the lower part of an almost through tree T^{β} and inherits its non-final vertices at the level $\propto$ from T^{β}, then $T^{\propto}$ is a strongly through tree.

Lemma 6. If T is a homogeneous (strongly homogeneous) tree and it has a through path (through path ended at a non-final upper-level vertex), then T is a through (strongly through) tree. If T is almost homogeneous, then it has the same cardinal number of child vertices for all non-final vertices of the same level and is almost through.

Lemma 7 (*the decomposition of isomorphism lemma*). Let T_a, T_b be isomorphic trees of the height $\propto$, *ism* be the isomorphism of T_a onto T_b, and $k < \propto$. Then *ism* is the union of the isomorphism of the tree $cut\left(T_a, k\right)$ to tree $cut\left(T_b, k\right)$ with isomorphisms of the subtrees generated by non-final vertices at the level k in the tree T_a to the corresponding subtrees in the tree T_b. The converse is also true: the union of the isomorphism of $cut\left(T_a, k\right)$ to $cut\left(T_b, k\right)$ with isomorphisms of subtrees generated by non-final vertices at the level k in the tree T_a to the corresponding subtrees in the tree T_b gives an isomorphism T_a to T_b. If T_a, T_b are homogeneous trees, then the decomposition of the isomorphism T_a to T_b described above is possible for any isomorphism of $cut\left(T_a, k\right)$ to $cut\left(T_b, k\right)$ where $k < \propto$.

The assertions of the lemma are the obvious consequence of the general properties of tree isomorphism and the definition of tree homogeneity.

Note that the assertion of lemma 7 carries over in an obvious way to the case of strong isomorphism (and strong homogeneity) of trees T_a, T_b.

Lemma 8. The limit level *l* is formed by the vertices that end all possible paths along the vertices below the level *l*.

See the item 3˚ of the definition of a tree.

As already been noted (see remark 1), it immediately follows from the definition of isomorphism and strong isomorphism that if trees T_a, T_b of height $\propto$ are isomorphic, then for $k < \propto$ the trees $cut(T_a, k)$ and $cut(T_b, k)$ are strongly isomorphic. Speaking about the isomorphism of these trees, we will mean by default strong isomorphism.

Lemma 9. Let trees T_a, T_b of the same height $\propto$ be almost isomorphic and almost homogeneous. For $i < \propto$ the trees $cut(T_a, i)$ and $cut(T_b, i)$ are strongly isomorphic and strongly homogeneous. Let $i < \propto$ and ism^i be an isomorphism of $cut(T_a, i)$ to $cut(T_b, i)$. The isomorphism ism^i can be extended to the isomorphism ism^j of the tree $cut(T_a, j)$ onto the tree $cut(T_b, j)$ for all *j* such that $i < j < \propto$.

The statement is a consequence of lemma 7.

Lemma 10. Let T_a, T_b be trees of the same height $\propto$, where $\propto$ is a limit ordinal, and $(ism^k, k < \propto)$ be a continuous sequence of isomorphisms T_a^k onto T_b^k (which are obviously strong isomorphisms). Then the trees T_a, T_b are isomorphic, and the sequence $(ism^k, k < \propto)$ introduces the isomorphism $ism^\propto$ of T_a onto T_b (not necessarily strong isomorphism) such that for all $k < \propto$ $ism^\propto$ is the extension of ism^k.

The isomorphism $ism^\propto$ is introduced as the limit of the continuing sequence of isomorphisms $(ism^k, k < \propto)$.

Setting $T = T_a = T_b$, we obtain the corresponding analogs of lemmas 7, 9, 10 for automorphisms in the tree *T*.

Lemma 11. Let $\propto < \omega_1$, $\propto$ be a limit ordinal and trees T_a, T_b of height $\propto$ be almost isomorphic and almost homogeneous. Then the trees T_a, T_b are isomorphic.

We will construct the required isomorphism as follows. Let $(i_k : 0 \le k < \omega)$ be a strictly increasing sequence of ordinals and $i_k \to \propto$ when $k \to \omega$. The tree $T_{a0} = cut(T_a, i_0)$ is isomorphic to the tree $T_{b0} = cut(T_b, i_0)$. Take some isomorphism T_{a0} to T_{b0}. This isomorphism (by lemmas 7, 9) can be extended to an isomorphism of $T_{a1} = cut(T_a, i_1)$ onto $T_{b1} = cut(T_b, i_1)$. Subsequent steps (of a similar type) allow us to construct isomorphisms of trees T_{a2} onto T_{b2}, T_{a3} onto T_{b3}, etc, every time the isomorphism of $T_{ak} = cut(T_a, i_k)$ onto $T_{bk} = cut(T_b, i_k)$ being the continuation of the isomophism of $T_{a,k-1}$ onto $T_{b,k-1}$. (Of course, we are always talking about strong isomorphisms.) As a result (see lemma 10), an isomorphism of T_a onto T_b will be constructed. The lemma is proved.

Lemma 12. Let a tree *T* of the height $\propto < \omega_1$ ($\propto$ is a limit ordinal) be almost homogeneous. Then it is homogeneous.

Indeed, consider non-final vertices v_a^k, v_b^k at the level $k < \propto$ in the tree *T*. The trees T_{ak}, T_{bk}, generated by them, are almost isomorphic (lemma 3) and therefore, by lemma 11, are isomorphic. Therefore, the tree is homogeneous.

Let a tree T of the height $\propto$ be given. Let us agree to mean by $cut\left(v^{\propto},k\right)$ the vertex v^k located at the level k on the path going from the root of the tree T to the vertex $v^{\propto}$ located at the level $\propto$.

Lemma 13. Let T_a, T_b be strongly isomorphic and strongly homogeneous trees of height $\propto < \omega_1$, and $\propto$ be a limit ordinal. Let $v_a^{\propto}, v_b^{\propto}$ be vertices at the upper level of these trees, which are non-final if there are vertices of two sorts at the upper level. There is an isomorphism of T_a onto T_b for which the vertex $v_a^{\propto}$ is mapped to $v_b^{\propto}$. (Note that in a case when there are vertices of one sort at the upper level, the word "strongly" can be omitted.)

It suffices to show that if the assertion of the lemma holds for all T_a, T_b of height $\beta < \propto$, then it also holds for T_a, T_b of height $\propto$. Let $\left(i_k : 0 \le k < \omega\right)$ be a strictly increasing sequence of ordinals and $i_k \to \propto$ as $k \to \omega$. T_a, T_b are considered as trees generated by root vertices v_a^0 and v_b^0. Let $T_a^1 = cut\left(T_a, i_1\right)$, $T_b^1 = cut\left(T_b, i_1\right)$ and ism^1 be an isomorphism mooving T_a^1 to T_b^1 in such a way that $v_a^1 = cut\left(v_a^{\propto}, i_1\right)$ goes to $v_b^1 = cut\left(v_b^{\propto}, i_1\right)$. Let, under the isomorphism ism^1, a non-final vertex u_a^1 (different from v_a^1) go to a non-final vertex u_b^1 (different from v_b^1). We introduce a strong isomorphism of the trees generated by these vertices (in the trees T_a, T_b) and perform this operation for all pairs $\left(u_a^1, u_b^1\right)$. At the next step, we turn to the trees generated by the vertices v_a^1 and v_b^1 and perform similar operations to obtain isomorphisms generated by the pairs $\left(u_a^2, u_b^2\right)$. Etc. After taking ω steps, we arrive at a strong isomorphism of trees T_a, T_b, under which the vertex $v_a^{\propto}$ goes to the vertex $v_b^{\propto}$.

Let $T^{\propto}$ be a through homogeneous tree of height $\propto$. Let $\tilde{M}^{\propto}$ denote *minimal sets of vertices* at the level $\propto$. Each minimal set is characterized by a function $\varphi : \left(v_i^k, k < \propto\right) \to \left(v_i^{\propto}\right)$, where v_i^k are non-final vertices at the level k, satisfying the conditions: 1) if $v^{\propto} = \varphi\left(v^k\right)$, then $v^k \le v^{\propto}$; 2) if $v^{\propto} = \varphi\left(v^k\right)$ and $v^k \le v^l \le v^{\propto}$, then $v^{\propto} = \varphi\left(v^l\right)$.

The functions φ are introduced using the following recursive *assignment process*. Assign some vertex $v^{\propto}$ to the root vertex v_R. With this, we have introduced $v^{\propto} = \varphi\left(v_R\right)$. Assign the same $v^{\propto}$ to each vertex v^k lying on the path from v_R to $v^{\propto}$. Let assignments be made for all non-final vertices on levels less than k. Then we make assignments for those non-final vertices of the level k that have not yet received assignments, keeping the rules 1) and 2). At the end of the described process, we obtain function φ.

Lemma 14. Let $T^{\propto}$ be a strongly through tree and $\bar{M}^{\propto}$ the set of non-final vertices at the level $\propto$. There exists $\tilde{M}^{\propto} \subseteq \bar{M}^{\propto}$.

To prove the lemma, we shall use the assignment process described above, restricting the assignments to non-final $v^{\propto}$.

Lemma 15. Let $\propto < \omega_1$, $T_a^{\propto}, T_b^{\propto}$ be isomorphic homogeneous through trees and $\tilde{M}_a^{\propto}, \tilde{M}_b^{\propto}$ be minimal sets. There is an isomorphism of trees $T_a^{\propto}, T_b^{\propto}$ such that $\tilde{M}_a^{\propto}$ goes to $\tilde{M}_b^{\propto}$.

Let us describe the process of constructing the required isomorphism. Let

φ_a, φ_b be functions associated with the sets $\tilde{M}_a^{\propto}, \tilde{M}_b^{\propto}$. Let $(i_k : 0 \le k < \omega)$ be a strictly increasing sequence of ordinals and $i_k \to \propto$ as $k \to \omega$. We introduce the future isomorphism correspondence between vertices located on the path from v_{Ra} to $\varphi_a(v_{Ra})$ and on the path from v_{Rb} to $\varphi_b(v_{Rb})$. Let $v_{a0}^{i_1}$ and $v_{b0}^{i_1}$ be vertices at level i_1 belonging to these paths (vertices $v_{a0}^{i_1}, v_{b0}^{i_1}$ are obviously non-final). We introduce a strong isomorphism of trees $T_a^{i_1}, T_b^{i_1}$ under which $v_{a0}^{i_1}$ becomes $v_b^{i_1}$ (lemma 13). Let $v_a^{i_1}$ be an arbitrary non-final vertex of the tree $T_a^{i_1}$, which is mapped by the introduced isomorphism to the non-final vertex $v_b^{i_1}$ of the tree $T_b^{i_1}$. Having performed operations similar to those for v_{Ra} and v_{Rb} for all such pairs of vertices, we arrive at a strong isomorphism of trees $T_a^{i_2}, T_b^{i_2}$, which continues the previously introduced isomorphism of trees $T_a^{i_1}, T_b^{i_1}$ (lemma 7). And then we proceed similarly to obtain a continuing sequence of isomorphisms of trees $\left(\left(T_a^{i_k}, T_b^{i_k}\right), k < \omega\right)$. Upon completion of the described process, we arrive at a tree isomorphism $T_a^{\propto}, T_b^{\propto}$, under which $\tilde{M}_a^{\propto}$ goes $\tilde{M}_b^{\propto}$ (lemma 10).

4. Existence of Strange Trees

Let us first consider the case when a tree T has the height $\propto < \omega_1$.

Lemma 16. Let $\propto < \omega_1$ and be a limit ordinal. There is a non-decreasing function $\varphi : \propto \to \propto$ for which (*a*) $\varphi(0) = 0, \varphi(\beta) < \beta$ for all $0 < \beta < \propto$, (*b*) $\varphi(\beta) \to \propto$ when $\beta \to \propto$.

Indeed, at first let $\propto_0 < \propto$ be a limit ordinal or 0 and there be no limit ordinals between $\propto_0$ and $\propto$. Let us define the function $\varphi(\beta)$ by the conditions: $\varphi(\beta) = 0$ for $\beta \le \propto_0$, $\varphi(\beta) = \beta - 1$ for $\beta > \propto_0$. Suppose now that there exists an increasing sequence of ordinals $(\propto_i, i < \omega)$ such that $\propto_0 = 0$, $\propto_i$ is a limit ordinal for $i > 0$ and $\propto_i \to \propto$ when $i \to \omega$. Then to determine $\varphi(\beta)$, we use the following conditions: $\varphi(0) = 0$, $\varphi(\beta) = \propto_i$ for $\propto_i < \beta \le \propto_{i+1}$. It is easy to see that the function $\varphi(\beta)$, introduced in this way, satisfies the requirements of the lemma.

Lemma 17. Let T be an almost through tree of height $\propto < \omega_1$ and $\propto$ be a limit ordinal. Then T is a through tree, and therefore it cannot be strange.

Indeed, let T be an almost through tree. Let us show first that there is a through path in it. Let $\varphi(k)$ be a function that satisfies the requirements of lemma 16. We will be constructing a through path in such a way that after k steps of construction, the constructed path ends with a vertex v^k located at the level k. Let k steps of the constructing be done and we have achieved a vertex v^k. If the level k is non-limit, then (supposing that the item 4° is fulfilled) we will go to a vertex v^{k+1} which is one of the child vertices of v^k. Now let k be a limit ordinal. If v^k is a non-final vertex, we obtain v^{k+1} in the same way (by going to a child vertex). If v^k is a final vertex, then first we shorten the constructed path to the path ending at the vertex u at the level $t = \varphi(k)$. And then we make the lengthening of the path to some vertex v^{k+1} at the level $k+1$. The possibility of this lengthening follows from the fact that T is an almost through tree. Since $\varphi(k) \to \propto$ when $k \to \propto$, the described construction leads to the selection of a through path in the tree *T*.

Let us now take a non-final vertex v at the level $k < \propto$. The tree it generates is obviously almost through. So, this tree has a through path. Thus, T is a through tree.

As for the trees of the height $\propto = \omega_1$, the situation is different.

Lemma 18. There is a strange homogeneous tree of height ω_1.

Let's build a strange tree in which there are ω transitions from each non-final vertex to its child ones.

Let $N = \{0,1,2,\cdots\}$ be the set of natural numbers. We will consider well-ordered sequences of natural numbers $X^k = (x_i, i < k)$, in which the numbers are not repeated, and $k < \omega_1$ is the length of the sequence. With each sequence X^k we associate the set $M(X^k)$ consisting of numbers that are not in X^k. The order relation between sequences X^k is the sequence continuation relation. Let us take the set of sequences X^k for which $M(X^k)$ is infinite. This set forms the set of non-final vertices of the tree under construction. We supplement this set with vertices with finite $M(X^k)$ that are path limits along non-final vertices. We get some tree T. The root of T is the empty set X^0 for which $M(X^0) = N$.

Since the set of natural numbers can be well-ordered according to the type determined by any transfinite ordinal less than ω_1, T is an almost through tree of height ω_1. And it has no vertices at the level ω_1. Otherwise, there would be a bijection of ω on ω_1. So *T* is a strange tree. Moreover, each tree, generated by a non-final vertex of *T*, is obviously isomorphic to *T*. Therefore, *T* is a homogeneous tree.

We denote by T_{str} the tree constructed above. In T_{str}, there are ω transitions to child vertices from each non-final one.

5. First Intuitive Consideration

The existence of strange trees looks like a paradox to intuition.

Let T be an almost through tree of the height ω_1 and $\boldsymbol{W}^k (k < \omega_1)$ be the set of paths in it that start at the root vertex and end at some non-final vertex of level *k*. It is easy to see that if $k < l$, then each path of the set $\boldsymbol{W}^k$ is an initial segment for some path of the set $\boldsymbol{W}^l$. Thus, the paths are lengthening with possible simultaneous reproduction. If there were, starting from some *k*, the simple lengthening of the constructing paths (lengthening without reproduction), then obviously at the end of the process we would have some set of through paths. It is natural to expect that lengthening with reproduction cannot worsen the situation with regard the appearance of through paths. But it is not the real case. When lengthening with reproduction takes place, such a "strange case" may occur when at the very end a "catastrophe" occurs: after the completion of the process (when ω_1 steps are made), all paths under construction disappear somewhere, and there is not a single through path in the tree *T*. It looks like a paradox of intuition that can induce a formal paradox.

Note that the strange tree paradox in its essence is associated with the introducing of uncountable infinity.

6. Strange Almost Isomorphism

Let us take the next step towards proving the existence of contradictions in set theory.

Theorem 1. Let T_a be an almost homogeneous tree of height ω_1. There is an almost homogeneous tree T_b that is almost isomorphic to T_a and contains a through path.

We will consider trees given in numeric form, and assume that from each non-final vertex at the level k there are m^k transitions to child ones. For simplicity, we will assume that item 4˚ of the tree definition has been fulfilled. In particular, it can be $T_a = T_{str}$.

The proof of the theorem is based on the following idea. Assuming that the tree T_a is a tree of sequences of ordinals, each of which for level k is less than the cardinal m^k, we choose a sequence of ordinals $\overline{v}$ of length ω_1 and "attach" to it a tree T_b, almost isomorphic to T_a, in which the sequence of ordinals $\overline{v}$ will describe a through path.

Let $\overline{v} = \left(\overline{x}_i, i < \omega_1\right)$. We denote $T_a^k = cut\left(T_a, k\right)$. The tree T_a^k is homogeneous for $k < \omega_1$. We will construct a continuing sequence of homogeneous trees in numeric form T_b^k such that T_b^k is isomorphic to T_a^k and T_b^k contains the vertex $\overline{v}^k = \left(\overline{x}_i, i < k\right)$. Suppose that we have already obtained a sequence $\left(T_b^i, i < k\right)$, where $k < \omega_1$, with the required properties. If k is a non-limit ordinal and there is the ordinal $k-2$, then for each vertex of level $k-1$ in the tree T_b^{k-1} we introduce m^k child vertices and obtain the tree T_b^k, which is obviously isomorphic to the tree T_a^k and contains the vertex $\overline{v}^k = \left(\overline{x}_i, i < k\right)$.

If k is a non-limit ordinal and $k-1$ is a limit one, then (using if necessary, lemma 13) we obtain the isomorphism T_a^{k-1} onto T_b^{k-1}, for which the vertex $\overline{v}^{k-1} = \left(\overline{x}_i, i < k-1\right)$ of the tree T_b^{k-1} is the image of some non-final vertex of the tree T_a^{k-1}, and then we do the same as in the previous case, but only in relation to the upper level vertices of the tree T_b^{k-1}, which are images of the non-final vertices of the tree T_a^{k-1}.

If $k < \omega_1$ is a limit ordinal, then for T_b^k we take the tree that is the limit for the continuing sequence $\left(T_b^i, i < k\right)$ (see lemmas 10, 11).

In this way, a continuing sequence $\left(T_b^k, k < \omega_1\right)$ can be obtained in which every T_b^k is isomorphic to T_a^k.

Let T_b be the limit of this sequence. The tree T_b is almost homogeneous, almost isomorphic to the tree T_a and contains the vertex $\overline{v} = \left(\overline{x}_i, i < \omega_1\right)$ at the level ω_1.

Theorem 1 can be strengthened.

Theorem 2. Let T_a be an almost homogeneous tree of height ω_1. There is an almost homogeneous through tree T_b, almost isomorphic to T_a.

The proof of theorem 2 will be similar the proof of theorem 1, but unlike the latter, future through paths will be entered for each non-final vertex of T_a under construction, not only for the root one. As before, we consider trees given in numeric form and assume that from each non-final vertex at the evel k there are m^k transitions to child ones. Also, we assume that item 4° of the tree definition is fulfilled.

We construct the tree T_b using elements of the assignment process described before the proof of lemma 14 (see Section 3). We choose $\overline{v} = (\overline{x}_i, i < \omega_1)$ and correlate $\overline{v}$ with the root vertex v_{Rb} of the future tree T_b. A future through path defined by the sequence $\overline{v}$ is thus assigned to the root vertex. Then we pass to the level 1. We assign a path defined by the same sequence $(\overline{x}_i, i < \omega_1)$ to the vertex $(\overline{x}_0)$ of the level 1 and a path defined by the sequence $(x_0, \overline{x}_1, \overline{x}_2, \cdots)$ to an arbitrary other vertex (x_0). Similarly, we introduce path assignments at each level k for vertices that have not previously received assignments. A path defined by the sequence $(x_i, i < k, \overline{x}_k, \overline{x}_{k+1}, \cdots)$ is assigned to the vertex $(x_i, i < k)$.

Next, we build a continuing sequence of homogeneous trees in numeric form T_b^k such that T_b^k is isomorphic to T_a^k and T_b^k contains the initial segments of all future through paths that have already been assigned to this moment.

If k is a non-limit ordinal and there exists an ordinal $k-2$, or if k is a limit ordinal, then our actions related to obtaining the tree T_b^k do not differ from those that took place in the proof of theorem 1. The only difference is for the case when k is a non-limit ordinal and $k-1$ is a limit one. We want all "assigned paths" to be real paths in T^b tree when its construction is finished. To do this, we need that all vertices of the level $k-1$ in the constructed tree T_b^{k-1} that received path assignments at earlier steps can be considered as non-final. In the proof of theorem 1, lemma 13 was used for this. Now, for the purposes we need, we will use lemmas 14 and 15. Using lemma 14, we single out $\tilde{M}_a^{k-1} \subseteq \overline{M}_a^{k-1}$, where $\overline{M}_a^{k-1}$ is the set of non-final vertices of the tree T_a^{k-1}. Let $\tilde{M}_b^{k-1}$ be the set of vertices of the level $k-1$ that received assignments of through paths at earlier steps. It is easy to see that $\tilde{M}_b^{k-1}$ is a minimal set (due to the similarity of the process of assigning future through paths to the process of assigning v^{∞} vertices introduced before the proof of lemma 14). Using lemma 15, we obtain isomorphism of the trees T_a^{k-1} and T_b^{k-1} under which $\tilde{M}_a^{k-1}$ goes over to $\tilde{M}_b^{k-1}$. Let $\overline{M}_a^{k-1}$ goes to $\overline{M}_b^{k-1}$. Then $\tilde{M}_b^{k-1} \subseteq \overline{M}_b^{k-1}$ is executed, and so the set $\overline{M}_b^{k-1}$ can be taken as the set of non-final vertices of the tree T_b at the level $k-1$.

Upon completion of the construction process, we get a tree T_b in which all assigned paths turn out to be through paths of the tree.

Remark 3. Let T_a be a homogeneous tree. Using the obvious homogeneity of the process of constructing the tree T_b, we can show that T_b is a homogeneous tree.

If it is proved that the almost isomorphism of trees that we consider implies their isomorphism, then by virtue of what has been proved, a contradiction will follow since in theorems 1 and 2 the strange tree T_{str} can be taken as T_a.

7. Second Intuitive Consideration

In our notion of tree, only its structure, which is determined by the order relation between vertices at different levels, is significant. Therefore, we can assume that vertices v_i^k (v_i^k is the *i*-th vertex at the level *k*) themselves (as mathematical things) can be either different or the same, and the real difference is determined only by their places in the tree. Let us consider trees in which all vertices v_i^k at a given level *k* are the same as things if they are removed from the tree. Let T_a and T_b be almost isomorphic and have the height $\propto$ ($\propto$ is a limit number), and at each level $k<\propto$ for all i $v_{ai}^k = v_{bi}^k = v^k$. Then the trees T_a^k and T_b^k for $k<\propto$ (having identical structures and identical vertices) are identical: $T_a^k = T_b^k$. But T_a and T_b are the limits of T_a^k and T_b^k as k tends to $\propto$. Therefore, $T_a = T_b$, and we arrive at a contradiction with theorems 1 and 2.

Note that this consideration has a certain connection with the following paradox of philosophical nature. Let *A* be any person who consists of a finite number of atoms. Let's start replacing his atoms with ones of the same type. All atoms of one type, according to physical concepts, are completely identical to each other. After a finite number of steps, we get the person *B*, who does not have a single old atom. Consequently, *B* is a person different from *A*. But at the same time, after each step of replacement, we have the same person *A* who was before the step (one atom was replaced by a completely identical one). Therefore, when the process ends, *A* remains the same. Paradox.

8. Spitting Trees

This section will introduce *splitting trees* and it will be shown that there are no strange trees in this class of trees. These considerations will reinforce the expectation of a real paradox in set theory. Besides, they are of some interest in themselves.

Let a *root set S* and ordinal $\propto$ be chosen. We will describe a recursive process that generates splitting the set *S* into disjoint subsets. In the beginning we have one (initial) set $S^0 = S$, which is the root of the tree under construction. We split S^0 into m^0 non-empty pairwise disjoint subsets and, as a result, obtain the sets $S_i^1, i<m^0$ (the vertices of the splitting tree under construction at the level 1). Further, we split each of the sets S_i^1 into m_i^1 non-empty pairwise disjoint subsets to obtain sets S_i^2 (vertices of the level 2), etc. After ω steps, we have the set of decreasing (more precisely, non-increasing) sequences $\left(S_{i_k}^k, k<\omega\right)$, in which $S_{i_k}^k \supseteq S_{i_{k+1}}^{k+1}$.

Let us introduce (for such sequences) $S_i^\omega = \bigcap_{k<\omega} S_{i_k}^k$. Then either S_i^ω is the empty set: $S_i^\omega = \varnothing$, or $S_i^\omega \neq \varnothing$. In the first case, by definition, S_i^ω is a final vertex of the splitting tree, in the second one it is non-final.

Then the process continues in a similar way for all $k<\propto$. Every $S_i^k \neq \varnothing$ will be splitted into $m_i^k \geq 1$ non-empty disjoint sets. (It is assumed that for each *k* a

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numbering of sets of this level is introduced.) Thus, the result of the splitting process is determined by specifying the set of splitting parameters m_i^k, $k < \propto$, $i < J^k$, where m_i^k, J^k are cardinals, and ways of splitting sets S_i^k into subsets.

Lemma 19. $\bar{S} = \bigcup_i S_i^k$ for each *k*, and non-empty sets S_i^k are pairwise disjoint. Similarly, $S_i^k = \bigcup_j S_j^l$ for all S_i^k and $l > k$, where the summation is over S_j^l for which $S_j^l \subseteq S_i^k$.

Let us introduce the order relation for the sets: $S_i^k \le S_j^l$ if and only if $k \le l$ and $S_j^l \subseteq S_i^k$.

A question arises: how are splitting trees representative as a subclass of the through tree class? This statement is true: all splitting trees are through trees and every through tree is isomorphic to some splitting tree (see theorem 3).

Theorem 3. The set of sets $\{S_i^k, k \le \propto\}$, obtained as a result of the process of splitting (starting from the set *S*), with the order relation introduced above, is a through tree of height ∝, and for all S_i^k for $l > k$ we have $S_i^k = \bigcup_j S_j^l$, where the summation is over all S_j^l for which $S_j^l \subseteq S_i^k$. For each non-final vertex, there is a through path that goes through it and ends at the upper-level vertex, which is a non-empty set. If all $S_i^\propto$ are non-empty sets, then any through path ends at a non-empty upper-level vertex. For every through tree $T^\propto$ there is a splitting tree $T_S = T_S^\propto$ which is isomorphic to $T^\propto$.

Indeed, let a non-final vertex $S_{i_l}^l$ and $x \in S_{i_l}^l$ be selected. For each *m*: $l < m < \propto$, there is $S_{i_m}^m$ for which $x \in S_{i_m}^m$. Then $\left(S_{i_k}^k, l \le k < \propto\right)$ is the upper segment of a through path passing through the vertex $S_{i_l}^l$ which has a terminating vertex located at level ∝ and containing *x*. In view of the arbitrariness of the choice of $S_{i_l}^l$, it follows that the constructed tree is through.

To prove the last statement, let us introduce a set *S* and pairwise disjoint non-empty sets $S_j^\propto$ in such a way that *S* is the union of sets $S_j^\propto$ and there is a one-to-one correspondence between the sets $S_j^\propto$ and the vertices $v_j^\propto$ of the tree (in particular, and it will be most natural, one can take various atomic sets as $S_j^\propto$). Further, for each non-final vertex v_i^k of the tree $T^\propto$ at the level $k < \propto$, we introduce the set $S_i^k = \bigcup_j S_j^\propto$, where the union is taken over all those $S_j^\propto$ for which there is a path in $T^\propto$ from the vertex v_i^k to the corresponding vertex $v_j^\propto$. Obviously, the constructed tree is isomorphic to $T^\propto$ and can be obtained as a result of the splitting process described above, with the appropriate choice of splitting parameters.

By virtue of theorem 3, a splitting tree cannot be strange. Thus, no splitting process can lead to the appearance of a strange tree.

9. Conclusion

The existence of strange trees is proven and it is shown that, as a result of their existence, a situation is created in set theory when its consistency is in serious question (theorems 1 - 3, first and second intuitive considerations). In the continuation of the work, an approach to proving the inconsistency of set theory will be formulated and, with its help, new strong results will be obtained.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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