Advances in Pure Mathematics, 2023, 13, 347-368
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

The Well-Posed Operators with Their Spectra in L_w^p-Spaces

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How to cite this paper: Ibrahim, S.E.-S. (2023) The Well-Posed Operators with Their Spectra in L_w^p-Spaces. *Advances in Pure Mathematics*, **13**, 347-368.
https://doi.org/10.4236/apm.2023.136023

Received: March 23, 2023
Accepted: June 24, 2023
Published: June 27, 2023

Abstract

In this paper, we have considered the general ordinary quasi-differential operators generated by a general quasi-differential expression $\tau_{p,q}$ in L_w^p-spaces of order n with complex coefficients and its formal adjoint $\tau_{q',p'}^{+}$ in L_w^q-spaces for arbitrary $p,q \in [1,\infty)$. We have proved in the case of one singular end-point that all well-posed extensions of the minimal operator $T_0(\tau_{p,q})$ generated by such expression $\tau_{p,q}$ and their formal adjoint on the interval $[a,b)$ with maximal deficiency indices have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum. This implies that all the regularly solvable operators have all the standard essential spectra to be empty. Also, a number of results concerning the location of the point spectra and regularity fields of the operators generated by such expressions can be obtained. Some of these results are extensions or generalizations of those in the symmetric case, while others are new.

Keywords

Quasi-Differential Expressions, Regular and Singular Endpoints, Minimal and Maximal Operators, Regularly Solvable Operators, Well-Posed Operators, Deficiency Indices

1. Introduction

In [1] Akhiezer and Glazman studied that the self-adjoint extension S of the minimal operator $T_0(\tau)$ generated by a formally symmetric differential expression τ with maximal deficiency indices have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum. In [2]-[5], the relationship between the square-integrable solutions for real values of the spectral parameter and the spectrum of self-adjoint ordinary differential operators of even order with real coefficients and arbitrary deficiency index are studied.

The main results of Evans, Sobhy El-Sayed and others in [6] [7] concerning the general ordinary quasi-differential operators are generalized to L_w^p-spaces with an arbitrary interval (a,b) (see [8] [9]). Also, the results includes those in [10]-[16].

The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression τ are those which are regularly solvable with respect to the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ generated by a general ordinary quasi-differential expression τ and its formal adjoint τ^+ respectively, the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ form an adjoint pair of closed, densely-defined operators in the underlying L_w^2-space, that is $T_0(\tau)\subset\left[T_0(\tau^+)\right]^*$. Such an operator S satisfies $T_0(\tau)\subset S\subset\left[T_0(\tau^+)\right]^*$ and for some $\lambda\in\mathbb{C}$, $(S-\lambda I)$ is a Fredholm operator of zero index, this means that S has the desirable Fredholm property that the equation $(S-\lambda I)u=f$ has a solution if and only if f is orthogonal to the solution space of $(S-\lambda I)u=0$ and furthermore the solution spaces of $(S-\lambda I)u=0$ and $\left(S^*-\bar{\lambda}I\right)v=0$ have the same finite dimension. This notion was originally and due to Visik [17]-[24].

Our objective in this paper, is to generalize the results in [11]-[16] for symmetric case and results of Sobhy El-sayed in [18]-[23] for general quasi-differential operators to L_w^p spaces in an analogue of Hilbert Frentzen in [8] [9]. A rather general class of quasi-differential expressions $\tau_{p,q}$ with matrix-valued coefficients and the associated maximal operators $T(\tau_{p,q})$ and minimal operator $T_0(\tau_{p,q})$ as maps of a subspace of L_w^p into L_w^q for arbitrary $p,q\in[1,\infty)$. Also, we have shown in the case of one singular end-point that all well-posed extensions of the minimal operator $T_0(\tau_{p,q})$ generated by such expression $\tau_{p,q}$ and their formal adjoint on the interval $[a,b)$ with maximal deficiency indices have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum. This implies that all the regularly solvable operators have all the standard essential spectra to be empty. The domains of these operators are described in terms of boundary conditions involving the p-integrable solutions of the quasi-differential equation $\tau_{p,q}[u]=\lambda uw$ and the adjoint equation $\tau^+_{q',p'}[v]=\bar{\lambda}vw(\lambda\in\mathbb{C})$ (see [21]).

For (formally) symmetric differential expressions τ and L_w^2 space much work has been done on the problem of finding self-adjoint differential operator $T(\tau)$ with the aid of boundary conditions as in A. N. Krall and A. Zettl [13], Naimark [14], D. Race [15] [16], Wang [25], Zettl [26] and Zhikhar [27] to mention only a few. If τ is not symmetric, one can ask for $T(\tau)$ which are Fredholm operators with index zero. For L_w^2 space and second order scalar differential expressions this question was answered by Evans and Edmunds [4] [5], for scalar differential expressions of general order on half-open intervals by Evans and Sobhy [6] and on open intervals by Sobhy El-sayed in [17]-[23].

The results herein include those of D. Race [15] [16], but also gives a description of the domain of the maximal symmetric extensions of $T_0(\tau_{p,q})$ in the case when $T_0(\tau_{p,q})$ is a symmetric operator of unequal deficiency indices. Another noteworthy special case of our result is that of Zhikhar [27] concerning the *J*-self-adjoint extensions of *J*-symmetric differential operators $T_0(\tau_{p,q})$, where *J* denotes complex conjugation.

The results include those of Jiong Sun [2], R. Agarwal [3] and Zettl [26] concerning self-adjoint realizations of symmetric operators when the minimal operator $T_0(\tau_{p,q})$ has equal deficiency indices. Also, includes those of Evans [4] [5], D. Race [15] [16] and N. A. Zhikhar [27] for the special case that concerns the *J*-self-adjoint operators, where *J* denotes complex conjugation. If the deficiency indices are unequal the maximal differential operators $T(\tau_{p,q})$ are determined by the results herein.

2. Notation and Preliminaries

We begin with a brief survey of adjoint pairs of operators and their associated regularly solvable operators; their full treatment can be found in ([5], Chapter III), ([6]-[25]). The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of T, written $nul(T)$, is the dimension of $N(T)$ and the deficiency of T, written $def(T)$, is the co-dimension of $R(T)$ in H; thus if T is densely defined and $R(T)$ is closed, then $def(T)=nul(T^*)$. The Fredholm domain of T is (in the notation of [4]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda\in\mathbb{C}$ which are such that $(T-\lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus $\lambda\in\Delta_3(T)$ if and only if $(T-\lambda I)$ has closed range and finite nullity and deficiency. The index of $(T-\lambda I)$ is the number $ind(T-\lambda I)=nul(T-\lambda I)-def(T-\lambda I)$, this being defined for $\lambda\in\Delta_3(T)$.

Two closed densely defined operators A and B acting in a Hilbert space H are said to form an adjoint pair if $A\subset B^*$ and, consequently $B\subset A^*$; equivalently, $(Ax,y)=(x,By)$ for all $x\in D(A)$ and $y\in D(B)$, where $(.,.)$ denotes the inner-product on H.

Definition 2.1: The field of regularity $\Pi(A)$ of A is the set of all $\lambda\in\mathbb{C}$ for which there exists a positive constant $K(\lambda)$ such that

$$\|(A-\lambda I)x\|\geq K(\lambda)\|x\| \text{ for all } x\in D(A), \tag{2.1}$$

or equivalently, on using the Closed Graph Theorem, $nul(A-\lambda I)=0$ and $R(A-\lambda I)$ is closed.

The joint field of regularity $\Pi(A,B)$ of A and B is the set of $\lambda\in\mathbb{C}$ which are such that $\lambda\in\Pi(A)$, $\bar{\lambda}\in\Pi(B)$ and both $def(A-\lambda I)$ and $def(B-\bar{\lambda}I)$ are finite. An adjoint pair A and B is said to be compatible if $\Pi(A,B)\neq\phi$.

Definition 2.2: A closed operator S in Hilbert space H is said to be regularly solvable with respect to the compatible adjoint pair of A and B if $A\subset S\subset B^*$

and $\Pi(A,B)\cap\Delta_4(S)\neq\phi$, where $\Delta_4(S)=\{\lambda:\lambda\in\Delta_3(S), ind(S-\lambda I)=0\}$.

Definition 2.3: The resolvent set $\rho(S)$ of a closed operator S in H consists of the complex numbers λ for which $(S-\lambda I)^{-1}$ exists, is defined on H and is bounded. The complement of $\rho(S)$ in $\mathbb{C}$ is called the spectrum of S and written $\sigma(S)$. The point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ are the following subsets of $\sigma(S)$ (see [5] [11] [12] [18] [19] [20] [23] [24] [25]).

$\sigma_p(S)=\{\lambda\in\sigma(S):(S-\lambda I) \text{ is not injective}\}$, *i.e.*, the set of eigenvalues of *S*;

$$\sigma_c(S)=\left\{\lambda\in\sigma(S):(S-\lambda I) \text{ is injective}, R(S-\lambda I)\subsetneq\overline{R(S-\lambda I)}=H\right\};$$

$$\sigma_r(S)=\left\{\lambda\in\sigma(S):(S-\lambda I) \text{ is injective}, \overline{R(S-\lambda I)}\neq H\right\}.$$

For a closed operator S we have,

$$\sigma(S)=\sigma_p(S)\cup\sigma_c(S)\cup\sigma_r(S). \tag{2.2}$$

An important subset of the spectrum of a closed densely defined operator S in H is the so-called essential spectrum. The various essential spectra of S are defined as in [5, Chapter 9] to be the sets:

$$\sigma_{ek}(S)=\mathbb{C}\setminus\Delta_k(S),(k=1,2,3,4,5), \tag{2.3}$$

where $\Delta_3(S)$ and $\Delta_4(S)$ have been defined earlier.

Definition 2.4: For two closed densely defined operators A and B acting in H, if $A\subset S\subset B^*$ and the resolvent set $\rho(S)$ of S is nonempty (see [5]), S is said to be well-posed with respect to A and B.

Note that, if $A\subset S\subset B^*$ and $\lambda\in\rho(S)$ then $\lambda\in\Pi(A)$ and $\bar{\lambda}\in\rho(S^*)\subset\Pi(B)$ so that if $def(A-\lambda I)$ and $def(B-\bar{\lambda}I)$ are finite, then A and B are compatible, in this case S is regularly solvable with respect to A and B. The terminology "regularly solvable" mentioned by Visik in [4] [5] [6] [23] and [24], while the notion of "well-posed" was introduced by Zhikhar in [27].

Theorem 2.5: (cf. ([5], Theorem III.3.1)). Let T be a closed, densely-defined operator in H with $\Pi(A)\neq\varnothing$. Then for any closed extension S of T and $\lambda\in\Pi(T)$ we have $D(S)=D(T)\dot{+}N([T^*-\bar{\lambda}I][S-\lambda I])$. If $\lambda\in\Pi(T)\cap\Delta_4(S)$ and $def(T-\lambda I)<\infty$, then

$$\dim\{D(S)/D(T)\}=ind(S-\lambda I)+def(T-\lambda I).$$

If A is symmetric operator, then with $B=A$, the operators S which conform to Definition 2.1 are the self-adjoint or maximal symmetric extensions of A. In this case when A is *J*-symmetric relative to complex conjugation *J*, A and $B=JAJ$ form an adjoint pair with $\Pi(A,B)=\Pi(A)$; any *J*-self-adjoint extension of A whose resolvent set is nonempty, is regularly solvable with respect to A and *JAJ*. For the above results, see (([5], Chapter III), [18] [19] [20] [23] [24] [25] and [27]).

Throughout this paper, let $\mathbb{K}$ denote either $\mathbb{R}$, the field of real numbers, or $\mathbb{C}$, the field of complex numbers. For a positive integers k and m let $\mathbb{M}_{k,m}$ denote the vector space of $k \times m$ matrices with $\mathbb{K}$-valued entries and GL_m the subset of $\mathbb{M}_m := \mathbb{M}_{m,m}$ consisting of all non-singular matrices. For $A \in \mathbb{M}_{k,m}$, let A^{T} denote the transpose and A^* the adjoint, *i.e.*, the complex conjugate transpose of.

If Z is a subset of $\mathbb{M}_{k,m}$ and I is an interval, $B(I,Z)$ denotes the set of Lebesgue measurable maps of I into Z and $AC_{loc}(I,Z)$ the set of locally absolutely continuous maps. Measurable maps are regarded as equal if they are equal almost everywhere on I. Further we define:

$$L^p(I,Z) := \left\{ y \in B(I,Z) \mid |y|^p \text{ is Lebesgue-integrable} \right\},$$

$$\|y\|_{p,I} := \left(\int_I |y|^p \right)^{\frac{1}{p}} \text{ for all } y \in L^p(I,Z) \text{ and } p \in [1,\infty),$$

$$L^\infty(I,Z) := \left\{ y \in B(I,Z) \mid y \text{ is essental bounded} \right\},$$

$$\|y\|_{\infty,I} := \operatorname{ess\,sup}_{x\in I} |y(x)| \text{ for all } y \in L^\infty(I,Z),$$

and

$$L^p_{loc}(I,Z) := \left\{ y \in B(I,Z) \mid y \mid K \in L^p(K,Z) \text{ for all compact subintervals } K \text{ of } I \right\}$$

for all $p \in [1,\infty)$.

If $r \in [1,\infty)$, then $r' \in [1,\infty)$ is always chosen such that $\frac{1}{r} + \frac{1}{r'} = 1$. We always assume that $p,q \in [1,\infty)$. If $L^p := L^p(I,\mathbb{K}^s)$ for some positive integer s, then $(L^p)^* = L^{p'}$ for $p \in [1,\infty)$ and L^1 is a subspace of $(L^\infty)^*$, where $(.)^*$ denotes the complex conjugate transpose. We refer to [8] and [9] for more details.

3. The Quasi-Differential Expressions

Let I be an interval with endpoints $a,b (-\infty \le a < b \le \infty)$, let n,s be positive integrs and $p,q \in [1,\infty)$. The quasi-differential expressions are defined in terms of a Shin-Zettl matrix F on the interval I.

Definition 3.1 [7] [8] [9]: The set $Z^{p,q}_{n,s}(I)$ of Shin-Zettl matrices on I consists of matrices is defined to be the set of all lower triangular matrices $F = \{f_{j,k}\}$ of the form

$$F = \begin{pmatrix} f_{0,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ f_{n,1} & \cdots & f_{n,n+1} \end{pmatrix},$$

whose entries are complex-valued functions on I which satisfy the following conditions:

$$\left.\begin{array}{l} f_{0,1} \in L^p_{loc}(I,\mathbb{M}_s) \text{ and } f_{n,n+1} \in L^{q'}_{loc}(I,\mathbb{M}_s), \\ f_{j,k} \in L^p_{loc}(I,\mathbb{M}_s) \text{ for all } 1 \le j \le n \text{ and } 1 \le k \le \min\{j+1,n\}, \\ f_{j,j+1}(x) \in GL_s \text{ for all } 0 \le j \le n \text{ and } x \in I. \end{array}\right\} \tag{3.1}$$

For $F \in Z^{p,q}_{n,s}(I)$ we define $\tilde{F}$ as the $(n \times n)$ matrix obtained from F by removing the first row and the last column, *i.e.*,

$$\tilde{F}=\begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ f_{n-1,1} & f_{n-1,2} & \cdots & f_{n-1,n} \\ f_{n,1} & f_{n,2} & \cdots & f_{n,n} \end{pmatrix},$$

Definition 3.2 [8] [9]: For $\tilde{F}\in Z_{n,s}^{p,q}(I)$, the quasi-derivatives associated with $\tilde{F}$ are defined by

$$y_{\tilde{F}}^{[0]}:=y_{\tilde{F}},$$

$$y_{\tilde{F}}^{[j]}:=\left(f_{j,j+1}\right)^{-1}\left\{\left(y_{\tilde{F}}^{[j-1]}\right)'-\sum_{k=1}^{j}f_{j,k}y_{\tilde{F}}^{[k-1]}\right\},\left(1\le j\le n-1\right) \tag{3.2}$$

$$y_{\tilde{F}}^{[n]}:=\left\{\left(y_{\tilde{F}}^{[n-1]}\right)'-\sum_{k=1}^{n}f_{j,k}y_{\tilde{F}}^{[k-1]}\right\},$$

where the prime' denotes differentiation.

The quasi-differential expression $\tau_{\tilde{F}}$ associated with $\tilde{F}$ is given by:

$$\tau_{\tilde{F}}[.]:=i^{n}y_{\tilde{F}}^{[n]},\ (n\ge 2), \tag{3.3}$$

this being defined on the set:

$$V\left(\tau_{\tilde{F}}\right):=\left\{y_{\tilde{F}}:y_{\tilde{F}}^{[j-1]}\in AC_{loc}\left(I,\mathbb{K}^{s}\right),1\le j\le n\right\}$$

where $AC_{loc}\left(I,\mathbb{K}^{s}\right)$, denotes the set of functions which are locally absolutely continuous on every compact subinterval of *I*.

For $y\in V\left(\tau_{\tilde{F}}\right)$, we define $Q_{\tilde{F}}y:=\begin{pmatrix} y_{\tilde{F}}^{[0]} \\ \vdots \\ y_{\tilde{F}}^{[n-1]} \end{pmatrix}$.

Clearly the maps $\tau_{\tilde{F}}:V\left(\tau_{\tilde{F}}\right)\to B\left(I,\mathbb{K}^{s}\right)$ and $Q_{\tilde{F}}:V\left(\tau_{\tilde{F}}\right)\to AC_{loc}\left(I,\mathbb{K}^{n,s}\right)$ are linear.

In analogy to the adjoint and the transpose of a matrix, there are two different "(formal) adjoint" of a quasi-differential expression, we refer to [8] [9] and [21] for more details.

In the following we always assume that $\tilde{F}\in Z_{n,s}^{p,q}$ and $\tau_{\tilde{F}}:=\tau_{p,q}$.

The formal adjoint $\tau_{p,q}^{+}$ of $\tau_{p,q}$ is defined by the matrix $\tilde{F}^{+}$ which given by:

$$\tilde{F}^{+}=-J_{n}^{-1}\tilde{F}^{*}J_{n} \tag{3.4}$$

where $\tilde{F}^{*}$ is the conjugate transpose of $\tilde{F}$ and J_{n} is the non-singular $(n\times n)$ matrix

$$J_{n}=\left((-1)^{j}\delta_{j,n+1-k}\right)_{\substack{1\le j\le n\\1\le k\le n}}, \tag{3.5}$$

δ being the Kronecker delta. If $\tilde{F}^{+}=f_{j,k}^{+}$ then it follows that

$$f_{j,k}^{+}=(-1)^{j+k+1}\overline{f}_{n-k+1,n-j+1}. \tag{3.6}$$

The quasi-derivatives associated with the matrix $\tilde{F}^{+}$ in $Z_{n,s}^{p,q}(I)$ are therefore

$$y_+^0 := y,$$

$$y_+^{[j]} := \left(\overline{f}_{n-j,n-j+1}\right)^{-1}\left\{\left(y_+^{[j-1]}\right)' - \sum_{k=1}^{j}(-1)^{j+k+1}\overline{f}_{n-k+1,n-j+1}y_+^{[k-1]}\right\}, \tag{3.7}$$

$$y_+^{[n]} := \left\{\left(y_+^{[n-1]}\right)' - \sum_{k=1}^{n}(-1)^{n+k+1}\overline{f}_{n-k+1,1}y_+^{[k-1]}\right\},$$

and

$$\tau_{q',p'}^{+}[.] := i^n y_+^{[n]}\left(n \geq 2\right) \text{ for all } y \in V\left(\tau_{q',p'}^{+}\right), \tag{3.8}$$

$$V\left(\tau_{q',p'}^{+}\right) := \left\{y : y_+^{[j-1]} \in AC_{loc}\left(I, \mathbb{K}^s\right), 1 \leq j \leq n\right\}, \tag{3.9}$$

Note that: $\left(\tilde{F}^{+}\right)^{+} = \tilde{F}$ and so $\left(\tau_{q',p'}^{+}\right)^{+} = \tau_{p,q}$. We refer to [5] [6] [7] [8] [9] and [21] for a full account of the above and subsequent results on quasi-differential expressions.

For $u \in V\left(\tau_{p,q}\right)$, $v \in V\left(\tau_{q',p'}^{+}\right)$ and $\alpha, \beta \in I$, we have Green's formula,

$$\int_{\alpha}^{\beta}\left\{\overline{v}\tau_{p,q}[u] - u\overline{\tau_{q',p'}^{+}[v]}\right\}\mathrm{d}x = [u,v](\beta) - [u,v](\alpha), \tag{3.10}$$

where,

$$\begin{aligned}
[u,v](x) &= i^n\left(\sum_{r=0}^{n-1}(-1)^{r+n+1}u^{[r]}\overline{v_+^{[n-r-1]}}\right)(x) \\
&= (-i)^n\left(Q_{\tilde{F}}^{\mathrm{T}}uJ_{n\times n}Q_{\tilde{F}}\overline{v}\right)(x) \\
&= (-i)^n\left(u, u^{[1]}, \cdots, u^{[n-1]}\right)J_{n\times n}\begin{pmatrix}\overline{v} \\ \vdots \\ \overline{v}_+^{[n-1]}\end{pmatrix}(x),
\end{aligned}$$

$$J_{n\times n} = \begin{pmatrix} 0 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ (-1)^n & \cdots & 0 \end{pmatrix}; \tag{3.11}$$

see [5] [6] [7] [8] [9] [17] [24] and [26]. Let $w: I \to \mathbb{R}$ be a non-negative weight function with $w \in L_{loc}^1(I)$ and $w > 0$ (for almost all $x \in I$). Then $H^r = L_w^r\left(I, \mathbb{K}^s\right)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that

$$\|y\|_{r,I} := \left(\int_I |y|^r w\right)^{\frac{1}{r}} \text{ for all } y \in L^r(I, Z) \text{ and } r \in [1, \infty). \tag{3.12}$$

The equation

$$\tau_{p,q}[u] - \lambda wu = 0\left(\lambda \in \mathbb{C}\right) \text{ on } I, \tag{3.13}$$

is said to be regular at the left end-point $a \in \mathbb{R}$, if for all $X \in (a,b)$,

$$a \in \mathbb{R}, w, f_{j,k} \in L^1(a, X), j,k = 1,2,\cdots,n,$$

otherwise (3.13) is said to be singular at *a*. If (3.13) is regular at both end-points, then it is said to be regular; in this case we have,

$$a,b\in\mathbb{R},\ w,f_{j,k}\in L^{1}(a,b),\ j,k=1,2,\cdots,n. \tag{3.14}$$

We shall be concerned with the case when a is a regular end-point of Equation (3.13), the end-point b being allowed to be either regular or singular. Note that, in view of (3.14), an end-point of I is regular for (3.13), if and only if it is regular for the equation

$$\tau^{+}_{p,q}[v]-\bar{\lambda}wv=0\ (\lambda\in\mathbb{C})\ \text{ on } I. \tag{3.15}$$

Definition 3.3 [5]-[21] [26]:

1) The maximal operators corresponding to $\tau_{p,q},\tau^{+}_{q',p'}$ are defined as operators from a subspaces of L^{p}_{w} into L^{q}_{w}, p,q are arbitrary.

$T(\tau_{p,q})=w^{-1}\tau[u]$ for all

$$u\in D\left[T(\tau_{p,q})\right]:=\left\{u\in V(\tau_{p,q})\cap L^{p}_{w}\mid w^{-1}\tau_{p,q}[u]\in L^{q}_{w}\right\},$$

$T(\tau^{+}_{q',p'})=w^{-1}\tau^{+}[v]$ for all

$$v\in D\left[T(\tau^{+}_{q',p'})\right]:=\left\{v\in V(\tau^{+}_{q',p'})\cap L^{p}_{w}\mid w^{-1}\tau^{+}_{q',p'}[v]\in L^{q}_{w}\right\}.$$

The subspaces $D\left[T(\tau_{p,q})\right]$ and $D\left[T(\tau^{+}_{q',p'})\right]$ of $L^{p}_{w}(I,\mathbb{K}^{s})$ are the domains of the so-called maximal operators $T(\tau_{p,q})$ and $T(\tau^{+}_{p,q})$ respectively.

2) For the regular problem the minimal operators $T_0(\tau_{p,q})$ and $T_0(\tau^{+}_{q',p'})$ are the restrictions of $w^{-1}\tau_{p,q}[u]$ and $w^{-1}\tau^{+}_{q',p'}[v]$ to the subspaces:

$$\left.\begin{aligned}D_0(\tau_{p,q})&:=\left\{u:u\in D(\tau_{p,q}),(Q_{\tilde{F}}u)(a)=(Q_{\tilde{F}}u)(b)=0\right\}\\ D_0(\tau^{+}_{q',p'})&:=\left\{v:v\in D(\tau^{+}_{q',p'}),(Q_{\tilde{F}}v_{+})(a)=(Q_{\tilde{F}}v_{+})(b)=0\right\}\end{aligned}\right\} \tag{3.16}$$

The subspaces $D_0(\tau_{p,q})$ and $D_0(\tau^{+}_{q',p'})$ are dense in $L^{p}_{w}(I,\mathbb{K}^{s})$ and $T_0(\tau_{p,q})$ and $T_0(\tau^{+}_{q',p'})$ are closed operators (see [4] ([5] Section 3) and [6]-[21]).

In the **singular problem** we first introduce the operators $T'_0(\tau_{p,q})$ and $T'_0(\tau^{+}_{q',p'})$; $T'_0(\tau_{p,q})$ being the restriction of $w^{-1}\tau_{p,q}[.]$ to the subspace

$$D'_0(\tau_{p,q}):=\left\{u:u\in D(\tau_{p,q})\mid supp(u)\text{ is compact and contained in the interior of } I\right\}.$$

and with $T'_0(\tau^{+}_{q',p'})$ defined similarly. These operators are densely-defined and closable in $L^{p}_{w}(I,\mathbb{K}^{s})$; and we define the minimal operators $T_0(\tau_{p,q})$ and $T_0(\tau^{+}_{q',p'})$ to be their respective closures (see [5]-[14] [21] and [26]). We denote the domains of $T_0(\tau_{p,q})$ and $T_0(\tau^{+}_{q',p'})$ by $D_0(\tau_{p,q})$ and $D_0(\tau^{+}_{q',p'})$ respectively. It can be shown that:

$$\left.\begin{aligned}u\in D_0(\tau_{p,q})&\Rightarrow(Q_{\tilde{F}}u)(a)=0\\ v\in D_0(\tau^{+}_{p,q})&\Rightarrow(Q_{\tilde{F}}v_{+})(a)=0\end{aligned}\right\}, \tag{3.17}$$

because we are assuming that a is a regular end-point. Clearly $T_0(\tau_{p,q})$ and $T(\tau_{p,q})$ are linear operators of L^{p}_{w} into L^{q}_{w} and $T_0(\tau_{p,q})\subset T(\tau_{p,q})$.

Moreover, in both regular and singular problems, we have

$$T^{*}_0(\tau_{p,q})=T(\tau^{+}_{q',p'}),\quad T(\tau_{p,q})=T^{*}_0(\tau^{+}_{q',p'}), \tag{3.18}$$

see [6]-[16] and ([26], Section 5) in the case when $\tau_{p,q}=\tau^{+}_{q',p'}$ and compare with treatment in ([5], Section III.10.3), [6] in general case. Also, we refer to [17]-[22] for more details.

Corollary 3.4 (cf. ([8], Corollary 3.10) and [9]):

a) If $\tau_{2,2}$ is symmetric, then $T_0(\tau_{2,2})$ is symmetric in the Hilbert space $L_w^2(a,b)$.

b) If $\tau_{2,2}$ is *J*-symmetric, then $T_0(\tau_{2,2})$ is *J*-symmetric in the Hilbert space $L_w^2(a,b)$.

4. L_w^p-Solutions

In this section, we shall concerned with L_w^p-Solutions of general ordinary quasi-differential equations, and we denote for $\tau_{p,q}$ by τ and $\tau_{p,q}^+$ by τ^+.

Denote by $S(\tau)$ and $S(\tau^+)$ the sets of all solutions of the equations

$$[\tau-\lambda_0 I]u=0\ (\lambda_0\in\mathbb{C}) \tag{4.1}$$

and

$$\left[\tau^+-\overline{\lambda_0}I\right]v=0\ (\lambda_0\in\mathbb{C}) \tag{4.2}$$

respectively. Let $\varphi_j(t,\lambda), j=1,2,\cdots,n$ be the solutions of the homogeneous equation $[\tau-\lambda I]u=0(\lambda\in\mathbb{C})$ satisfying:

$\varphi_j^{[r]}(t_0,\lambda)=\delta_{j,r+1}$ for all $t_0\in[a,b)$ $(j=1,2,\cdots,n; r=0,1,\cdots,n-1)$

for fixed t_0, $a<t_0<b$. Then $\varphi_j(t,\lambda)$ is continuous in (t,λ) for $0<t<b$, $|\lambda|<\infty$, and for fixed *t* it is entire in λ. Let $\varphi_k^+(t,\lambda), k=1,2,\cdots,n$ denote the solutions of the adjoint homogeneous equation $\left[\tau^+-\overline{\lambda}I\right]v=0(\lambda\in\mathbb{C})$ satisfying:

$\left(\varphi_k^+\right)^{[r]}(t_0,\lambda)=(-1)^{k+r}\delta_{k,n-r}$ for all $t_0\in[a,b)$ $(k=1,2,\cdots,n; r=0,1,\cdots,n-1)$.

Suppose $a<c<b$, by [5] [21] [22] [23] [24] and [26], a solution of the quasi-differential equation

$$[\tau-\lambda I]\varphi=wf\ (\lambda\in\mathbb{C}),\ f\in L_w^1(a,b) \tag{4.3}$$

satisfying $\varphi^{[r]}(c)=0, r=0,1,\cdots,n-1$ is giving by:

$$\varphi(t,\lambda)=\left(\frac{\lambda-\lambda_0}{i^n}\right)\sum_{j,k=1}^n\xi^{jk}\varphi_j(t,\lambda)\int_a^t\overline{\varphi_k^+(s,\lambda)}f(s)w(s)\mathrm{d}s,$$

where $\varphi_k^+(s,\lambda)$ stands for the complex conjugate of $\varphi_k(t,\lambda)$ and for each j,k, ξ^{jk} is constant which is independent of t,λ (but does depend in general on *t*).

The next lemma is a form of the variation of parameters formula for a general quasi-differential equation is giving by the following Lemma.

Lemma 4.1: Suppose $f\in L_w^1(0,b)$ locally integrable function and $\varphi(t,\lambda)$ is the solution of Equation (4.3) satisfying:

$$\varphi^{[r]}(t_0,\lambda)=\alpha_{r+1}\ \text{ for }\ r=0,1,\cdots,n-1,\ t_0\in[a,b).$$

Then

$$\varphi(t,\lambda)=\sum_{j=1}^{n}\alpha_j(\lambda)\varphi_j(t,\lambda_0)+\left((\lambda-\lambda_0)/i^n\right)\sum_{j,k=1}^{n}\xi^{jk}\varphi_j(t,\lambda_0)\times\int_a^t\overline{\varphi_k^+(s,\lambda_0)}f(s)w(s)\mathrm{d}s \tag{4.4}$$

for some constants $\alpha_1(\lambda),\alpha_2(\lambda),\cdots,\alpha_n(\lambda)\in\mathbb{C}$, where $\varphi_j(t,\lambda_0)$ and $\varphi_k^+(t,\lambda_0), j,k=1,2,\cdots,n$ are solutions of Equations (4.1) and (4.2) respectively, ξ^{jk} is a constant which is independent of *t*.

Theorem 4.2: (cf. [5] [6]). Let τ be a regular quasi-differential expression of order n on the interval [a, b]. For $f\in L_w^2(a,b)$, the equation $\tau[\varphi]=wf$ has a solution $\varphi\in V(\tau)$ satisfying

$$\varphi^{[r]}(a)=\varphi^{[r]}(b)=0,\ r=0,1,2,\cdots,n-1$$

If and only if *f* is orthogonal in $L_w^2(a,b)$ to solution space of $\tau^+[\psi]=0$, *i.e.*,

$$R\left[T_0(\tau)-\lambda I\right]=N\left[T(\tau^+)-\bar{\lambda}I\right]^{\perp}.$$

Corollary 4.3 (cf. [19] [20] [21]), As a result from Theorem 4.2, we have that

$$R\left[T_0(\tau)-\lambda I\right]^{\perp}=N\left[T(\tau^+)-\bar{\lambda}I\right].$$

Lemma 4.4 [22]: (Gronwall's inequality). Let $u(t)$ and $v(t)$ be two real-functions defined, non-negative and *u*, $v\in L^1(t_0,t)$ for $t>t_0$, and if

$$u(t)\le c+\int_{t_0}^t u(s)v(s)\mathrm{d}s,\ c>0,$$

for some positive constant *c*, then

$$u(t)\le c\exp\left(\int_0^t v(s)\mathrm{d}s\right). \tag{4.5}$$

Lemma 4.5: Suppose $f\in L_{loc}^1(I)$, and suppose that the Conditions (3.1) are satisfied. Then, given any complex numbers $c_r\in\mathbb{C}$, $r=0,1,2,\cdots,n-1$ and $x_0\in(a,b)$, there exists a unique solutions of $(\tau-\lambda)\varphi=wf$ $(\lambda\in\mathbb{C})$ which satisfies

$$\varphi^{[r]}(x_0)=c_j,\ r=0,1,2,\cdots,n-1.$$

Proof: The proof is similar to that in (14], part II, Theorem 16.2.2) and therefore omitted.

Lemma 4.6: Suppose that for some $\lambda_0\in\mathbb{C}$ all solutions of Equations (4.1) and (4.2) are in $L_w^2(a,b)$. then all solutions of Equations (4.1) and (4.2) are in $L_w^2(a,b)$ for every complex number $\lambda\in\mathbb{C}$.

Proof: The proof is similar to that in [21] [22] [23] [24], and therefore omitted.

Lemma 4.7: If all solutions of the equation $[\tau-\lambda_0 w]u=0$ are bounded on $[a,b]$ and $\varphi_k^+(t,\lambda_0)\in L_w^1(a,b)$ for some $\lambda_0\in\mathbb{C},k=1,\cdots,n$. Then all solutions of the equation $[\tau-\lambda w]u=0$ are also bounded on $[a,b]$ for every complex number $\lambda\in\mathbb{C}$.

Lemma 4.8: Suppose that for some complex number $\lambda_0\in\mathbb{C}$ all solutions of Equation (4.1) are in $L_w^p(a,b)$ and all solutions of (4.2) are in $L_w^q(a,b)$. Suppose $f \in L_w^p(a,b)$, then all solutions of Equation (4.3) are in $L_w^p(a,b)$ for all $\lambda \in \mathbb{C}$

Proof: Let $\{\varphi_1(t,\lambda_0),\cdots,\varphi_n(t,\lambda_0)\}$, $\{\varphi_1^+(s,\lambda_0),\cdots,\varphi_n^+(s,\lambda_0)\}$ be two sets of linearly independent solutions of Equations (4.1) and (4.2) respectively. Then for any solutions $\varphi(t,\lambda)$ of the equation $[\tau-\lambda I]\varphi = wf\ (\lambda \in \mathbb{C})$ which may be written as follows

$$[\tau-\lambda_0 w]\varphi = (\lambda-\lambda_0)w\varphi + wf,\ \lambda_0 \in \mathbb{C}$$

and it follows from (4.4) that

$$\begin{aligned}\varphi(t,\lambda) = \sum\nolimits_{j=1}^{n}\alpha_j(\lambda)\varphi_j(t,\lambda_0) + \frac{1}{i^n}\sum\nolimits_{j,k=1}^{n}\xi^{jk}\varphi_j(t,\lambda_0)\\ \times\int_a^t \overline{\varphi_k^+(t,\lambda_0)}\left[(\lambda-\lambda_0)\varphi(s,\lambda)+f(s)\right]w(s)\mathrm{d}s,\end{aligned} \tag{4.6}$$

for some constants $\alpha_1(\lambda),\alpha_2(\lambda),\cdots,\alpha_n(\lambda) \in \mathbb{C}$. Hence

$$\begin{aligned}|\varphi(t,\lambda)| \le \sum\nolimits_{j=1}^{n}\left(|\alpha_j(\lambda)||\varphi_j(t,\lambda_0)|\right) + \sum\nolimits_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)|\\ \times\int_a^t \overline{\varphi_k^+(t,\lambda_0)}\left[|\lambda-\lambda_0||\varphi(s,\lambda)|+|f(s)|\right]w(s)\mathrm{d}s.\end{aligned} \tag{4.7}$$

Since $f \in L_w^p(a,b)$ and $\varphi_k^+(.,\lambda_0) \in L_w^q(a,b)$ for some $\lambda_0 \in \mathbb{C}$, then $\varphi_k^+(.,\lambda_0)f \in L_w^1(a,b)$ for some $\lambda_0 \in \mathbb{C}$ and $k=1,\cdots,n$. Setting:

$$C_j(\lambda) = \sum\nolimits_{j,k=1}^{n}|\xi^{jk}|\left|\int_a^t\left|\overline{\varphi_k^+(s,\lambda_0)}\right||f(s)|w(s)\mathrm{d}s\right|,\ j=1,2,\cdots,n, \tag{4.8}$$

then

$$\begin{aligned}|\varphi(t,\lambda)| \le \sum\nolimits_{j=1}^{n}\left(|\alpha_j(\lambda)|+C_j(\lambda)\right)|\varphi_j(t,\lambda_0)| + |\lambda-\lambda_0|\\ \times\left(\sum\nolimits_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)|\left|\int_a^t\left|\overline{\varphi_k^+(s,\lambda_0)}\right||\varphi(s,\lambda)|w(s)\mathrm{d}s\right|\right).\end{aligned} \tag{4.9}$$

On application of the Cauchy-Schwartz inequality to the integral in (4.9), we get

$$\begin{aligned}|\varphi(t,\lambda)| \le \sum\nolimits_{j=1}^{n}\left(|\alpha_j(\lambda)|+C_j(\lambda)\right)|\varphi_j(t,\lambda_0)| + |\lambda-\lambda_0|\sum\nolimits_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)|\\ \times\left(\int_a^t\left|\overline{\varphi_k^+(t,\lambda_0)}\right|^q w(s)\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_a^t|\varphi(s,\lambda)|^p w(s)\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned} \tag{4.10}$$

From the inequality $(u+v)^p \le 2^{p-1}(u^p+v^p)$, it follows that

$$\begin{aligned}|\varphi(t,\lambda)|^p \le 2^{2(p-1)}\sum\nolimits_{j=1}^{n}\left(|\alpha_j(\lambda)|+C_j(\lambda)\right)^p|\varphi_j(t,\lambda_0)|^p\\ +2^{2(p-1)}|\lambda-\lambda_0|^p\sum\nolimits_{j,k=1}^{n}|\xi^{jk}|^p|\varphi_j(t,\lambda_0)|^p\\ \times\left(\int_a^t\left|\overline{\varphi_k^+(t,\lambda_0)}\right|^q w(s)\mathrm{d}s\right)^{\frac{p}{q}}\left(\int_a^t|\varphi(s,\lambda)|^p w(s)\mathrm{d}s\right).\end{aligned} \tag{4.11}$$

By hypothesis there exist positive constant K_0 and K_1 such that

$$\|\varphi_j(t,\lambda_0)\|_{L_w^p(a,b)} \le K_0 \text{ and } \left\|\overline{\varphi_k^+(s,\lambda_0)}\right\|_{L_w^q(a,b)} \le K_1,\quad j,k=1,2,\cdots,n. \tag{4.12}$$

Hence

$$\left|\varphi(t,\lambda)\right|^p \le 2^{2(p-1)}\sum_{j=1}^n\left(\left|\alpha_j(\lambda)\right|+C_j(\lambda)\right)^p\left|\varphi_j(t,\lambda_0)\right|^p + 2^{2(p-1)}K_1^p\left|\lambda-\lambda_0\right|^p\sum_{j,k=1}^n\left|\xi^{jk}\right|^p\left|\varphi_j(t,\lambda_0)\right|^p\left(\int_a^t\left|\varphi(s,\lambda)\right|^p w(s)\mathrm{d}s\right). \tag{4.13}$$

Integrating the inequality in (4.13) between a and t, we obtain

$$\int_a^t\left|\varphi(s,\lambda)\right|^p w(s)\mathrm{d}s \le K_2 + \left(2^{2(p-1)}\left|\lambda-\lambda_0\right|^p\sum_{j,k=1}^n\left|\xi^{jk}\right|^p\right) \times\int_a^t\left|\varphi_j(s,\lambda_0)\right|^p\left(\int_a^s\left|\varphi(x,\lambda)\right|^p w(x)\mathrm{d}x\right)w(s)\mathrm{d}s, \tag{4.14}$$

where

$$K_2 = 2^{2(p-1)}K_0^p\sum_{j=1}^n\left(\left|\alpha_j(\lambda)\right|+C_j(\lambda)\right)^p. \tag{4.15}$$

Now, on using Gronwall's inequality (Lemma 4.4), it follows that

$$\int_a^t\left|\varphi(s,\lambda)\right|^p w(s)\mathrm{d}s \le K_2\exp\left(2^{2(p-1)}K_1^p\left|\lambda-\lambda_0\right|^p\sum_{j,k=1}^n\left|\xi^{jk}\right|^p\int_a^t\left|\varphi_j(s,\lambda_0)\right|^p w(s)\mathrm{d}s\right). \tag{4.16}$$

Since, $\varphi_j(t,\lambda_0)\in L_w^p(a,b)$ for some $\lambda_0\in\mathbb{C}$ and for $j=1,\cdots,n$, then $\varphi(t,\lambda)\in L_w^p(0,b)$ for all $\lambda\in\mathbb{C}$.

Remark: Lemma 4.8 also holds if the function f is bounded on $[a,b)$.

Lemma 4.9: Let $f\in L_w^p(0,b)$. Suppose for some $\lambda_0\in\mathbb{C}$ that:

(i) All solutions of $\left(\tau^+-\bar{\lambda}I\right)\varphi^+=0$ are in $L_w^q(a,b)$.

(ii) $\varphi_j(t,\lambda_0), j=1,\cdots,n$ are bounded on $[0,b)$.

Then all solutions $\varphi(t,\lambda)$ of Equation (4.3) are in $L_w^p(a,b)$ for all $\lambda\in\mathbb{C}$.

Lemma 4.10: Let $f\in L_w^p(a,b)$. Suppose for some $\lambda_0\in\mathbb{C}$ that:

(i) All solutions of $\left(\tau^+-\bar{\lambda}I\right)\varphi^+=0$ are in $L_w^q(a,b)$.

(ii) $\varphi_j^{[r]}(t,\lambda_0), j=1,\cdots,n$ are bounded on $[a,b)$ for some $r=0,1,\cdots,n-1$.

Then $\varphi^{[r]}(t,\lambda)\in L_w^p(a,b)$ for any solution $\varphi(t,\lambda)$ of the equation $(\tau-\lambda I)\varphi=wf$ for all $\lambda\in\mathbb{C}$

5. The Regularly Solvable Operators

We see from (3.18) that $T_0\left(\tau_{p,q}\right)\subset T\left(\tau_{p,q}\right)=\left[T_0\left(\tau_{q',p'}^+\right)\right]^*$ and hence $T_0\left(\tau_{p,q}\right)$, $T_0\left(\tau_{q',p'}^+\right)$ form an adjoint pair of closed-densely operators in $L_w^p\left(I,\mathbb{K}^s\right)$.

Lemma 5.1 [17] [18]: For $\lambda\in\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau_{q',p'}^+\right)\right]$,

$def\left[T_0\left(\tau_{p,q}\right)-\lambda I\right]+def\left[T_0\left(\tau_{q',p'}^+\right)-\bar{\lambda}I\right]$

is constant and

$$0\le def\left[T_0\left(\tau_{p,q}\right)-\lambda I\right]+def\left[T_0\left(\tau_{q',p'}^+\right)-\bar{\lambda}I\right]\le 2n.$$

In the problem with one singular end-point,

$n\le def\left[T_0\left(\tau_{p,q}\right)-\lambda I\right]+def\left[T_0\left(\tau_{q',p'}^+\right)-\bar{\lambda}I\right]\le 2n$, for all $\lambda\in\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau_{q',p'}^+\right)\right]$.

In the regular problem,

$def\left[T_0\left(\tau_{p,q}\right)-\lambda I\right]+def\left[T_0\left(\tau^+_{q',p'}\right)-\bar{\lambda} I\right]=2n$, for all $\lambda\in\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau^+_{q',p'}\right)\right]$.

Proof: The proof is similar to that in ([4] [5] [6]) ([17] [18] [19] [20]) and ([22] [23] [24]) and therefore omitted.

For $\lambda\in\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau^+_{q',p'}\right)\right]$, we define r, s and m as follows:

$$\left.\begin{aligned}&r=r(\lambda):=def\left[T_0\left(\tau_{p,q}\right)-\lambda I\right]=nul\left[T\left(\tau^+_{q',p'}\right)-\bar{\lambda} I\right],\\&s=s(\lambda):=def\left[T_0\left(\tau^+_{q',p'}\right)-\bar{\lambda} I\right]=nul\left[T\left(\tau_{p,q}\right)-\lambda I\right],\\&m=r+s.\end{aligned}\right\}, \tag{5.1}$$

then, $0\le r,s\le n$ and by Lemma 5.1, m is constant on $\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau^+_{q',p'}\right)\right]$ and

$$n\le m\le 2n. \tag{5.2}$$

For $\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau^+_{q',p'}\right)\right]\ne\varnothing$ the operators which are regularly solvable with respect to the minimal operators $T_0\left(\tau_{p,q}\right)$ and $T_0\left(\tau^+_{q',p'}\right)$ are characterized by the following theorem which proved for a general quasi-differential operator in ([5], Theorem 10.15).

Theorem 5.2 ([6], Theorem 3.2): For $\lambda\in\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau^+_{q',p'}\right)\right]$. Let r and m be defined by (5.1), and let $\psi^{p,q}_j\left(j=1,2,\cdots,r\right),\Phi^{p,q}_k\left(k=r+1,\cdots,m\right)$ be arbitrary functions satisfying:

(i) $\psi^{p,q}_j\left(j=1,2,\cdots,r\right)\subset D\left[T\left(\tau_{p,q}\right)\right]$ are linearly independent modulo $D\left[T_0\left(\tau_{p,q}\right)\right]$ and

$\Phi^{p,q}_k\left(k=r+1,\cdots,m\right)\subset D\left[T\left(\tau^+_{q',p'}\right)\right]$ are linearly independent modulo $D\left[T_0\left(\tau^+_{q',p'}\right)\right]$.

(ii) $\left[\psi^{p,q}_j,\Phi^{p,q}_k\right](b)-\left[\psi^{p,q}_j,\Phi^{p,q}_k\right](a)=0$, $\left(j=1,2,\cdots,r;k=r+1,\cdots,m\right)$.

Then the set

$$\left\{u:u\in D\left[T\left(\tau_{p,q}\right)\right],\left[u,\Phi^{p,q}_k\right](b)-\left[u,\Phi^{p,q}_k\right](a)=0,k=r+1,\cdots,m\right\}, \tag{5.3}$$

is the domain of an operator S which is regularly solvable with respect to $T_0\left(\tau_{p,q}\right)$ and $T_0\left(\tau^+_{q',p'}\right)$ and the set

$$\left\{v:v\in D\left[T\left(\tau^+_{q',p'}\right)\right],\left[\psi^{p,q}_j,v\right](b)-\left[\psi^{p,q}_j,v\right](a)=0,j=1,2,\cdots,r\right\} \tag{5.4}$$

is the domain of the operator S^*; moreover $\lambda\in\Delta_4(S)$.

Conversely, if S is regularly solvable with respect to $T_0\left(\tau_{p,q}\right)$ and $T_0\left(\tau^+_{q',p'}\right)$ and $\lambda\in\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau^+_{q',p'}\right)\right]\cap\Delta_4(S)$, then with r and s defined by (5.1) there exist functions $\psi^{p,q}_j\left(j=1,2,\cdots,r\right),\Phi^{p,q}_k\left(k=r+1,\cdots,m\right)$ which satisfy (i) and (ii) and are such that (5.3) and (5.4) are the domains of S and S^* respectively.

S is self-adjoint if, and only if, $\tau_{p,q}=\tau^+_{q',p'}$, $r=s$ and

$\Phi_k^{p,q} = \psi_{k-r}^{p,q}(k = r+1,\cdots,m)$; S is J-self-adjoint if $\tau_{p,q} = J\tau_{p,q}J$ (J is a complex conjugate), $r = s$ and $\Phi_k^{p,q} = \bar{\psi}_{k-r}^{p,q}(k = r+1,\cdots,m)$.

Proof: The proof is entirely similar to that in [6]. We refer also to [17]-[24] for more details.

6. The Spectra of General Differential Operators

In this subsection we deal with the various components of the spectra of quasi-differential operators $T_0(\tau_{p,q})$ and $T_0(\tau^+_{q',p'})$.

We see from (3.18) and Theorem 4.2 that $T_0(\tau_{p,q}) \subset T(\tau_{p,q}) = \left[T_0(\tau^+_{q',p'})\right]^*$ and hence $T_0(\tau_{p,q})$ and $T_0(\tau^+_{q',p'})$ form an adjoint pair of closed, closed-densely operators in $L^p_w(I,\mathbb{K}^s)$.

We shall now investigate in the case of one singular end-point that the resolvent of all well-posed extensions of the minimal operator $T_0(\tau_{p,q})$ and we show that in the maximal case, *i.e.*, when

$def\left[T_0(\tau_{p,q}) - \lambda I\right] = def\left[T_0(\tau^+_{q',p'}) - \bar{\lambda} I\right] = n$, for all

$\lambda \in \Pi\left[T_0(\tau_{p,q}), T_0(\tau^+_{q',p'})\right]$

that these resolvents are integral operators, in fact they are Hilbert-Schmidt integral operators by considering that the function f to be in $L^p_w(a,b)$, *i.e.*, is p-integrable over the interval $[a,b)$.

Theorem 6.1: Suppose for an operator $T_0(\tau_{p,q})$ with one singular end-point that,

$def\left[T_0(\tau_{p,q}) - \lambda I\right] = def\left[T_0(\tau^+_{q',p'}) - \bar{\lambda} I\right] = n$, for all

$\lambda \in \Pi\left[T_0(\tau_{p,q}), T_0(\tau^+_{q',p'})\right]$,

and let S be an arbitrary closed operator which is a well-posed extension of the minimal operator $T_0(\tau_{p,q})$ and $\lambda \in \rho(S)$, then the resolvents $R^{p,q}_\lambda$ and $R^{*(p,q)}_{\bar{\lambda}}$ of S and S^* respectively are Hilbert-Schmidt integral operators whose kernels $K_{p,q}(t,s,\lambda)$ and $K^+_{q',p'}(s,t,\lambda)$ are continuous functions on $[a,b)\times[a,b)$ and satisfy:

$$K_{p,q}(t,s,\lambda) = \overline{K^+_{q',p'}(s,t,\bar{\lambda})} \text{ and } \int_a^b\int_a^b \left|K_{p,q}(t,s,\lambda)\right|^p w(s)w(t)\mathrm{d}s\mathrm{d}t < \infty. \quad (6.1)$$

where,

$R^{p,q}_\lambda f(t) = \int_a^b K_{p,q}(t,s,\lambda) f(s) w(s)\mathrm{d}s$, for all $t \in [a,b)$, $f \in L^p_w(a,b)$.

$R^{*(p,q)}_{\bar{\lambda}} g(t) = \int_a^b \overline{K^+_{q',p'}(s,t,\bar{\lambda})} g(t) w(t)\mathrm{d}t$ for all $s \in [a,b)$, $g \in L^q_w(a,b)$.

Remark An example of a closed operator which is a well-posed with respect to a compatible adjoint pair is given by the Visik extension ([5], Theorem III.3.3) (see ([18], Theorem 1) [19] and [20]). Note that if S is well-posed, then $T_0(\tau_{p,q})$ and $T_0(\tau^+_{q',p'})$ are compatible adjoint pair and S is regularly solvable with respect to $T_0(\tau_{p,q})$ and $T_0(\tau^+_{q',p'})$.

Proof: Let $def\left[T_0(\tau_{p,q}) - \lambda I\right] = def\left[T_0(\tau^+_{q',p'}) - \bar{\lambda} I\right] = n$ for all $\lambda \in \Pi\left[T_0(\tau_{p,q}), T_0(\tau^+_{q',p'})\right]$, then we choose a fundamental system of solutions

$\{\varphi_1(t,\lambda),\varphi_2(t,\lambda),\cdots,\varphi_n(t,\lambda)\}$, $\{\varphi_1^+(t,\lambda),\varphi_2^+(t,\lambda),\cdots,\varphi_n^+(t,\lambda)\}$ of the equations,

$$\left[T_0(\tau_{p,q})-\lambda I\right]\varphi_j=0,\ \left[T_0(\tau_{q',p'}^+)-\bar{\lambda} I\right]\varphi_k^+=0\quad (j,k=1,\cdots,n)\ \text{ on } [a,b), \tag{6.2}$$

so that $\{\varphi_1(t,\lambda),\varphi_2(t,\lambda),\cdots,\varphi_n(t,\lambda)\}$, $\{\varphi_1^+(t,\lambda),\varphi_2^+(t,\lambda),\cdots,\varphi_n^+(t,\lambda)\}$ belong to $L_w^p(a,b)$ and $L_w^q(a,b)$ repectively, *i.e.*, they are p and q-integrable on the interval $[a,b)$. Let $R_\lambda^{p,q}=(S-\lambda I)^{-1}$ be the resolvent of any well-posed extension of the minimal operator $T_0(\tau_{p,q})$. For $f\in L_w^p(a,b)$ we put $\varphi(t,\lambda)=R_\lambda^{p,q}f(t)$ then $\left[T(\tau_{p,q})-\lambda I\right]\varphi=wf$ and consequently has a solution $\varphi(t,\lambda)$ in the form,

$$\begin{aligned}\varphi(t,\lambda)=&\sum\nolimits_{j=1}^{n}\alpha_j(\lambda)\varphi_j(t,\lambda_0)+\frac{1}{i^n}(\lambda-\lambda_0)\sum\nolimits_{j,k=1}^{n}\xi^{jk}\varphi_j(t,\lambda_0)\\&\times\int_a^t\overline{\varphi_k^+(s,\lambda_0)}f(s)w(s)\mathrm{d}s\end{aligned} \tag{6.3}$$

for some constants $\alpha_1(\lambda),\alpha_2(\lambda),\cdots,\alpha_n(\lambda)\in\mathbb{C}$ (see Lemma 4.5). Since $f\in L_w^p(a,b)$ and $\varphi_k^+(.,\lambda_0)\in L_w^q(a,b)$ for some $\lambda_0\in\mathbb{C}$, then $\varphi_k^+(.,\lambda_0)f\in L_w^1(a,b)$, $k=1,\cdots,n$ for some $\lambda_0\in\mathbb{C}$ and hence the integral in the right-hand of (6.3) will be finite.

To determine the constants $\alpha_j(\lambda),j=1,\cdots,n$, let $\varphi_k^+(t,\lambda),k=1,\cdots,n$ be a basis for $\left\{D(S^*)/D_o\left[T(\tau_{q',p'}^+)\right]\right\}$, then because $\varphi(t,\lambda)\in D(S)\subset\rho(S)\subset\Delta_4(S)$, we have from Theorem 5.2 that,

$$\left[\varphi,\varphi_k^+\right](b)-\left[\varphi,\varphi_k^+\right](a)=0,\ (k=1,2,\cdots,n)\ \text{ on } [a,b) \tag{6.4}$$

and hence from (6.3), (6.4) and on using Lemma 4.1, we have:

$$\begin{aligned}&\left[\varphi,\varphi_k^+\right](b)\\&=\sum\nolimits_{j=1}^{n}\left[\alpha_j(\lambda)+\frac{1}{i^n}(\lambda-\lambda_0)\sum\nolimits_{j,k=1}^{n}\xi^{jk}\int_a^t\overline{\varphi_k^+(s,\lambda_0)}f(s)w(s)\mathrm{d}s\right]\left[\varphi_j,\varphi_k^+\right](b),\end{aligned}$$

$$\left[\varphi,\varphi_k^+\right](a)=\sum\nolimits_{j=1}^{n}\alpha_j(\lambda)\left[\varphi_j,\varphi_k^+\right](a),\ k=1,2,\cdots,n. \tag{6.5}$$

By substituting these expressions into the Conditions (6.4), we get:

$$\begin{aligned}&\sum\nolimits_{j=1}^{n}\left[\alpha_j(\lambda)+\frac{1}{i^n}(\lambda-\lambda_0)\sum\nolimits_{j,k=1}^{n}\xi^{jk}\int_a^t\overline{\varphi_k^+(s,\lambda_0)}f(s)w(s)\mathrm{d}s\right]\left[\varphi_j,\varphi_k^+\right](b)\\&=\sum\nolimits_{j=1}^{n}\alpha_j(\lambda)\left[\varphi_j,\varphi_k^+\right](a).\end{aligned}$$

This implies that the system

$$\sum\nolimits_{j=1}^{n}\alpha_j(\lambda)\left[\varphi_j,\varphi_k^+\right]_a^b=-\frac{\lambda-\lambda_0}{i^n}\left(\sum\nolimits_{j,k=1}^{n}\xi^{jk}\int_a^t\overline{\varphi_k^+(s,\lambda_0)}f(s)w(s)\mathrm{d}s\right), \tag{6.6}$$

in the variable $\alpha_j(\lambda),j=1,2,\cdots,n$. The determinant of this system does not vanish (see [9] and [12]). If we solve the System (6.6) we obtain:

$$\alpha_j(\lambda)=\frac{\lambda-\lambda_0}{i^n}\left(\sum\nolimits_{j,k=1}^{n}\xi^{jk}\int_a^b h_j^{p,q}(s,\lambda)f(s)w(s)\mathrm{d}s\right),\ j=1,2,\cdots,n, \tag{6.7}$$

where $h_j^{p,q}(s,\lambda)$ is a solution of the system:

$$\sum_{j=1}^{n} h_j^{p,q}(s,\lambda)\left(\left[\varphi_j,\varphi_k^+\right]\right)_a^b = -\sum_{j,k=1}^{n} \xi^{jk}\overline{\varphi_k^+(s,\lambda_0)}\left[\varphi_j,\varphi_k^+\right](b). \tag{6.8}$$

Since, the determinant of the above System (6.8) does not vanish, and the functions $\varphi_k^+(s,\lambda_0), k=1,2,\cdots,n$ are continuous in the interval $[a,b)$, then the functions $h_j^{p,q}(s,\lambda)$ are also continuous in the interval $[a,b)$. By substituting in Formula (6.3) for the expressions $\alpha_j(\lambda), j=1,2,\cdots,n$ we get,

$$\begin{aligned} R_\lambda^{p,q} f(t) &= \varphi(t,\lambda) \\ &= \frac{\lambda-\lambda_0}{i^n}\left[\sum_{j,k=1}^{n}\alpha_j(\lambda)\varphi_j(t,\lambda_0)\int_a^t\left[\xi^{jk}\overline{\varphi_k^+(s,\lambda_0)}+h_j^{p,q}(s,\lambda)\right]f(s)w(s)\mathrm{d}s\right. \\ &\quad \left.+\sum_{j=1}^{n}\varphi_j(t,\lambda_0)\int_t^b h_j^{p,q}(s,\lambda)f(s)w(s)\mathrm{d}s\right] \end{aligned} \tag{6.9}$$

Now, we put

$$K_{p,q}(t,s,\lambda) = \begin{cases} \dfrac{\lambda-\lambda_0}{i^n}\left(\sum_{j=1}^{n}\varphi_j(t,\lambda_0)h_j^{p,q}(s,\lambda)\right) & \text{for } t<s \\ \dfrac{\lambda-\lambda_0}{i^n}\left(\sum_{j,k=1}^{n}\xi^{jk}\varphi_j(t,\lambda_0)\left(\overline{\varphi_k^+(s,\lambda_0)}+h_j^{p,q}(s,\lambda)\right)\right) & \text{for } t>s \end{cases} \tag{6.10}$$

Formula (6.9) then takes the form

$$R_\lambda^{p,q} f(t) = \int_a^b K_{p,q}(t,s,\lambda)f(s)w(s)\mathrm{d}s \quad \text{for all } t\in[a,b), \tag{6.11}$$

i.e., $R_\lambda^{p,q}$ is an integral operator with the kernel $K_{p,q}(t,s,\lambda)$ operating on the functions $f\in L_w^p(a,b)$. Similarly, the solutions $\varphi^+(t,\lambda)$ of the equation $\left[T\left(\tau_{q',p'}^+\right)-\bar{\lambda}I\right]\varphi^+ = wg$ has the form:

$$\begin{aligned} \varphi^+(s,\lambda) &= \sum_{j=1}^{n}\alpha_j(\lambda)\varphi_j^+(s,\lambda_0)+\frac{\bar{\lambda}-\bar{\lambda}_0}{i^n}\sum_{j,k=1}^{n}\xi^{jk}\varphi_j^+(s,\lambda_0) \\ &\quad \times\int_a^s\overline{\varphi_k(t,\lambda_0)}g(t)w(t)\mathrm{d}t, \end{aligned} \tag{6.12}$$

where $\varphi_k(t,\lambda_0)$ and $\varphi_j^+(s,\lambda_0), k,j=1,2,\cdots,n$ are solutions of the equations in (6.2). The argument as before leads to,

$$R_{\bar{\lambda}}^{*(p,q)} g(t) = \int_a^b \overline{K_{q',p'}^+(s,t,\bar{\lambda})}g(t)w(t)\mathrm{d}t \quad \text{for } g\in L_w^q(a,b), \tag{6.13}$$

i.e., $R_{\bar{\lambda}}^{*(p,q)}$ is an integral operator with the kernel $K_{q',p'}^+(s,t,\bar{\lambda})$ operating on the function $g\in L_w^q(a,b)$, where

$$K_{q',p'}^+(s,t,\bar{\lambda}) = \begin{cases} \dfrac{\bar{\lambda}-\bar{\lambda}_0}{i^n}\left(\sum_{j=1}^{n}\varphi_j^+(s,\lambda_0)h_j^{+(p,q)}(t,\lambda)\right) & \text{for } s<t \\ \dfrac{\bar{\lambda}-\bar{\lambda}_0}{i^n}\left(\sum_{j,k=1}^{n}\xi^{jk}\varphi_j^+(s,\lambda_0)\left(\overline{\varphi_k(t,\lambda_0)}+h_j^{+(p,q)}(t,\lambda)\right)\right) & \text{for } s>t \end{cases} \tag{6.14}$$

and $h_j^{+(p,q)}(t,\lambda)$ is a solution of the system

$$\sum_{j=1}^{n}\overline{h_j^{+(p,q)}(s,\lambda)}\left(\left[\varphi_j,\varphi_k^+\right]\right)_a^b = -\sum_{j,k=1}^{n}\xi^{jk}\varphi_j(t,\lambda_0)\left[\varphi_j,\varphi_k^+\right](b). \tag{6.15}$$

From definitions of $R_\lambda^{p,q}$ and $R_{\bar{\lambda}}^{*(p,q)}$, it follows that

$$\begin{aligned}\left(R_{\lambda}^{p,q} f, g\right) &= \int_0^b \left\{\int_0^b K_{p,q}(t,s,\lambda) f(s) w(s)\mathrm{d}s\right\} \overline{g(t)} w(t)\mathrm{d}t \\ &= \int_0^b \left\{\int_0^b K_{p,q}(t,s,\lambda) \overline{g(t)} w(t)\mathrm{d}t\right\} f(s) w(s)\mathrm{d}s \\ &= \left(f, R_{\bar{\lambda}}^{*(p,q)} g\right),\end{aligned} \tag{6.16}$$

for any continuous functions $f, g \in H$ and by construction (see (6.10) and (6.14)), $K_{p,q}(t,s,\lambda)$ and $K_{q',p'}^{+}(s,t,\bar{\lambda})$ are continuous functions on $[a,b)\times[a,b)$ and (6.16) gives us

$$K_{p,q}(t,s,\lambda) = \overline{K_{q',p'}^{+}(s,t,\bar{\lambda})} \quad \text{for all} \quad t,s \in [a,b)\times[a,b). \tag{6.17}$$

Since $\varphi_j(t,\lambda) \in L_w^p(a,b)$, $\varphi_k^{+}(s,\lambda) \in L_w^q(a,b)$ for $j,k = 1,2,\cdots,n$ and for fixed s, $K_{p,q}(t,s,\lambda)$ is a linear combination of $\varphi_j(t,\lambda)$ while, for fixed t, $\overline{K_{q',p'}^{+}(s,t,\bar{\lambda})}$ is a linear combination of $\varphi_k^{+}(s,\lambda)$. Then we have

$$\int_a^b \left|K_{p,q}(t,s,\lambda)\right|^p w(t)\mathrm{d}t < \infty, \int_a^b \left|K_{q',p'}^{+}(s,t,\bar{\lambda})\right|^q w(s)\mathrm{d}s < \infty, a < s,t < b,$$

and (6.17) implies that,

$$\int_a^b \left|K_{p,q}(t,s,\lambda)\right|^p w(s)\mathrm{d}s = \int_a^b \left|K_{q',p'}^{+}(s,t,\bar{\lambda})\right|^q w(s)\mathrm{d}s < \infty,$$

$$\int_a^b \left|K_{q',p'}^{+}(s,t,\bar{\lambda})\right|^q w(t)\mathrm{d}t = \int_a^b \left|K_{p,q}(t,s,\lambda)\right|^p w(t)\mathrm{d}t < \infty.$$

Now, it is clear from (6.8) that the functions $h_j^{p,q}(s,\lambda),(j = 1,2,\cdots,n)$ belong to $L_w^p(a,b)$ since $h_j^{p,q}(s,\lambda)$ is a linear combination of the functions $\varphi_j^{+}(s,\lambda)$ which lie in $L_w^q(a,b)$ and hence $h_j^{p,q}(t,\lambda)$ belong to $L_w^p(a,b)$. Similarly $h_j^{+(p,q)}(t,\lambda)$ belong to $L_w^q(a,b)$. By the upper half of the formula (6.10) and (6.14), we have:

$$\int_a^b \left(\int_a^b \left|K_{p,q}(t,s,\lambda)\right|^p w(s)\mathrm{d}s\right) w(t)\mathrm{d}t < \infty,$$

for the inner integral exists and is a linear combination of the products $\varphi_j(t,\lambda)\varphi_k^{+}(s,\lambda),(j,k = 1,2,\cdots,n)$ and these products are integrable because each of the factors belongs to $L_w^1(a,b)$. Then by (6.17), and by the upper half of (6.14),

$$\begin{aligned}&\int_a^b \left(\int_a^b \left|K_{p,q}(t,s,\lambda)\right|^p w(s)\mathrm{d}s\right) w(t)\mathrm{d}t \\ &= \int_a^b \left(\int_a^b \left|K_{q',p'}^{+}(s,t,\bar{\lambda})\right|^q w(s)\mathrm{d}s\right) w(t)\mathrm{d}t < \infty.\end{aligned}$$

Hence, we also have:

$$\int_a^b \int_a^b \left|K_{p,q}(t,s,\lambda)\right|^p w(t) w(s)\mathrm{d}t\mathrm{d}s < \infty,$$

and the theorem is completely proved for any well-posed extension.

Remark: It follows immediately from Theorem 6.1 that, if for an operator $T_0(\tau_{p,q})$ with one singular end-point that $def\left[T_0(\tau_{p,q}) - \lambda I\right] = def\left[T_0(\tau_{q',p'}^{+}) - \bar{\lambda} I\right] = n$, for all $\lambda \in \Pi\left[T_0(\tau_{p,q}), T_0(\tau_{q',p'}^{+})\right]$ and S is well-posed with respect to $T_0(\tau)$ and $T_0(\tau_{q',p'}^{+})$ with $\lambda \in \rho(S)$ then

$R_{\lambda}^{p,q} = (S - \lambda I)^{-1}$ is a Hilbert-Schmidt integral operator. Thus it is a completely continuous operator, and consequently its spectrum is discrete and consists of isolated eigenvalues having finite algebraic (so geometric) multiplicity with zero as the only possible point of accumulation. Hence, the spectra of all well-posed operators S are discrete, *i.e.*,

$$\sigma_{ek}(S) = \varnothing, \text{ for } k = 1,2,3,4,5. \tag{6.18}$$

We refer to ([5], Theorem IX.3.1), [14] [15] [16] [18] [19] [20] and [23] for more details.

An example of a closed operator which is a well-posed with respect to a compatible adjoint pair is given by the Visik extension ([5], Theorem III.3.3) (See ([18], Theorem 1) [19] and [20]). Note that if S is well-posed, then $T_0(\tau_{p,q})$ and $T_0(\tau^{+}_{q',p'})$ are compatible adjoint pair and S is regularly solvable with respect to $T_0(\tau_{p,q})$ and $T_0(\tau^{+}_{q',p'})$.

Lemma 6.2: The point spectra $\sigma_p[T_0(\tau_{p,q})]$ and $\sigma_p[T_0(\tau^{+}_{q',p'})]$ of the operators $T_0(\tau_{p,q})$ and $T_0(\tau^{+}_{q',p'})$ are empty.

Proof: Let $\lambda \in \sigma_p[T_0(\tau_{p,q})]$. Then there exists a nonzero element $\varphi \in D[T_0(\tau_{p,q})]$, such that

$$[T_0(\tau_{p,q}) - \lambda I]\varphi = 0.$$

In particular, this gives

$$(\tau_{p,q} - \lambda w)\varphi = 0, \varphi^{[r]}(a) = \varphi^{[r]}(b) = 0, r = 0,1,2,\cdots,n-1.$$

From Lemma 4.2, it follows that $\varphi \equiv 0$ and hence $\sigma_p[T_0(\tau_{p,q})] = \varnothing$.

Similarly $\sigma_p[T_0(\tau^{+}_{q',p'})] = \varnothing$.

Theorem 6.3: (i) $\rho[T_0(\tau_{p,q})] = \varnothing$, (ii) $\sigma_p[T_0(\tau_{p,q})] = \sigma_c[T_0(\tau_{p,q})] = \varnothing$, (iii) $\sigma[T_0(\tau_{p,q})] = \sigma_r[T_0(\tau_{p,q})] = \mathbb{C}$.

Proof: (i) Since $R[T_0(\tau_{p,q}) - \lambda I]$ is a proper closed subspace of $L_w^p(a,b)$, then the resolvent set $\rho[T_0(\tau_{p,q})]$ is empty.

(ii) Since $R[T_0(\tau_{p,q}) - \lambda I]$ is closed, then the continuous spectrum of $T_0(\tau_{p,q})$ is empty set, *i.e.*, $\sigma_c[T_0(\tau_{p,q})] = \varnothing$.

(iii) From (i) and (ii) and Lemma 6.2, it follows that $\sigma[T_0(\tau_{p,q})] = \sigma_r[T_0(\tau_{p,q})] = \mathbb{C}$.

Corollary 6.4: (i) $\sigma_c[T(\tau_{p,q})] = \sigma_r[T(\tau_{p,q})] = \varnothing$,

(ii) $\sigma[T(\tau_{p,q})] = \sigma_p[T(\tau_{p,q})] = \mathbb{C}$ and $\rho[T(\tau_{p,q})] = \varnothing$.

Proof: From Theorem 4.2 and since $T(\tau_{p,q}) = [T_0(\tau^{+}_{q',p'})]^*$, it follows that $R[T(\tau_{p,q}) - \lambda I]$ is closed for every $\lambda \in \mathbb{C}$, see [3, Theorem 1.3.7]. Also, we have

$$null[T(\tau_{p,q}) - \lambda I] = def[T_0(\tau^{+}_{q',p'}) - \bar{\lambda} I] = n,$$

and

$$def[T(\tau_{p,q}) - \lambda I] = null[T_0(\tau^{+}_{q',p'}) - \bar{\lambda} I] = n.$$

(i) Since $R\left[T\left(\tau_{p,q}\right)-\lambda I\right]$ is closed and $def\left[T\left(\tau_{p,q}\right)-\lambda I\right]=0$, then $R\left[T\left(\tau_{p,q}\right)-\lambda I\right]=H$
and this yields that

$$\sigma_c\left[T\left(\tau_{p,q}\right)\right]=\sigma_r\left[T\left(\tau_{p,q}\right)\right]=\varnothing .$$

(ii) Since $null\left[T\left(\tau_{p,q}\right)-\lambda I\right]=n$ for every $\lambda\in\mathbb{C}$, then we have $\sigma_p\left[T\left(\tau_{p,q}\right)\right]=\mathbb{C}$. Also, it follows that $\sigma\left[T\left(\tau_{p,q}\right)\right]=\mathbb{C}$ and hence $\rho\left[T\left(\tau_{p,q}\right)\right]=\varnothing$.

Lemma 6.5: (cf. ([5], Lemma IX.9.1). If $I=[a,b]$, with $-\infty<a<b<\infty$ then for any $\lambda\in\mathbb{C}$, the operator $T_0\left(\tau_{p,q}\right)$ has closed range, zero nullity and deficiency. Hence,

$$\sigma_{ek}\left[T_0\left(\tau_{p,q}\right)\right]=\begin{cases}\varnothing & (k=1,2,3)\\ \mathbb{C} & (k=4,5)\end{cases}. \tag{6.19}$$

Proof: The proof is similar to that in [19] [20] and ([23], Lemma 4.9).

Corollary 6.6: Let $\lambda\in\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau^+_{q',p'}\right)\right]$ with

$$def\left[T_0\left(\tau_{p,q}\right)-\lambda I\right]=def\left[T_0\left(\tau^+_{q',p'}\right)-\bar{\lambda} I\right]=n. \tag{6.20}$$

Then,

$$\sigma_{ek}(S)=\varnothing \text{, for } k=1,2,3, \tag{6.21}$$

of all regularly solvable extensions S with respect to the compatible adjoint pair $T_0\left(\tau_{p,q}\right)$ and $T_0\left(\tau^+_{q',p'}\right)$.

Proof: Since
$def\left[T_0\left(\tau_{p,q}\right)-\lambda I\right]=def\left[T_0\left(\tau^+_{q',p'}\right)-\bar{\lambda} I\right]=n$, for all $\lambda\in\Pi\left[T_0\left(\tau_{p,q}\right),T_0\left(\tau^+_{q',p'}\right)\right]$.

Then we have from ([5], Theorem III.3.5) that,

$$dim\left\{D(S)/D_0\left[\left(\tau_{p,q}\right)\right]\right\}=def\left[T_0\left(\tau_{p,q}\right)-\lambda I\right]=n,$$

$$dim\left\{D(S^*)/D_0\left[\left(\tau^+_{q',p'}\right)\right]\right\}=def\left[T_0\left(\tau^+_{q',p'}\right)-\bar{\lambda} I\right]=n.$$

Thus S is an n-dimensional extension of $T_0\left(\tau_{p,q}\right)$ and so by ([5], Corollary IX.4.2),

$$\sigma_{ek}(S)=\sigma_{ek}\left[T_0\left(\tau_{p,q}\right)\right],\ (k=1,2,3). \tag{6.22}$$

From Lemma 6.2 and Lemma 6.5, we get,

$$\sigma_{ek}\left[T_0\left(\tau_{p,q}\right)\right]=\varnothing,\ (k=1,2,3). \tag{6.23}$$

Hence, by (6.22) and (6.23) we have that,

$$\sigma_{ek}(S)=\varnothing,\ (k=1,2,3).$$

Remark: If S is well-posed (say the Visik's extension, see [15]-[20]), we get from (6.19) and (6.22) that

$$\sigma_{ek}\left[T_0\left(\tau_{p,q}\right)\right]=\varnothing,\ (k=1,2,3).$$

On applying (6.22) again to any regularly solvable extensions S under consideration, hence (6.21).

Corollary 6.7: If for some $\lambda_0 \in \mathbb{C}$, there are n linearly independent solutions of the equations

$$\left(\tau_{p,q} - \lambda_0 w\right)u = 0,\ \left(\tau^{+}_{q',p'} - \overline{\lambda_0} w\right)v = 0,\ \lambda_0 \in \Pi\left[T_0\left(\tau\right), T_0\left(\tau^{+}_{q',p'}\right)\right] \tag{6.24}$$

in $L^p_w\left(a,b\right)$, $L^q_w\left(a,b\right)$ and hence,

$$\Pi\left[T_0\left(\tau\right), T_0\left(\tau^{+}_{q',p'}\right)\right] = \mathbb{C} \text{ and } \sigma_{ek}\left[T_0\left(\tau\right), T_0\left(\tau^{+}_{q',p'}\right)\right] = \varnothing,\ k = 1,2,3,$$

where $\sigma_{ek}\left[T_0\left(\tau_{p,q}\right), T_0\left(\tau^{+}_{q',p'}\right)\right]$ is the joint essential spectra of $T_0\left(\tau_{p,q}\right), T_0\left(\tau^{+}_{q',p'}\right)$ defined as the joint field of regularity $\Pi\left[T_0\left(\tau\right), T_0\left(\tau^{+}_{q',p'}\right)\right]$.

Proof: Since all solutions of the equations in (6.24) are in $L^p_w\left(a,b\right)$ and $L^q_w\left(a,b\right)$ respectively for some $\lambda_0 \in \mathbb{C}$, then,

$def\left[T_0\left(\tau_{p,q}\right) - \lambda_0 I\right] = def\left[T_0\left(\tau^{+}_{q',p'}\right) - \overline{\lambda_0} I\right] = n$, for some $\lambda_0 \in \Pi\left[T_0\left(\tau_{p,q}\right), T_0\left(\tau^{+}_{q',p'}\right)\right]$.

From Lemma 4.2, we have that $T_0\left(\tau_{p,q}\right)$ has no eigenvalues and so $\left[T_0\left(\tau_{p,q}\right) - \lambda_0 I\right]^{-1}$ exists and its domain $R\left[T_0\left(\tau_{p,q}\right) - \lambda_0 I\right]$ is a closed subspace of $L^q_w\left(a,b\right)$. Hence, since $T_0\left(\tau_{p,q}\right)$ is a closed operator, then $\left[T_0\left(\tau_{p,q}\right) - \lambda_0 I\right]^{-1}$ is bounded and hence $\Pi\left[T_0\left(\tau_{p,q}\right)\right] = \mathbb{C}$. Similarly $\Pi\left[T_0\left(\tau^{+}_{q',p'}\right)\right] = \mathbb{C}$. Therefore $\Pi\left[T_0\left(\tau_{p,q}\right), T_0\left(\tau^{+}_{q',p'}\right)\right] = \mathbb{C}$, and hence, $def\left[T_0\left(\tau_{p,q}\right) - \lambda I\right] = def\left[T_0\left(\tau^{+}_{q',p'}\right) - \overline{\lambda} I\right] = n$ for all $\lambda \in \Pi\left[T_0\left(\tau_{p,q}\right), T_0\left(\tau^{+}_{q',p'}\right)\right]$.

From Corollary 6.6, we have for any regularly solvable extension S of $T_0\left(\tau_{p,q}\right)$ that

$\sigma_{ek}\left(S\right) = \varnothing$, $k = 1,2,3$ and by (6.22) we get $\sigma_{ek}\left[T_0\left(\tau_{p,q}\right)\right] = \varnothing$, $k = 1,2,3$.

Similarly $\sigma_{ek}\left[T_0\left(\tau^{+}_{q',p'}\right)\right] = \varnothing$, $k = 1,2,3$. Hence,

$$\sigma_{ek}\left[T_0\left(\tau_{p,q}\right), T_0\left(\tau^{+}_{q',p'}\right)\right] = \varnothing,\ k = 1,2,3.$$

Remark: If there are n linearly independent solutions of Equations (6.24) in $L^p_w\left(a,b\right)$ and $L^q_w\left(a,b\right)$ for some $\lambda_0 \in \mathbb{C}$ then the complex plane can be divided into two disjoint sets:

$$\mathbb{C} = \Pi\left[T_0\left(\tau_{p,q}\right), T_0\left(\tau^{+}_{q',p'}\right)\right] \cup \sigma_{ek}\left[T_0\left(\tau_{p,q}\right), T_0\left(\tau^{+}_{q',p'}\right)\right],\ k = 1,2,3.$$

We refer to [5] [6] [12]-[20] and [23] for more details.

Conclusion: It has been shown that all the well-posed extensions of the minimal operator $T_0\left(\tau_{p,q}\right)$ generated by a general ordinary quasi-differential expression $\tau_{p,q}$ of order n with complex coefficients and their formal adjoints on the interval $\left[a,b\right)$ with maximal deficiency indices have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum. This implies that all the regularly solvable operators have all the standard essential spectra to be empty. Also, the location of the point spectra and regularity fields of these operators are investigated in the case of one singular end-point and when all solutions of the equations $\left(\tau_{p,q}-\lambda w\right)u=0$ and its adjoint $\left(\tau_{p,q}{}^{+}-\bar{\lambda}w\right)v=0$ are p-integrable.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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