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Economy Equilibrium and Sustainable Development

Nicholas Simon Gonchar

Bogolyubov Institute for Theoretical Physics of NAS of Ukraine, Kyiv, Ukraine
Email: mhonchar@i.ua

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Abstract

The structure of technological mappings is studied and it is shown that for an important class of technological mappings there exists a continuous strategy of the behavior of firms as arbitrarily close in terms of profit. The latter is important for building the theory of economic equilibrium, which is an important tool for building the theory of sustainable economic development. Theorems of the existence of economic equilibrium under conditions of uncertainty and under general assumptions about the structure of supply and demand have been proved.

Keywords

Technological Mapping, Economic Equilibrium, Structure of Supply, Structure of Demand, Sustainable Economic Development

1. Introduction

This article is the first in formulating the principles of sustainable economic development. In this formulation, the question of the existence of economic equilibrium under fairly general assumptions regarding the structure of supply and the structure of demand plays a significant role. The economic reality generates the problems the solution of that is only possible by a new paradigm of the description of economy system. The classical mathematical economics is based on a notion of the rational consumer choice generated by a certain preference relation on some set of goods a consumer wanted and the concept of maximization of the firm profit. The sense of the notion of the rational consumer choice is that

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it is determined by a certain utility function, defining the choice of a consumer by maximization of it on a certain budget set of goods. Moreover, choices of consumers are independent. In the reality choices of consumers are not independent because they depend on the firms supply.

Except the firms supply, the consumer choice is also determined by information about the state of the economy system that the consumer has and respectively evaluates at the moment of the choice. In turn, the firms supply is made on the basis of needs of the consumers and their buying power. By information about the state of the economy system we understand a certain information about the equilibrium price vector and productive processes realized in the economy system under the equilibrium price vector.

The new concept of the description of the economy system is to construct the stochastic model of the economy system based on the principle that the firms supply is primary and the consumers choice is secondary. Consumers make their choice by having information about the state of the economy system that is taken into account by them under the choice. Firms make decisions relative to productive processes on the basis of information about the real needs of consumers and this principle is called the agreement of the supply structure with the choice structure.

To construct such a theory, it is necessary to formulate adequate to the reality the theory of consumers choice and decisions making by firms. Thus, the main foundations of the stochastic model of economy system are the notions of consumers choice and decisions making by firms relative to productive processes. Under uncertainty these two notions have the stochastic nature. Besides, the theory of consumers choice must take into account the structure of supply and mutual dependence of the consumers choice.

So, the sense of the new paradigm of the description of the economic phenomena is to construct models of the economy systems, describing adequately consumers choice and decisions making by firms relative to productive processes, that give possibility to use them for finding the conditions of the stable growth of real economy systems (see [1]).

This work is the implementation of the ideas laid down in [1] for the formulation of the concept of sustainable economic development at the microeconomic level. In the next paper, we clarify the issue of the influence of spontaneous output of goods by firms on the phenomenon of recession, formulate the concept of sustainable economic development.

The Section 2 is devoted to the investigation of technological maps that describe production of firms. The most important among them are technological maps from the CTM class (compact technological maps) and from the CTM class in a wide sense. The notion of optimal strategy of firm behavior is introduced and the proposition 1 of the existence of optimal strategy of firm behavior for technological maps from the CTM class in a wide sense is proved. A technological map given by the formula (3) is introduced, the Lemma 3 on the belonging of this technological map to the CTM class in a wide sense and its convexity down is proved. In the Lemma 4 an algorithm of the construction of optimal strategy of firm behaviour for the technological map given by the formula (3) is presented. The Lemma 6 is fundamental for further investigation and proves for the technological map, given by the formula (3), the existence of continuous strategy of firm behavior that is arbitrary close in income to the optimal one. By a convex down technological map from the CTM class in a wide sense, a technological map given by the formula (4) is constructed and the Lemma 7 about the belonging of the constructed technological map to the CTM class and its convexity down is proved. In the Lemma 8 the structure of optimal strategy of firm behavior for a technological map given by the formula (4) is found. The Lemma 9 guarantee the convergence of the sequence of optimal incomes for technological maps of the type (4) to optimal income of the technological map by that the sequence of technological maps is constructed. The basic result is the Theorem 1, in which under wide assumptions the existence of continuous strategy of firm behavior that is arbitrarily close in income to optimal strategy of firm behavior is proved.

Section 3 contains Theorems of the existence of economic equilibrium in the case of the general form of consumer demand and firm supply. In Theorem 2, the existence of a solution to the system of Equation (9) is established under fairly general assumptions about the structure of technological mappings, consumers income functions, and the production economic process. Using this result and auxiliary Theorem 3, Theorem 4 establishes the existence of an economic equilibrium. The disadvantage of Theorem 4 is the fact that each economic agent must own all types of goods in order to fulfill the condition of having a strictly positive income for each economic agent.

In this work, which continues and develops the works of [2]-[6] for the case of the economy under conditions of uncertainty, new mathematical methods of proving theorems of the existence of economic equilibrium and constructive methods of constructing equilibrium states have been developed. In the works [2], [3] the existence of economic equilibrium was established for the first time under the assumption that the choice of consumers takes place in accordance with their preference for a set of products, and firms maximize their profits and do not influence the formation of prices in the economic system. The last principle is called the principle of perfect competition. It may turn out that firms that have maximized their profit are zero for a certain number of firms. Such a model is ideal and its application to real systems is problematic. In [1], another approach to the description of economic equilibrium is proposed. There, the axioms of choice and decision-making by firms based on realistic postulates are proposed. One of the principles is that the supply of firms is primary and the consumers choice is secondary. This principle, supplemented by the consistency of the proposal with the structure of the choice, made it possible to obtain algorithms for finding equilibrium states. Our approach to establishing economic equilibrium starts with proving the existence of solutions to the system of nonlinear equations, and only then the existence of economic equilibrium is established. The latter makes it possible to reduce the problem of finding equilibrium states to solving a system of nonlinear equations. The advantage of the methods developed by us is that in practically important cases the problem is reduced to a linear problem. The effectiveness of the developed methods was illustrated in the works [7] [8] [9]. Based on fairly general assumptions about the structure of supply and the structure of demand, theorems for the existence of economic equilibrium and algorithms for constructing equilibrium states have been established.

2. The Structure of Production Technology

In the contemporary economy system, the large number of participants of the economic process act interacting between themselves. The chain of economic relations between participants of the economic process and the interaction between them are not always foreseen. The main aim of the modeling of economic processes is to construct a mathematical model in that nonessential factors are neglected. A model describes a real state of an economy if it describes observable facts both qualitatively and quantitatively. Goods are all that is produced for sale. So, if it has a consumption value then the measure of the consumption value is its price. The participants of the economic process are firms and consumers. We call them economic agents. By the economic process we understand the activities of the economic agents to produce goods and services. Economic agents interact between themselves to transform one set of goods into another one with the aim to satisfy needs of society. A set of goods transformed is called input vector and the set of goods to that the input vector is transformed is called output vector. A firm is a set of the productive processes and the aim of the firm is to product goods and services for consumption by firms and consumers. A goods, for example, is consumed if it is transformed into the other goods. From this point of view any firm is a consumer. The main subject of the economic process is a person. To restore a labour force and intelligence, the person has to consume various goods. Under goods we understand the material and intelligent values, technologies, a labour force, services, and so on. Financial operations of banks, intermediary activity, trade, lease are kinds of services. In further consideration, we assume that the set of possible goods is ordered. Every set of goods we describe by a vector $x=\left(x_1,\cdots,x_n\right)$, where x_i is a quantity of units of the i-th goods, e_i is a unit of its measurement, $x_i e_i$ is the natural quantity of the goods. If p_i is the price of the unit of the goods e_i, $i=\overline{1,n}$, then $p=\left(p_1,\cdots,p_n\right)$ is the price vector that corresponds to the vector of goods $\left(e_1,\cdots,e_n\right)$. The price vector of goods $x=\left(x_1,\cdots,x_n\right)$ is given by the formula $\langle p,x\rangle=\sum_{i=1}^{n} p_i x_i$. The set of possible goods in the considered period of the economy system operation is denoted by S. We assume that S is a convex subset of the set R_{+}^{n}. Since for further consideration only the property of convexity is important, we assume, without loss of generality, that S is a certain n-dimensional parallelepiped that can coincide with R_+^n. Thus, we assume that in the economy system the set of possible goods S is a convex subset of the non-negative orthant R_+^n of n-dimensional arithmetic space R^n, the set of possible prices is a certain cone K_+^n, contained in $R_+^n \setminus \{0\}$, and that can coincide with $R_+^n \setminus \{0\}$. Here and further $R_+^n \setminus \{0\}$ is the cone formed from the nonnegative orthant R_+^n by ejection of the null vector $\{0\} = \{0, \cdots, 0\}$. Further, the cone $R_+^n \setminus \{0\}$ is denoted by $\bar{R}_+^n$. The fact that the set of possible prices is not $\bar{R}_+^n$ obligatory will be understand when we will consider random fields discontinuous on $\bar{R}_+^n$ whose contraction on a certain cone K_+^n is yet continuous random fields of choice. We assume that the Euclidean metric is introduced in the cone K_+^n.

Definition 1. *A set $K_+^n \subseteq \bar{R}_+^n$ is called a nonnegative cone if together with a point $u \in K_+^n$ the point tu belongs to the set K_+^n for every real $t > 0$.*

To describe an economic system, it is necessary to describe the behaviour of consumers and firms and their interaction. We put that in the economy system there are l consumers, m firms, and n kinds of goods.

Definition 2. *By a technological map $F(x), x \in X$, we understand a multi-valued map defined on a set $X \subseteq S$ taking values in the set of all subsets of the set S. Any vector $x \in X$ is called an input vector and its image $F(x)$ is called a set of plans of the technological map $F(x)$. The set X is called the expenditure set and the pair $z = (x, y)$, where $x \in X$, $y \in F(x)$, is called the productive process. The set $\Gamma = \{(x, y), x \in X, y \in F(x)\}$ we call the set of possible productive processes for the technological map $F(x)$.*

Any i-th firm activity we describe by a technological map $F_i(x)$ defined on the expenditure set X_i and taking values in 2^S. We introduce denotation $\Gamma_i = \{(x, y), x \in X_i, y \in F_i(x)\}$ and the direct product of the sets

$$\Gamma_i = \{z = \{x, y\}, x \in X_i, y \in F_i(x)\}, \quad i = \overline{1, m},$$

that is, $\Gamma^m = \prod_{i=1}^{m} \Gamma_i$.

A wide class of compact technological maps (CTM) is introduced. For technological map from CTM class in wide sense and convex down, the Theorem guaranteeing existence of continuous strategy of firm behaviour arbitrary close, in income, to optimal one is established.

This section is devoted to the investigation of technological maps that describe production of firms. The most important among them are technological maps from the CTM class and from the CTM class in a wide sense. The notion of optimal strategy of firm behavior is introduced and the proposition 1 of the existence of optimal strategy of firm behavior for technological maps from the CTM class in a wide sense is proved.

Define a set of functions that we call income pre-functions of consumers.

Definition 3. *A set of functions $K_i^0(p, u)$, $i = \overline{1, l}$, given on the set $K_+^n \times \Gamma^m$ with values in the set R^1 we call income pre-functions of consumers if they sa-*

tisfy conditions:

1) *For every* $p \in K_+^n$ $K_i^0(p,u), i=\overline{1,l}$, *is a measurable mapping of the space* $\{\Gamma^m, \mathcal{B}(\Gamma^m)\}$ *into the space* $\{R^1, \mathcal{B}(R^1)\}$;

2) *For every* $p \in K_+^n$ *the set* $D(p)=\bigcap_{i=1}^{l} D_i(p)$ *is not empty, where* $D_i(p)=\{u \in \Gamma^m, K_i^0(p,u) \geq 0\}, i=\overline{1,l}$;

3) $K_i^0(tp,u)=tK_i^0(p,u), t>0, (p,u) \in K_+^n \times \Gamma^m, i=\overline{1,l}$.

Definition 4. *A technological map* $F(x), x \in X$, *is the Kakutani continuous from above if for every sequence* $x_n \in X$, $x_n \to x$, $x \in X$, *and for any sequence* $y_n \in F(x_n)$ *such that* $y_n \to y$, *it follows that* $y \in F(x)$.

Definition 5. *A technological map* $F(x), x \in X$, *belongs to the CTM (compact technological maps) class in a wide sense if the domain of its definition* $X \subseteq S$ *is a closed bounded convex set,* $F(x)$ *is Kakutani continuous from above, takes values in the set of closed bounded convex subsets of the set S, moreover, there exists a compact set* $Y \subseteq S$ *such that* $F(x) \subseteq Y, x \in X$. *The technological map* $F(x), x \in X$, *belongs to the CTM class if it belongs to the CTM class in a wide sense and, in addition,* $0 \in X, 0 \in F(0)$.

Lemma 1. *Let a technological map* $F(x), x \in X$, *belong to the CTM class in a wide sense, then the set*

$$\Gamma=\{(x,y) \in S^2, x \in X, y \in F(x)\}$$

is a closed subset of the set S^2.

Lemma 2. *Let technological maps* $F_i(x), x \in X_i, i=\overline{1,m}$, *belong to the CTM class in a wide sense, and let nonnegative property vectors* $b_i(p,z)=\{b_{ki}(p,z)\}_{k=1}^{n}, i=\overline{1,l}$, *be measurable maps of the measurable space* $\{\Gamma^m, \mathcal{B}(\Gamma^m)\}$ *into the measurable space* $\{S, \mathcal{B}(S)\}$ *for every* $p \in K_+^n$, *then the set*

$$G(p)=\{z \in \Gamma^m, R(p,z) \in S\}$$

belongs to $\mathcal{B}(\Gamma^m)$ *for every* $p \in K_+^n$, *where*

$$R(p,z)=\sum_{i=1}^{m}[y_i-x_i]+\sum_{k=1}^{l} b_k(p,z),\ z^i=(x_i,y_i) \in \Gamma_i, z=\{z^i\}_{i=1}^{m}.$$

The proof of Lemmas 1 and 2 see in [1].

Let us associate with the *i*-th consumer a certain nonnegative vector $b_i(p,z)=\{b_{ki}(p,z)\}_{k=1}^{n}, (p,z) \in K_+^n \times \Gamma^m$, that we call the supply vector of goods which the *i*-th consumer has at the beginning of the period of the economy operation and that is a measurable map of the space $\{\Gamma^m, \mathcal{B}(\Gamma^m)\}$ into the space $\{S, \mathcal{B}(S)\}$ for every $p \in K_+^n$ and such that $b_i(tp,z)=b_i(p,z), t>0, i=\overline{1,l}$. The vector $b_i(p,z)=\{b_{ki}(p,z)\}_{k=1}^{n}, (p,z) \in K_+^n \times \Gamma^m$, we call the property vector of the *i*-th consumer under the realized price vector $p \in R_+^n$ and the set of productive processes $z \in \Gamma^m$.

Definition 6. *Let the production of the i-th firm in the economy system be described by technological map* $F_i(x), x \in X_i, i=\overline{1,m}$, *the i-th consumer have* *property vector* $b_i(p,z), i=\overline{1,l}$, *and for every* $(p,z)\in K_+^n\times\Gamma^m$ *the set of productive processes*

$$\left(X_i(p,z),Y_i(p,z)\right),\quad X_i(p,z)\in X_i,\quad Y_i(p,z)\in F_i\left(X_i(p,z)\right),\quad i=\overline{1,m},$$

satisfy conditions:

1) $\left(X_i(p,z),Y_i(p,z)\right)$ *is a measurable map of the space* $\left\{\Gamma^m,\mathcal{B}\left(\Gamma^m\right)\right\}$ *into the space* $\left\{\Gamma_i,\mathcal{B}(\Gamma_i)\right\}, i=\overline{1,m}$, *for every* $p\in K_+^n$, *where the set* $\Gamma_i=\left\{(x,y)\in S^2, x\in X_i, y\in F_i(x)\right\}$;

2) $\left(X_i(tp,z),Y_i(tp,z)\right)=\left(X_i(p,z),Y_i(p,z)\right), t>0,(p,z)\in K_+^n\times\Gamma^m$.

A measurable map $Q(p,z)$ *of the space* $\left\{\Gamma^m,\mathcal{B}\left(\Gamma^m\right)\right\}$ *into itself for every* $p\in K_+^n$ *defined by the formula*

$$Q(p,z)=\left\{\left(X_i(p,z),Y_i(p,z)\right)\right\}_{i=1}^m \tag{1}$$

we call productive economic process if for every $p\in K_+^n$ *the set of values* $Q\left(p,\Gamma^m\right)$ *of the map* $Q(p,z)$ *belongs to the set* $G(p)$ *constructed in the Lemma* 2, *where as measurable map* $b_i(p,z), i=\overline{1,l}$, *the set of supply vectors of goods of consumers* $b_i(p,z), i=\overline{1,l}$, *is chosen at the beginning of the economy operation.*

Definition 7. *A set of functions* $K_i(p,z), i=\overline{1,l}$, *defined on the set* $K_+^n\times\Gamma^m$ *being measurable maps of the space* $\left\{\Gamma^m,\mathcal{B}\left(\Gamma^m\right)\right\}$ *into the space* $\left\{R_+^1,\mathcal{B}\left(R_+^1\right)\right\}$ *for every* $p\in K_+^n$ *is named income functions if there exist a set of income pre-functions* $K_i^0(p,z), i=\overline{1,l}$, *a productive economic process* $Q(p,z)$ *defined on the set* $K_+^n\times\Gamma^m$ *and such that* $Q\left(p,\Gamma^m\right)$ *belongs to the set* $D(p)$ *from the Definition 3 for every* $p\in K_+^n$ *and the equalities*

1) $K_i(p,z)=K_i^0\left(p,Q(p,z)\right),(p,z)\in K_+^n\times\Gamma^m, i=\overline{1,l}$;

2)
$$\begin{aligned}\sum_{i=1}^{l}K_i(p,z)=&\left\langle p,\sum_{i=1}^{m}\left[Y_i(p,z)-X_i(p,z)\right]\right\rangle\\&+\left\langle p,\sum_{k=1}^{l}b_k\left(p,Q(p,z)\right)\right\rangle,(p,z)\in K_+^n\times\Gamma^m,\end{aligned} \tag{2}$$

are valid.

Let $F(x), x\in X$, be a technological map describing the production structure of the firm. Strategy of firm behaviour is a map of the set possible prices K_+^n into the set of possible productive processes of the firm $\Gamma=\left\{(x,y)\in S^2, x\in X, y\in F(x)\right\}$. We denote strategy of firm behavior by $\left(x(p),y(p)\right), p\in K_+^n$.

Definition 8. *A strategy of firm behavior* $\left(x(p),y(p)\right), p\in K_+^n$, *is optimal one if*

$$\sup_{x\in X}\sup_{y\in F(x)}\langle y-x,p\rangle=\left\langle y(p)-x(p),p\right\rangle,\quad p\in K_+^n.$$

Proposition 1. *For technological map* $F(x), x\in X$, *belonging to the CTM class in a wide sense an optimal strategy of firm behaviour exists.*

The proof of Lemma 1 see in [1].

Definition 9. *A set of points* $(x_1,\cdots,x_k)$, *where* $x_i \in R_+^n, i=\overline{1,k}$, *generates a set X, if the set X coincides with linear convex span constructed by the set of points* $(x_1,\cdots,x_k)$, *that is,*

$$X=\left\{x \in R_+^n, x=\sum_{i=1}^{k}\alpha_i x_i, \alpha \in P_1\right\},$$

where

$$P_1=\left\{\alpha=\{\alpha_i\}_{i=1}^{k} \in R_+^k, \sum_{i=1}^{k}\alpha_i=1\right\}.$$

We call the set X the linear convex span generated by the set of points $(x_1,\cdots,x_k)$.

Remark 1. *The same set X can be generated by various sets of points. If to the set of points that generates X to add any set of points that belongs to X, then they will also generate the set X. But there always exists the minimal number of points from the set of points* $(x_1,\cdots,x_k)$ *that generates X. These points are called extreme points.*

Let $X \subset R_+^n$ be a convex closed bounded polyhedron and the set of points $(x_1,\cdots,x_k)$ generates it. Further, let $Y_i, i=\overline{1,k}$, be a convex closed bounded polyhedrons from R_+^n and $\left\{y_j^{(i)}\right\}_{j=1}^{m(i)}$ be a set of points that generates the polyhedron $Y_i, i=\overline{1,k}$. Let us give a technological map $F(x)$ on X by the formula

$$F(x)=\bigcup_{\alpha\in\Delta(x)}\sum_{i=1}^{k}\alpha_i F_1(x_i), \quad x \in X, \tag{3}$$

where $F_1(x_i)=Y_i$, and by $\Delta(x)$ we denote the set

$$\Delta(x)=\left\{\alpha=\{\alpha_i\}_{i=1}^{k} \in P_1, x=\sum_{i=1}^{k}\alpha_i x_i\right\}.$$

The set $\sum_{i=1}^{k}\alpha_i F_1(x_i)$ is the set of all points of the kind $\sum_{i=1}^{k}\alpha_i y_i$, where the point y_i runs over the set $F_1(x_i), i=\overline{1,k}$.

Lemma 3. *A technological map defined by the formula* (3) *takes values in the set of closed bounded and convex sets, is convex down, and Kakutani continuous from above, that is, it belongs to the CTM class in a wide sense and is convex down.*

The proof of Lemma 3 see in [1].

Lemma 4. *Let* $X \subset R_+^n$ *be a convex bounded closed polyhedron and the set of points* $(x_1,\cdots,x_k)$ *generate it. Further, let* $Y_i, i=\overline{1,k}$, *be convex bounded closed polyhedrons from* R_+^n, *and a technological map* $F(x)$ *given on X be defined by the formula* (3). *If the set of points* $\left\{y_1^{(i)},\cdots,y_{m(i)}^{(i)}\right\}$ *generates the set* $Y_i, i=\overline{1,k}$, *then*

$$\max_{x\in X}\max_{y\in F(x)}\langle y-x,p\rangle=\max_{1\le i\le k}\max_{1\le s\le m(i)}\left\langle p, y_s^{(i)}-x_i\right\rangle.$$

The proof of Lemma 4 see in [1].

Lemma 5. *Let* $X \subset R_+^n$ *be a compact set, a technological map* $F(x)$ *maps X into bounded closed subsets of* R_+^n. *There exists a compact* $Y \subset R_+^n$ *such that*

$$F(x) \subseteq Y, \quad x \in X.$$

Then

$$\varphi(p) = \sup_{x \in X} \sup_{y \in F(x)} \langle y - x, p \rangle, \quad p \in R_+^n,$$

is positively homogeneous subadditive continuous function on R_+^n.

The proof of Lemma 5 see in [1].

Hereinafter, we are only interested in the restriction of $\varphi(p)$ on the simplex

$$P = \left\{ p = (p_1, \cdots, p_n) \in R_+^n, \sum_{i=1}^n p_i = 1 \right\},$$

due to positive homogeneity of $\varphi(p)$.

Definition 10. *We call a productive process* $(x(p), y(p))$ *of technological map* $F(x)$ *given on X optimal one for the price vector p if* $x(p) \in X$, $y(p) \in F(x(p))$ *and the equality*

$$\langle y(p) - x(p), p \rangle = \varphi(p)$$

is valid, where

$$\varphi(p) = \sup_{x \in X} \sup_{y \in F(x)} \langle y - x, p \rangle, \quad p \in \bar{R}_+^n.$$

In the next Lemma, we give the sufficient conditions for a technological map of the firm under the realization of that there exists a continuous strategy of firm behavior which is arbitrary close to optimal one. Further, this Lemma will be generalized onto a wide class of technological maps. The need in such assertion exists at least because the optimal behavior strategies for a quite wide class of technological maps are not continuous. In further construction, this result will play the very important role.

Lemma 6. [1] *Let points* $x_i, i = \overline{1,k}$, *generate convex bounded closed polyhedron X and* $Y_i, i = \overline{1,k}$, *be convex bounded closed polyhedrons from* R_+^n *generated, correspondingly, by points* $\left\{ y_1^{(i)}, \cdots, y_{m(i)}^{(i)} \right\}, i = \overline{1,k}$. *If technological map* $F(x)$ *given on X by the formula* (3), *then for every sufficiently small* $\delta > 0$ *there exists a firm behavior strategy* $\left(x^0(p), y^0(p) \right)$, $y^0(p) \in F\left(x^0(p) \right)$, *where* $x^0(p)$ *is an input vector and* $y^0(p)$ *is an output vector such that* $\left(x^0(p), y^0(p) \right)$ *is a continuous map of*

$$P = \left\{ p = (p_1, \cdots, p_n) \in R_+^n, \sum_{i=1}^n p_i = 1 \right\}$$

into R_+^{2n} *and, furthermore,*

$$\sup_{p \in P} \left| \varphi(p) - \left\langle y^0(p) - x^0(p), p \right\rangle \right| < \delta.$$

The proof of Lemma 6 the reader can find in [1].

Let X be a convex linear span of the set of points $(x_1,\cdots,x_k)$, the set of that is not obligatory minimal, and let $F(x)$ be a convex down technological map given on X_1 belonging to the CTM class in a wide sense. We assume that every point of X is an internal point of the set X_1. Let us give on X a technological map

$$F_1(x)=\bigcup_{\alpha\in\Delta(x)}\sum_{i=1}^{k}\alpha_i F(x_i),\quad x\in X, \tag{4}$$

where by $\Delta(x)$ we denote the set

$$\Delta(x)=\left\{\alpha=\{\alpha_i\}_{i=1}^{k}\in P_1, x=\sum_{i=1}^{k}\alpha_i x_i\right\}.$$

The set $\sum_{i=1}^{k}\alpha_i F(x_i)$ is the set of all points of the kind $\sum_{i=1}^{k}\alpha_i y_i$, where the point $y_i\in F(x_i)$, and

$$P_1=\left\{\alpha=\{\alpha_i\}_{i=1}^{k}\in R_+^k,\sum_{i=1}^{k}\alpha_i=1\right\}.$$

Lemma 7. *The technological map* $F_1(x)$ *defined on X by the formula* (4) *is convex down and belongs to the CTM class in a wide sense if* $F(x)$ *belongs to the CTM class in a wide sense and is convex down, furthermore,* $F_1(x)\subseteq F(x), x\in X$.

Proof. Prove the convexity of $F_1(x)$ for every $x\in X$. Let y_1 and y_2 belong to $F_1(x)$. It means that there exist

$$\alpha'=\{\alpha_i'\}_{i=1}^{k},\quad \alpha''=\{\alpha_i''\}_{i=1}^{k}\in\Delta(x)$$

such that $x=\sum_{i=1}^{k}\alpha_i' x_i, x=\sum_{i=1}^{k}\alpha_i'' x_i$, and for y_1 and y_2 the representations

$$y_1=\sum_{i=1}^{k}\alpha_i' y_j^1,\quad y_2=\sum_{i=1}^{k}\alpha_i'' y_i^2,\quad y_i^1,y_i^2\in F(x_i),\quad i=\overline{1,k},$$

are valid. For arbitrary $0<\alpha<1$ and for those i for which α_i',α_i'' do not equal zero simultaneously

$$\alpha y_1+(1-\alpha)y_2=\sum_{i=1}^{k}\alpha_i'''\left[\frac{\alpha\alpha_i'}{\alpha\alpha_i'+(1-\alpha)\alpha_i''}y_i^1+\frac{(1-\alpha)\alpha_i''}{\alpha\alpha_i'+(1-\alpha)\alpha_i''}y_i^2\right],$$

where $\alpha_i'''=\alpha\alpha_i'+(1-\alpha)\alpha_i''$. Due to the convexity of the set $F(x_i)$, the point

$$\frac{\alpha\alpha_i'}{\alpha\alpha_i'+(1-\alpha)\alpha_i''}y_i^1+\frac{(1-\alpha)\alpha_i''}{\alpha\alpha_i'+(1-\alpha)\alpha_i''}y_i^2$$

belongs to this set too. Because

$$x=\sum_{i=1}^{k}\alpha_i'''x_i,\quad \sum_{i=1}^{k}\alpha_i'''=1,$$

we have $\alpha y_1+(1-\alpha)y_2\in F_1(x)$ for any $\alpha\in(0,1)$.

Prove the completeness of $F_1(x)$.

Let the sequence $y_n\in F_1(x)$ and $y_n\to y_0$. Prove that $y_0\in F_1(x)$. From that $y_n \in F_1(x)$ the existence of sequences follows

$$\alpha^n = \{\alpha_i^n\}_{i=1}^k \in P_1, \quad \bar{y}_n = \{y_i^n\}_{i=1}^k, \quad y_i^n \in F(x_i), \quad i = \overline{1,k},$$

that satisfy conditions: for $y_n \in F_1(x)$ and $x \in X$ the representations

$$x = \sum_{i=1}^k \alpha_i^n x_i, \quad y_n = \sum_{i=1}^k \alpha_i^n y_i^n$$

hold. From the compactness of the considered sequences, the existence of a subsequence n_m follows such that $\alpha^{n_m} = \{\alpha_i^{n_m}\}_{i=1}^k, \bar{y}_{n_m} = \{y_i^{n_m}\}_{i=1}^k$ are convergent correspondingly to $\alpha^0 = \{\alpha_i^0\}_{i=1}^k, \bar{y}_0 = \{y_i^0\}_{i=1}^k$, as $m \to \infty$, and for the limit point y_0 of the subsequence y_{n_m} the representation

$$y_0 = \sum_{i=1}^k \alpha_i^0 y_i^0, \quad y_i^0 \in F(x_i), \quad i = \overline{1,k},$$

holds, where

$$x = \sum_{i=1}^k \alpha_i^0 x_i, \quad \alpha^0 = \{\alpha_i^0\}_{i=1}^k \in P_1.$$

The latter means that $y_0 \in F_1(x)$. The closure of $F_1(x)$ follows from that y_0 is an arbitrary limit point of the sequence y_n. The boundedness of $F_1(x)$ is obvious.

Prove the Kakutani continuity from above of $F_1(x)$. Let the sequence $y_n \in F_1(x_n)$ and $x_n \to x_0$, $y_n \to y_0$. Show that $y_0 \in F_1(x_0)$. From that $y_n \in F_1(x_n)$, the existence of such sequences $\alpha^n = \{\alpha_i^n\}_{i=1}^k \in P_1$ and $\bar{y}_n = \{y_i^n\}_{i=1}^k$ follows that for $y_n \in F_1(x_n)$ and $x_n \in X$ the representations

$$x_n = \sum_{i=1}^k \alpha_i^n x_i, \quad y_n = \sum_{i=1}^k \alpha_i^n y_i^n$$

hold. From the compactness of the considered sequences, the existence of such subsequence n_m follows that $\alpha^{n_m} = \{\alpha_i^{n_m}\}_{i=1}^k$, and $\bar{y}_{n_m} = \{y_i^{n_m}\}_{i=1}^k$ are correspondingly convergent to $\alpha^0 = \{\alpha_i^0\}_{i=1}^k$ and $\bar{y}_0 = \{y_i^0\}_{i=1}^k$, as $m \to \infty$, and for the limit points y_0 and x_0 of subsequences y_{n_m} and x_{n_m}, correspondingly, the representations

$$y_0 = \sum_{i=1}^k \alpha_i^0 y_i^0, \quad y_i^0 \in F(x_i), \quad i = \overline{1,k},$$

$$x_0 = \sum_{i=1}^k \alpha_i^0 x_i, \quad \alpha^0 = \{\alpha_i^0\}_{i=1}^k \in P_1,$$

are valid. The latter means that $y_0 \in F_1(x_0)$.

Prove that $F_1(x)$ is a convex down technological map. Let

$$x_1 = \sum_{i=1}^k \alpha_i' x_i, \quad x_2 = \sum_{i=1}^k \alpha_i'' x_i \in X$$

$$\alpha' = \{\alpha_i'\}_{i=1}^k \in P_1, \quad \alpha'' = \{\alpha_i''\}_{i=1}^k \in P_1,$$

$$y_1=\sum_{i=1}^{k}\alpha_i' y_i^1 \in F_1(x_1), \quad y_2=\sum_{i=1}^{k}\alpha_i'' y_i^2 \in F_1(x_2).$$

Then for arbitrary $0<\alpha<1$,

$$\alpha y_1+(1-\alpha)y_2=\sum_{i=1}^{k}\alpha_i'''\left[\frac{\alpha\alpha_i'}{\alpha\alpha_i'+(1-\alpha)\alpha_i''}y_i^1+\frac{(1-\alpha)\alpha_i''}{\alpha\alpha_i'+(1-\alpha)\alpha_i''}y_i^1\right].$$

Due to the convexity of the set $F(x_i)$, the point

$$y_i^3=\frac{\alpha\alpha_i'}{\alpha\alpha_i'+(1-\alpha)\alpha_i''}y_i^1+\frac{(1-\alpha)\alpha_i''}{\alpha\alpha_i'+(1-\alpha)\alpha_i''}y_i^2$$

belongs to this set too, where $\alpha_i'''=\alpha\alpha_i'+(1-\alpha)\alpha_i''$ for those i for which α_i' or α_i'' do not equal zero. So,

$$\alpha y_1+(1-\alpha)y_2=\sum_{i=1}^{k}\alpha_i''' y_i^3, \quad 0<\alpha<1.$$

Since

$$\alpha x_1+(1-\alpha)x_2=\sum_{i=1}^{k}\alpha_i''' x_i, \quad \sum_{i=1}^{k}\alpha_i'''=1,$$

we have $\alpha y_1+(1-\alpha)y_2\in F_1(\alpha x_1+(1-\alpha)x_2)$ for any $\alpha\in(0,1)$.

The latter means that $\alpha y_1+(1-\alpha)y_2\in F_1(\alpha x_1+(1-\alpha)x_2)$, or the same that $\alpha F_1(x_1)+(1-\alpha)F_1(x_2)\subseteq F_1(\alpha x_1+(1-\alpha)x_2)$. At last,

$$F_1(x)=\bigcup_{\alpha\in\Delta(x)}\sum_{i=1}^{k}\alpha_i F(x_i)\subseteq\bigcup_{\alpha\in\Delta(x)}F\left(\sum_{i=1}^{k}\alpha_i x_i\right)=F(x),$$

$$x=\sum_{i=1}^{k}\alpha_i x_i, \quad x\in X.$$

□

Lemma 8. *Let points* $(x_1,\cdots,x_k)$ *generate a convex bounded closed polyhedron* $X\subset R_+^n$ *and a technological map* $F_1(x)$, *given on X, be defined by the formula* (4). *Then*

$$\sup_{x\in X}\sup_{y\in F_1(x)}\langle p,y-x\rangle=\max_{1\le i\le k}\sup_{y\in F(x_i)}\langle p,y-x_i\rangle.$$

Proof. Any points $x\in X$ and $y\in F_1(x)$ can be represented in the form

$$x=\sum_{i=1}^{k}\alpha_i x_i, \quad y=\sum_{i=1}^{k}\alpha_i y_i, \tag{5}$$

$$\sum_{i=1}^{k}\alpha_i=1, \quad \alpha_i\ge 0, \quad i=\overline{1,k}.$$

Substituting the representations for x and y into the expression

$$\psi(y,x)=\langle y-x,p\rangle,$$

we obtain

$$\langle y-x,p\rangle=\sum_{i=1}^{k}\alpha_i\langle y_i-x_i,p\rangle.$$

There hold the inequalities

$$\langle y-x_i,p\rangle \le \sup_{y\in F(x_i)}\langle p,y-x_i\rangle \le \max_{1\le i\le k}\sup_{y\in F(x_i)}\langle p,y-x_i\rangle.$$

Therefore, for any $x\in X$ and $y\in F_1(x)$

$$\langle y-x,p\rangle \le \max_{1\le i\le k}\sup_{y\in F(x_i)}\langle p,y-x_i\rangle,$$

or

$$\sup_{x\in X}\sup_{y\in F_1(x)}\langle p,y-x\rangle \le \max_{1\le i\le k}\sup_{y\in F(x_i)}\langle p,y-x_i\rangle.$$

Prove the inverse inequality. It is obvious that $F_1(x_i)=F(x_i), i=\overline{1,k}$. Therefore,

$$\sup_{x\in X}\sup_{y\in F_1(x)}\langle p,y-x\rangle \ge \sup_{y\in F_1(x_i)}\langle p,y-x_i\rangle = \sup_{y\in F(x_i)}\langle p,y-x_i\rangle, \quad i=\overline{1,k}.$$

Taking maximum over all $1\le i\le k$ from the left and right side of the last inequality, we obtain the needed inequality. □

Definition 11. *A set of points* $\{x_i, i=\overline{1,\infty}\}, x_i\in R_+^n$, *generates a set* $X\subseteq R_+^n$ *if the set X is the closure of the set of points of the form*

$$V=\left\{x=\sum_{i=1}^{k}\alpha_i x_i\in R_+^n, \alpha=\{\alpha_i\}_{i=1}^{k}\in P_1^k, k=\overline{1,\infty}\right\},$$

where

$$P_1^k=\left\{\alpha=\{\alpha_i\}_{i=1}^{k}\in R_+^k, \sum_{i=1}^{k}\alpha_i=1\right\}.$$

Lemma 9. *Let* $X\subset R_+^n$ *be a convex bounded closed set whose every point is internal for a set* X_1 *and* $\{x_i, i=\overline{1,\infty}\}$ *be dense in X set of points that generate it. If the technological map* $F_1^k(x)$ *is given by the formula* (4) *on the set* X_k *generated by the first k points* $\{x_i, i=\overline{1,k}\}$ *from the set of points* $\{x_i, i=\overline{1,\infty}\}$, *that generate X, then*

$$\lim_{k\to\infty}\sup_{x\in X_k}\sup_{y\in F_1^k(x)}\langle p,y-x\rangle = \sup_{x\in X}\sup_{y\in F(x)}\langle p,y-x\rangle.$$

Proof. We assume that the set of points $\{x_i, i=\overline{1,\infty}\}$ are ordered and the set X_{k+1} is generated by the set of points $\{x_i, i=\overline{1,k+1}\}$. There holds the inclusion

$$F_1^k(x)=\bigcup_{\alpha\in\Delta(x)}\sum_{i=1}^{k}\alpha_i F(x_i)\subseteq \bigcup_{\alpha\in\Delta_1(x)}\sum_{i=1}^{k+1}\alpha_i F(x_i)=F_1^{k+1}(x),$$

where

$$\Delta(x)=\left\{\alpha=\{\alpha_i\}_{i=1}^{k}\in P_1^k, x=\sum_{i=1}^{k}\alpha_i x_i\right\},$$

$$\Delta_1(x)=\left\{\alpha=\{\alpha_i\}_{i=1}^{k+1}\in P_1^{k+1}, x=\sum_{i=1}^{k+1}\alpha_i x_i\right\},$$

$$P_1^k = \left\{ \alpha = \{\alpha_i\}_{i=1}^k \in R_+^k, \sum_{i=1}^k \alpha_i = 1 \right\},$$

$$P_1^{k+1} = \left\{ \alpha = \{\alpha_i\}_{i=1}^{k+1} \in R_+^{k+1}, \sum_{i=1}^{k+1} \alpha_i = 1 \right\}.$$

So, the sequence of functions $\varphi_k(p) = \sup_{x \in X_k} \sup_{y \in F_1^k(x)} \langle y - x, p \rangle$ is monotonously non decreasing, that is, $\varphi_k(p) \le \varphi_{k+1}(p)$. Consider $\sup_{y \in F(x)} \langle y - x, p \rangle$. Due to the continuity of the function $\langle y - x, p \rangle$ in argument y and the compactness of the set $F(x)$, there exists a point $y(x, p) \in F(x)$ such that

$$\sup_{y \in F(x)} \langle y - x, p \rangle = \langle y(x, p) - x, p \rangle.$$

From the convexity down of $F(x)$, the function $\langle y(x, p) - x, p \rangle$ is a convex up function of the argument x and so it is continuous one by argument x on the set X. On the basis of the Weierstrass theorem,

$$\sup_{x \in X} \sup_{y \in F(x)} \langle y - x, p \rangle = \sup_{x \in X} \langle y(x, p) - x, p \rangle = \langle y(x_0, p) - x_0, p \rangle, \quad x_0 \in X.$$

Consider the sequence $\varphi_k(p) = \sup_{x \in X_k} \sup_{y \in F_1^k(x)} \langle y - x, p \rangle$ and show that for every $p \in P$

$$\lim_{k \to \infty} \varphi_k(p) = \sup_{x \in X} \sup_{y \in F(x)} \langle y - x, p \rangle,$$

where

$$P = \left\{ p = \{p_i\}_{i=1}^n \in R_+^n, \sum_{i=1}^n p_i = 1 \right\}.$$

Because the sequence of points $\{x_i, i = \overline{1, \infty}\}$ of the set X is dense in X, there exists subsequence x_{n_m} of this sequence that converges to a point x_0 in which the supremum is realized $\sup_{x \in X} \langle y(x, p) - x, p \rangle = \langle y(x_0, p) - x_0, p \rangle$. It is evident that for $k > n_m$ there holds the inequality

$$\varphi_k(p) = \left\langle y\left(x_{i_0^k}, p\right) - x_{i_0^k}, p \right\rangle \ge \left\langle y\left(x_{n_m}, p\right) - x_{n_m}, p \right\rangle, \quad x_{n_m}, x_{i_0^k} \in X_k.$$

Tending from the beginning k and then m to the infinity, we obtain

$$\lim_{k \to \infty} \varphi_k(p) \ge \langle y(x_0, p) - x_0, p \rangle = \sup_{x \in X} \langle y(x, p) - x, p \rangle.$$

On the other hand,

$$\varphi_k(p) = \sup_{x \in X_k} \sup_{y \in F_1^k(x)} \langle y - x, p \rangle \le \sup_{x \in X} \sup_{y \in F(x)} \langle y - x, p \rangle = \varphi(p).$$

Therefore, $\lim_{k \to \infty} \varphi_k(p) = \varphi(p)$. □

Let $F(x)$ be a convex down technological map from the CTM class in a wide sense defined on X_1. Cover the set P by balls

$$C(\delta, \bar{p}) = \{ p \in P \mid p - \bar{p} < \delta \}$$

of a radius $\delta > 0$. Thanks to the compactness of P, there exists a finite subcovering with the center at the points $\{\tilde{p}_1,\cdots,\tilde{p}_{m(\delta)}\}$, that is,

$$\bigcup_{i=1}^{m(\delta)} C(\delta,\tilde{p}_i)=P.$$

Denote by $y(x_s,\tilde{p}_i)$ the point of the set $F(x_s)$, in which the maximum of the problem

$$\sup_{y\in F(x_s)}\langle y-x_s,\tilde{p}_i\rangle=\langle y(x_s,\tilde{p}_i)-x_s,\tilde{p}_i\rangle$$

is reached, where the set of points $\{x_i, i=\overline{1,k}\}$ generates the set $X\subset X_1$.

Let $\{y(x_s,\tilde{p}_i), i=\overline{1,m(\delta)}\}$ be a set of points that generates the set $Y_s^\delta, s=\overline{1,k}$, and $F_1(x)$ be a technological map given on the set X by the formula (4). Define on X a technological map

$$F_1^\delta(x)=\bigcup_{\alpha\in\Delta(x)}\sum_{i=1}^{k}\alpha_i Y_i^\delta,\quad x\in X,\tag{6}$$

where by $\Delta(x)$ we denote the set

$$\Delta(x)=\left\{\alpha=\{\alpha_i\}_{i=1}^k\in P_1, x=\sum_{i=1}^k\alpha_i x_i\right\}.$$

The set $\sum_{i=1}^k\alpha_i Y_i^\delta$ is the set of points of the form $\sum_{i=1}^k\alpha_i y_i$, where the point $y_i\in Y_i^\delta$ and the vector $\alpha=\{\alpha_i\}_{i=1}^k$ runs over the set

$$P_1=\left\{\alpha=\{\alpha_i\}_{i=1}^k\in R_+^k,\sum_{i=1}^k\alpha_i=1\right\}.$$

Owing to the points $\{y(x_s,\tilde{p}_i), i=\overline{1,m(\delta)}\}$ generate the set $Y_s^\delta, s=\overline{1,k}$, and $Y_s^\delta\subseteq F(x_s)$, then

$$\sum_{i=1}^k\alpha_i Y_i^\delta\subseteq\sum_{i=1}^k\alpha_i F(x_i)\subseteq F(x).$$

Thus, $F_1^\delta(x)\subseteq F(x)$.

Lemma 10. *Let technological maps* $F_1(x)$ *and* $F_1^\delta(x)$ *be given, respectively, by the formulas* (4) *and* (6) *on the set X generated by the set of points* $(x_1,\cdots,x_k)$ *that non obligatory is minimal and every point of the set X be an internal for the set* X_1, *on which a convex down technological map* $F(x)$ *from the CTM class in a wide sense is given. Then, for every* $\varepsilon>0$, *there exists* $\delta>0$ *such that*

$$\sup_{p\in P}\left|\sup_{x\in X}\sup_{y\in F_1(x)}\langle y-x,p\rangle-\sup_{x\in X}\sup_{y\in F_1^\delta(x)}\langle y-x,p\rangle\right|<\varepsilon.$$

Proof. Let us show the equality

$$\sup_{x\in X}\sup_{y\in F_1(x)}\langle y-x,\tilde{p}_i\rangle=\sup_{x\in X}\sup_{y\in F_1^\delta(x)}\langle y-x,\tilde{p}_i\rangle,\quad i=\overline{1,m(\delta)}.$$

At first, show that

$$\sup_{x\in X}\sup_{y\in F_1(x)}\langle y-x,\tilde{p}_i\rangle=\max_{1\le s\le k}\langle y(x_s,\tilde{p}_i)-x_s,\tilde{p}_i\rangle,\quad i=\overline{1,m(\delta)}.$$

This equality follows from the Lemma 8 and the fact that

$$\sup_{y\in F(x_s)}\langle y-x_s,\tilde{p}_i\rangle=\langle y(x_s,\tilde{p}_i)-x_s,\tilde{p}_i\rangle.$$

At last, on the basis of the Lemma 4

$$\sup_{x\in X}\sup_{y\in F_1^{\delta}(x)}\langle y-x,p\rangle=\max_{1\le s\le k}\max_{1\le j\le m(\delta)}\langle p,y(x_s,\tilde{p}_j)-x_s\rangle.$$

Since

$$\langle y(x_s,\tilde{p}_j)-x_s,\tilde{p}_i\rangle\le\sup_{y\in F(x_s)}\langle y-x_s,\tilde{p}_i\rangle=\langle y(x_s,\tilde{p}_i)-x_s,\tilde{p}_i\rangle,\quad \tilde{p}_j\neq\tilde{p}_i,$$

we have

$$\max_{1\le j\le m(\delta)}\langle y(x_s,\tilde{p}_j)-x_s,\tilde{p}_i\rangle=\langle y(x_s,\tilde{p}_i)-x_s,\tilde{p}_i\rangle,\quad i=\overline{1,m(\delta)}.$$

So,

$$\sup_{x\in X}\sup_{y\in F_1^{\delta}(x)}\langle y-x,\tilde{p}_i\rangle=\max_{1\le s\le k}\langle y(x_s,\tilde{p}_i)-x_s,\tilde{p}_i\rangle,\quad i=\overline{1,m(\delta)}.$$

The needed equality is proved. Let us estimate the difference

$$\begin{aligned}T&=\sup_{p\in P}\left|\sup_{x\in X}\sup_{y\in F_1(x)}\langle y-x,p\rangle-\sup_{x\in X}\sup_{y\in F_1^{\delta}(x)}\langle y-x,p\rangle\right|\\&\le\sup_{p\in P}\left|\sup_{x\in X}\sup_{y\in F_1(x)}\langle y-x,p\rangle-\sup_{x\in X}\sup_{y\in F_1(x)}\langle y-x,\tilde{p}_i\rangle\right|\\&\quad+\sup_{p\in P}\left|\sup_{x\in X}\sup_{y\in F_1^{\delta}(x)}\langle y-x,p\rangle-\sup_{x\in X}\sup_{y\in F_1^{\delta}(x)}\langle y-x,\tilde{p}_i\rangle\right|.\end{aligned}$$

Since $F_1(x)$ and $F_1^{\delta}(x)$ satisfy conditions of the Lemma 5, then for $p\in C(\delta,\tilde{p}_i)$, $T\le\delta A$, $A=4\sup_{x\in X}\sup_{y\in F(x)}\langle y+x,e\rangle$. Choosing $\delta>0$ such that for given $\varepsilon>0$ the inequality $\delta A<\varepsilon$ would be valid, we obtain the proof of the Lemma. □

Now, prove the main statement of this Section.

Theorem 1. [1] *Let X be a bounded closed convex set whose every point is internal for* X_1 *and* $F(x)$ *be a convex down technological map from the CTM class in a wide sense given on a convex compact set* X_1. *Then, for every sufficiently small* $\varepsilon>0$, *there exists a continuous firm behavior strategy* $(x^0(p),y^0(p))$, $y^0(p)\in F(x^0(p))$ *such that*

$$\sup_{p\in P}\left|\varphi(p)-\langle y^0(p)-x^0(p),p\rangle\right|<\varepsilon,$$

where

$$\varphi(p)=\sup_{x\in X}\sup_{y\in F(x)}\langle y-x,p\rangle.$$

Proof. Let $\{x_i,i=\overline{1,\infty}\}$ be a dense countable set of points that generates *X* and such that it contains a dense set of extreme points of *X*. Further, let X_k be a polyhedron generated by the first *k* points $(x_1,\cdots,x_k)$. There hold such enclosures $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_k \subseteq \cdots \subseteq X$. On the basis of the Lemma 9, the sequence of continuous functions on P $\varphi_k(p) = \sup_{x \in X_k} \sup_{y \in F_1^k(x)} \langle y - x, p \rangle$ is monotonically non decreasing and converges to a continuous function $\varphi(p) = \sup_{x \in X} \sup_{y \in F(x)} \langle y - x, p \rangle$. According to the Dini theorem, the sequence $\varphi_k(p)$ converges to $\varphi(p)$ uniformly. Therefore, there exists a number k_0 such that for every $k \ge k_0$

$$\sup_{p \in P} \left| \varphi_k(p) - \varphi(p) \right| < \varepsilon/3.$$

From the Lemma 10

$$\sup_{p \in P} \left| \varphi_k(p) - \varphi_k^{\delta}(p) \right| < \varepsilon/3$$

for sufficiently small δ, where

$$\varphi_k^{\delta}(p) = \sup_{x \in X_k} \sup_{y \in F_1^{\delta}(x)} \langle y - x, p \rangle.$$

According to the Lemma 6, there exists a continuous firm behavior strategy $(x_0(p), y_0(p))$, $x_0(p) \in X_k \subseteq X$, $y_0(p) \in F_1^{\delta}(x_0(p)) \subseteq F(x_0(p))$ such that

$$\sup_{p \in P} \left| \varphi_k^{\delta}(p) - \langle y_0(p) - x_0(p), p \rangle \right| < \varepsilon/3.$$

The latter proves the Theorem. □

3. Equilibrium State Existence

This section examines the conditions for the existence of economic equilibrium in the presence of production of goods. Production is described by technological mappings belonging to the class of compact technological mappings.

We suppose that economy system models described non-aggregately behavior strategies of consumers choice (realizations of random fields of consumers choice) are not continuous functions on the simplex P. If the consumer consumes not all the goods the economy system produces, but some part of them, then components of the demand vector corresponding to goods he does not consume equal zero. As a result, the demand vector of such consumer can not be given unambiguously on the whole simplex P such that to be continuous on this simplex. To describe discontinuous behavior strategies of consumers choice, define random fields of consumers choice not on the whole cone $\bar{R}_+^n$, but on a certain cone $K_+^n \subset \bar{R}_+^n$ being a subcone of the cone $\bar{R}_+^n$ on which the conditions of the Theorem 1.4.6 and the Theorem 1.4.7 on the existence of random fields of consumers choice are valid [1]. Let $C = \| c_{ik} \|_{i,k=1}^{n,l}$ be a certain $n \times l$ - dimensional matrix satisfying conditions: $\sum_{k=1}^{n} c_{ki} > 0, i = \overline{1,l}$, and $\min_{k,s c_{ks} \neq 0} c_{ks} = 1$. Introduce the cone K_+^n built by the rule

$$K_+^n = \left\{ p \in \bar{R}_+^n, \sum_{k=1}^{n} c_{ki} p_k > 0, i = \overline{1,l} \right\}. \tag{7}$$

First, let us consider the case of all insatiable consumers. Suppose random fields of information evaluation by consumers satisfy the condition: for every i-th consumer a random field of information evaluation by the i-th consumer $\eta_i^0(p,z,\omega_i)=\{\eta_{ik}^0(p,z,\omega_i)\}_{k=1}^n$ on a probability space $\{\Omega_i,F_i,P_i\}$ satisfies the inequality

$$\eta_i^0(p,z,\omega_i)\geq m_iC_i,\quad C_i=\{c_{ki}\}_{k=1}^n,\quad (p,z,\omega_i)\in K_+^n\times\Gamma^m\times\Omega_i,\quad i=\overline{1,l},$$

where the real numbers $m_i>0$, $i=\overline{1,l}$, and components $\eta_{ik}^0(p,z,\omega_i)=0$ if and only if $c_{ki}=0$.

Under these additional assumptions about the random fields $\eta_i^0(p,z,\omega_i)$, $i=\overline{1,l}$, and assumptions about the matrix C on the above built cone K_+^n (with the rest conditions of Theorems 1.4.4 and 1.4.6 hold (see [1]), there exist random fields of consumers choice and decisions making by firms for insatiable consumers.

In what follows, we assume the conditions the above stated hold for the random fields on the above built cone.

Note that if a certain components of the vector $C_i=\{c_{ki}\}_{k=1}^n$ equal zero, then the i-th consumer does not consume goods numbered by these components.

Let $z(p)$ be a certain continuous realization of a random field $\zeta(p,\omega_0)$, and $\mu_i(p)=\{\mu_{ki}(p)\}_{k=1}^n$ be a continuous realization of a random field

$$\eta_i(p,\zeta_0(p,\omega_0),\omega_i)=\eta_i^0(p,\zeta(p,\omega_0),\omega_i)=\{\eta_{ik}^0(p,\zeta(p,\omega_0),\omega_i)\}_{k=1}^n.$$

Therefore, $\mu_{ki}(p)=\eta_{ik}^0(p,\zeta(p,\omega_0),\omega_i)$ for a certain ω_0 and ω_i. Denote by

$$\gamma_{ik}(p)=\frac{\mu_{ki}(p)p_k}{\sum_{j=1}^n\mu_{ji}(p)p_j},\quad k=\overline{1,n},\quad i=\overline{1,l},$$

$$\psi(p)=\{\psi_k(p)\}_{k=1}^n,$$

$$\psi_k(p)=\sum_{i=1}^m\left[y_{ki}(p)-x_{ki}(p)\right]+\sum_{i=1}^l b_{ki}(p,z(p)),\quad k=\overline{1,n},$$

$$\Gamma_k(p)=\frac{1}{\psi_k(p)}\sum_{i=1}^l\frac{\mu_{ki}(p)p_kD_i(p)}{\sum_{j=1}^n\mu_{ji}(p)p_j},\quad k=\overline{1,n},$$

$$\Gamma(p)=\{\Gamma_k(p)\}_{k=1}^n.$$

Introduce the set

$$C_\delta=\left\{p\in R_+^n,\sum_{s=1}^n c_{si}p_s\geq\delta,i=\overline{1,l},\sum_{i=1}^n p_i=1\right\}$$

and the next notations

$$\sup_{k,p\in P\cap K_+^n}\psi_k(p)=R_1,\quad \inf_{k,p\in P\cap K_+^n}\psi_k(p)=R_0.$$

Theorem 2. *Let technological maps* $F_i(x),x\in X_i^1,i=\overline{1,m}$, *describing the* *economy system production be convex down, belong to the CTM class, and let a productive economic process* $Q(p,z)$, *a family of income pre-functions* $K_i^0(p,z), i=\overline{1,l}$, *and property vectors* $b_i(p,z), i=\overline{1,l}$, *be continuous maps of variables* $(p,z)\in K_+^n\times\Gamma^m$, *where* K_+^n *is a cone given by the formula* (7), *and also let*

$$D_i(p)=K_i(p,\zeta_0(p,\omega_0))>a>0,\quad p\in P\cap K_+^n,\quad \omega_0\in\Omega_0,\quad i=\overline{1,l},$$

where a does not depend on (p,ω_0). *Assume that random fields of information evaluation by consumers and decisions making by firms satisfy the conditions of the Theorem* 1.4.6 [1] *and the productive economic process* $Q(p,z)$ *satisfies the condition*

$$R(p,Q(p,z))>r,\quad p\in K_+^n,\quad z\in\Gamma^m, \tag{8}$$

where

$$R(p,z)=\sum_{i=1}^m[y_i-x_i]+\sum_{j=1}^l b_j(p,z),\quad r=\{r_i\}_{i=1}^n,\quad r_i>0,\quad i=\overline{1,n}.$$

If the set C_δ *is non-empty for some* $\delta>0$, *the inequality*

$$\min_{\{k,s,f_{ks}=1\}}\inf_{p\in C_\delta}\frac{D_s(p)}{\psi_k(p)}\ge\frac{R_1}{R_0}\delta$$

holds, where

$$f_{ks}=\begin{cases}1, & \text{if } c_{ks}\ne 0,\\ 0, & \text{if } c_{ks}=0,\end{cases}$$

then for every continuous demand matrix $\|\gamma_{ik}(p)\|_{i=1,k=1}^{l,n}$ *and continuous realization of random field of decisions making by firms* $z(p)$ *on* K_+^n, *i.e., with probability* 1, *there exists a price vector* $\bar{p}$ *corresponding to them satisfying the set of equations*

$$\sum_{i=1}^l\gamma_{ik}(p)D_i(p)=p_k\left[\sum_{i=1}^m[y_{ki}(p)-x_{ki}(p)]+\sum_{i=1}^l b_{ki}(p,z(p))\right],\quad k=\overline{1,n}. \tag{9}$$

Proof. On the closed bounded convex set C_δ, let us consider the non-linear operator

$$f(p)=\{f_k(p)\}_{k=1}^n,$$

where

$$f_k(p)=\frac{\Gamma_k(p)}{\sum_{i=1}^n\Gamma_i(p)},\quad k=\overline{1,n}.$$

Under the conditions of the Theorem, $R_0>\min_{1\le i\le n}r_i>0$, and $R_1<\infty$. The operator $f(p)$ transforms the set C_δ into itself and is a continuous map. It is sufficient to check the inequalities

$$\sum_{k=1}^n c_{ki}f_k(p)\ge\delta,\quad i=\overline{1,l},$$

and show the continuity of $f(p)$. First, prove the continuity of $f(p)$. In view of the continuity of $\mu_{ki}(p)$ for every realization and the condition of the Theorem,

$$\sum_{j=1}^{n} \mu_{ji}(p) p_j \geq m_i \sum_{j=1}^{n} c_{ji} p_j > m_i \delta > 0.$$

So, we have that $\sum_{j=1}^{n} \mu_{ji}(p) p_j$ nowhere vanishes on the set C_δ. From the fact that $\psi_k(p) > r_k, k = \overline{1,n}$, we have that $\psi_k(p)$ is a continuous function of $p \in K_+^n$ and $\Gamma_k(p)$ is a continuous map on C_δ. Find lower bound for $\sum_{k=1}^{n} \Gamma_k(p)$. We have

$$\sum_{k=1}^{n} \Gamma_k(p) \geq \frac{1}{R_1} \sum_{i=1}^{l} \frac{\sum_{k=1}^{n} \mu_{ki}(p) p_k}{\sum_{j=1}^{n} \mu_{ji}(p) p_j} D_i(p) = \frac{1}{R_1} \sum_{i=1}^{n} \psi_i(p) p_i \geq \frac{R_0}{R_1}.$$

Therefore, $f(p)$ is a continuous map on C_δ. For $\sum_{k=1}^{n} \Gamma_k(p)$, the upper bound

$$\sup_{p \in P \cap K_+^n} \sum_{r=1}^{n} \Gamma_r(p) \leq \frac{1}{R_0} \sup_{p \in P \cap K_+^n} \sum_{i=1}^{l} D_i(p) = \frac{1}{R_0} \sup_{p \in P \cap K_+^n} \sum_{i=1}^{n} \psi_i(p) p_i \leq \frac{R_1}{R_0}$$

is valid. Finally, show that

$$\sum_{k=1}^{n} c_{ki} f_k(p) \geq \delta, \quad i = \overline{1,l}.$$

Really,

$$\sum_{k=1}^{n} c_{ki} f_k(p) = \sum_{k=1}^{n} c_{ki} \frac{1}{\psi_k(p) \sum_{r=1}^{n} \Gamma_r(p)} \sum_{s=1}^{l} \frac{\mu_{ks}(p) p_k D_s(p)}{\sum_{j=1}^{n} \mu_{js}(p) p_j}$$

$$\geq \delta \sum_{k=1}^{n} c_{ki} \sum_{s=1}^{l} \frac{f_{ks} \mu_{ks}(p) p_k}{\sum_{j=1}^{n} \mu_{js}(p) p_j} = \delta \sum_{s=1}^{l} \frac{\sum_{k=1}^{n} c_{ki} f_{ks} \mu_{ks}(p) p_k}{\sum_{j=1}^{n} \mu_{js}(p) p_j}.$$

Because of the assumptions for the matrix elements c_{ki}, for $s = i$

$$\frac{\sum_{k=1}^{n} c_{ki} f_{ki} \mu_{ki}(p) p_k}{\sum_{j=1}^{n} \mu_{ji}(p) p_j} \geq \min_{\{k,i,f_{ki}=1\}} c_{ki} \frac{\sum_{k=1}^{n} \mu_{ki}(p) p_k}{\sum_{j=1}^{n} \mu_{ji}(p) p_j} = 1.$$

So,

$$\sum_{k=1}^{n} c_{ki} f_k(p) \geq \delta, \quad i = \overline{1,l}.$$

By the Schauder Theorem [10], there exists a fixed point of the map $f(p)$. This point is also a fixed point for the map $\Gamma(p)$. Really, as p^* is a fixed point of the map $f(p)$, we have

$$\Gamma_k(p^*) = p_k^* \sum_{s=1}^n \Gamma_s(p^*).$$

Multiplying by $\psi_k(p^*)$ the last equality and summing up over *k*, we have

$$\sum_{k=1}^n \psi_k(p^*)\Gamma_k(p^*) = \langle \psi(p^*), p^* \rangle \sum_{s=1}^n \Gamma_s(p^*).$$

However,

$$\sum_{k=1}^n \psi_k(p^*)\Gamma_k(p^*) = \sum_{i=1}^l D_i(p^*) = \langle \psi(p^*), p^* \rangle.$$

Reducing by $\langle \psi(p^*), p^* \rangle$, in view of the inequality $\langle \psi(p^*), p^* \rangle \neq 0$, we obtain $\sum_{s=1}^n \Gamma_s(p^*) = 1$. The last means that p^* solves the set of Equation (9). □

Establish sufficient conditions for the previous Theorem conditions to hold.

Lemma 11. *If the matrix elements* c_{ki} *satisfy the conditions stated above and the conditions of the Theorem* 2 *are valid and also*

$$\min_{1\le i\le l} \inf_{p\in P\cap K_+^n} D_i(p) = d > 0,$$

then for all δ *satisfying the inequalities*

$$0 < \delta \le \min\left\{ \frac{R_0 d}{R_1^2}, \frac{1}{n} \min_{1\le i\le l} \sum_{j=1}^n c_{ji} \right\}$$

the set C_δ *is non-empty and the inequality*

$$\min_{\{k,s,f_{ks}=1\}} \inf_{p\in C_\delta} \frac{D_s(p)}{\psi_k(p)} \ge \frac{R_1}{R_0}\delta$$

holds.

Proof. There hold bounds

$$\min_{\{k,s,f_{ks}=1\}} \inf_{p\in C_\delta} \frac{D_s(p)}{\psi_k(p)} \ge \frac{d}{R_1} \ge \frac{R_1}{R_0}\delta.$$

Finally, the set C_δ is non-empty because it contains the vector $\{1/n, \cdots, 1/n\}$.

□

Theorem 3. *Let technological maps* $F_i(x), x \in X_i^1, i = \overline{1,m}$, *describing the economy system production structure, be convex down, belong to the CTM class, and let a productive economic process* $Q(p,z)$, *a family of income pre-functions* $K_i^0(p,z), i = \overline{1,l}$, *and property vectors* $b_i(p,z), i = \overline{1,l}$, *be continuous maps of variables* $(p,z) \in K_+^n \times \Gamma^m$, *where* K_+^n *is a cone given by the formula* (7), *and also let the conditions of the Lemma* 11 *hold.*

Suppose that random fields of information evaluation by consumers and decisions making by firms satisfy the conditions of the Theorem 1.4.6 [1] *and a productive economic process* $Q(p,z)$ *satisfies the condition*

$$R(p, Q(p,z)) > r, \quad p \in K_+^n, z \in \Gamma^m, \tag{10}$$

where

$$R(p,z)=\sum_{i=1}^{m}[y_i-x_i]+\sum_{j=1}^{l}b_j(p,z),\quad r=\{r_i\}_{i=1}^{n},\quad r_i>0,\quad i=\overline{1,n}.$$

If $\sum_{i=1}^{l} f_{ki}>0, k=\overline{1,n}$ *, then for every continuous demand matrix* $\left\|\gamma_{ik}(p)\right\|_{i=1,k=1}^{l,n}$ *and continuous realization of random field of decisions making by firms* $z(p)$ *on* K_+^n *there exists a strictly positive price vector* p^{ε} *corresponding to them and satisfying the set of equations*

$$\sum_{i=1}^{l}\gamma_{ik}^{\varepsilon}(p)D_i(p)=p_k\left[\sum_{i=1}^{m}\left[y_{ki}(p)-x_{ki}(p)\right]+\sum_{i=1}^{l}b_{ki}(p,z(p))\right],\quad k=\overline{1,n},\quad (11)$$

where

$$\gamma_{ik}^{\varepsilon}(p)=\frac{\mu_{ki}(p)p_k+\varepsilon f_{ki}}{\sum_{j=1}^{n}\mu_{ji}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{ji}},\quad k=\overline{1,n},\quad i=\overline{1,l},\quad 0<\varepsilon<1.$$

Proof. Denote

$$\Gamma_k^{\varepsilon}(p)=\frac{1}{\psi_k(p)}\sum_{i=1}^{l}\frac{\left(\mu_{ki}(p)p_k+\varepsilon f_{ki}\right)D_i(p)}{\sum_{j=1}^{n}\mu_{ji}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{ji}},\quad k=\overline{1,n},$$

$$\Gamma^{\varepsilon}(p)=\left\{\Gamma_k^{\varepsilon}(p)\right\}_{k=1}^{n}.$$

On the closed bounded convex set C_δ, where $\delta>0$ satisfies the conditions of the Lemma 11, let us consider the non-linear map

$$f^{\varepsilon}(p)=\left\{f_k^{\varepsilon}(p)\right\}_{k=1}^{n},$$

where

$$f_k^{\varepsilon}(p)=\frac{\Gamma_k^{\varepsilon}(p)}{\sum_{i=1}^{n}\Gamma_i^{\varepsilon}(p)},\quad k=\overline{1,n}.$$

The map $f^{\varepsilon}(p)$ transforms the set C_δ into itself and is a continuous map. It is sufficient to check the inequalities

$$\sum_{k=1}^{n}c_{ki}f_k^{\varepsilon}(p)\geq\delta,\quad i=\overline{1,l},$$

and show the continuity of $f^{\varepsilon}(p)$. First, let us prove the continuity of $f^{\varepsilon}(p)$. In view of the continuity of $\mu_{ki}(p)$ for every realization and the condition of the Theorem,

$$\sum_{j=1}^{n}\mu_{ji}(p)p_j\geq m_i\sum_{j=1}^{n}c_{ji}p_j>m_i\delta>0.$$

Thus, $\sum_{j=1}^{n}\mu_{ji}(p)p_j$ nowhere vanishes on the set C_δ. Therefore, $\Gamma_k^{\varepsilon}(p)$ is a continuous map on C_δ.

There hold the estimates

$$\sum_{k=1}^{n}\Gamma_k^{\varepsilon}(p)\geq\frac{1}{R_1}\sum_{i=1}^{l}D_i(p)=\frac{1}{R_1}\sum_{i=1}^{n}\psi_i(p)p_i\geq\frac{R_0}{R_1},$$

$$\sup_{p\in P\cap K_+^n}\sum_{r=1}^{n}\Gamma_r^{\varepsilon}(p)\leq\frac{1}{R_0}\sup_{p\in P\cap K_+^n}\sum_{i=1}^{l}D_i(p)=\frac{1}{R_0}\sup_{p\in P\cap K_+^n}\sum_{i=1}^{n}\psi_i(p)p_i\leq\frac{R_1}{R_0}.$$

Therefore, $f(p)$ is a continuous map on C_δ.
Finally, show that

$$\sum_{k=1}^{n}c_{ki}f_k^{\varepsilon}(p)\geq\delta,\quad i=\overline{1,l}.$$

Really,

$$\sum_{k=1}^{n}c_{ki}f_k^{\varepsilon}(p)=\sum_{k=1}^{n}c_{ki}\frac{1}{\psi_k(p)\sum_{r=1}^{n}\Gamma_r^{\varepsilon}(p)}\sum_{s=1}^{l}\frac{(\mu_{ks}(p)p_k+\varepsilon f_{ks})D_s(p)}{\sum_{j=1}^{n}\mu_{js}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{js}}$$

$$\geq\delta\sum_{k=1}^{n}c_{ki}\sum_{s=1}^{l}\frac{f_{ks}(\mu_{ks}(p)p_k+\varepsilon f_{ks})}{\sum_{j=1}^{n}\mu_{js}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{js}}$$

$$=\delta\sum_{s=1}^{l}\frac{\sum_{k=1}^{n}c_{ki}f_{ks}(\mu_{ks}(p)p_k+\varepsilon f_{ks})}{\sum_{j=1}^{n}\mu_{js}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{js}}.$$

In view of the assumptions about matrix elements c_{ki}, for $s=i$

$$\frac{\sum_{k=1}^{n}c_{ki}f_{ki}(\mu_{ki}(p)p_k+\varepsilon f_{ki})}{\sum_{j=1}^{n}\mu_{ji}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{ji}}\geq\min_{\{k,i,f_{ki}=1\}}c_{ki}\frac{\sum_{k=1}^{n}(\mu_{ki}(p)p_k+\varepsilon f_{ki})}{\sum_{j=1}^{n}\mu_{ji}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{ji}}=1.$$

Therefore,

$$\sum_{k=1}^{n}c_{ki}f_k^{\varepsilon}(0)\geq\delta,\quad i=\overline{1,l}.$$

By the Schauder Theorem [10], there exists a fixed point of the map $f^{\varepsilon}(p)$. This point is also a fixed point for the map $\Gamma^{\varepsilon}(p)$. Really, as p^{ε} is the fixed point of the map $f^{\varepsilon}(p)$, we have

$$\Gamma_k(p^{\varepsilon})=p_k^{\varepsilon}\sum_{s=1}^{n}\Gamma_s(p^{\varepsilon}).$$

Multiplying the last equality by $\psi_k(p^{\varepsilon})$ and summing up over *k*, we have

$$\sum_{k=1}^{n}\psi_k(p^{\varepsilon})\Gamma_k(p^{\varepsilon})=\langle\psi(p^{\varepsilon}),p^{\varepsilon}\rangle\sum_{s=1}^{n}\Gamma_s(p^{\varepsilon}).$$

However,

$$\sum_{k=1}^{n}\psi_k(p^{\varepsilon})\Gamma_k(p^{\varepsilon})=\sum_{i=1}^{l}D_i(p^{\varepsilon})=\langle\psi(p^{\varepsilon}),p^{\varepsilon}\rangle.$$

Reducing by $\langle\psi(p^{\varepsilon}),p^{\varepsilon}\rangle$, in view of $\langle\psi(p^{\varepsilon}),p^{\varepsilon}\rangle\neq 0$, we obtain

$\sum_{s=1}^{n}\Gamma_s\left(p^{\varepsilon}\right)=1$. The latter means that p^{ε} solves the set of Equation (11). As the price vector p^{ε} solves the set of Equation (11) and the conditions of the Lemma 11 hold, the inequalities for components p_k^{ε}

$$p_k^{\varepsilon} \geq \frac{a\varepsilon\sum_{i=1}^{l} f_{ki}}{R_1\left(\mu+\max_i\sum_{j=1}^{n} f_{ji}\varepsilon\right)} > 0, \quad k=\overline{1,n},$$

hold, where

$$\mu=\max_i \sup_{p\in P\cap K_+^n}\sum_{s=1}^{n}\mu_{si}(p)p_s<\infty.$$

□

Theorem 4. *Let the conditions of the Theorem* 2, *of the Lemma* 11 *and the inequalities* $\sum_{i=1}^{l} f_{ki}>0, k=\overline{1,n}$, *hold. Then there exists an equilibrium price vector* $p^0\in C_\delta$, *under which the demand does not exceed the supply, i.e., the set of inequalities holds*

$$\begin{aligned}&\sum_{i=1}^{l}\frac{\mu_{ki}\left(p^0\right)}{\sum_{s=1}^{n}\mu_{si}\left(p^0\right)p_s^0}D_i\left(p^0\right)\\&\leq\sum_{i=1}^{m}\left[y_{ki}\left(p^0\right)-x_{ki}\left(p^0\right)\right]+\sum_{i=1}^{l}b_{ki}\left(p^0,z\left(p^0\right)\right), \quad k=\overline{1,n}.\end{aligned} \tag{12}$$

Every equilibrium price vector satisfies the set of Equation (9).

Proof. Consider the auxiliary set of equations

$$\begin{aligned}&\sum_{i=1}^{l}\frac{p_k\mu_{ki}(p)+\varepsilon f_{ki}}{\sum_{s=1}^{n}\mu_{si}(p)p_s+\varepsilon\sum_{s=1}^{n}f_{si}}D_i(p)\\&=p_k\left[\sum_{i=1}^{m}\left[y_{ki}(p)-x_{ki}(p)\right]+\sum_{i=1}^{l}b_{ki}(p,z(p))\right], \quad k=\overline{1,n}, \quad 0<\varepsilon<1,\end{aligned} \tag{13}$$

built after the set of Equation (9). Every component of the solution $p^{\varepsilon}=\left\{p_k^{\varepsilon}\right\}_{k=1}^{n}$ to the set of Equation (13), by the Theorem 3, is strictly positive.

As p^{ε} solves the set of Equation (13) and $p_k^{\varepsilon}>0, k=\overline{1,n}$, we obtain the set of inequalities

$$\begin{aligned}&\sum_{i=1}^{l}\frac{\mu_{ki}\left(p^{\varepsilon}\right)}{\sum_{s=1}^{n}\mu_{si}\left(p^{\varepsilon}\right)p_s^{\varepsilon}+\varepsilon\sum_{s=1}^{n}f_{si}}D_i\left(p^{\varepsilon}\right)\\&\leq\sum_{i=1}^{m}\left[y_{ki}\left(p^{\varepsilon}\right)-x_{ki}\left(p^{\varepsilon}\right)\right]+\sum_{i=1}^{l}b_{ki}\left(p^{\varepsilon},z\left(p^{\varepsilon}\right)\right), \quad k=\overline{1,n}.\end{aligned} \tag{14}$$

The sequence p^{ε}, when $\varepsilon\to 0$, is compact one, because it belongs to the set C_δ. Due to the continuity of $\sum_{s=1}^{n}\mu_{si}(p)p_s$ on the set P and the inequality

$$\sum_{s=1}^{n}\mu_{si}(p)p_s > m_i\delta, \quad p\in C_\delta,$$

one can go to the limit in the set of Inequalities (14). Denote one of the possible limit points of the sequence p^ε by p^0. Then p^0 solves the set of inequalities

$$\sum_{i=1}^{l}\frac{\mu_{ki}(p^0)}{\sum_{s=1}^{n}\mu_{si}(p^0)p_s^0}D_i(p^0) \le \sum_{i=1}^{m}\left[y_{ki}(p^0)-x_{ki}(p^0)\right]+\sum_{i=1}^{l}b_{ki}\left(p^0,z(p^0)\right), \quad k=\overline{1,n}. \tag{15}$$

It is obvious that the vector p^0 belongs to the set C_δ and solves the set of Equation (9). □

4. Economy Equilibrium with Fixed Profits of Consumers

We introduced a model of consumption economy with fixed profits and studied it in the papers and monographs [1] [11] [12] [13]. Under rather simple restrictions on the consumption structure, supply vector, and consumers profits, we proved the existence Theorem for equilibrium price vector. Suppose that in a certain economic system there are n kinds of goods and l consumers. Consider that the i-th consumer has fund $D_i>0, i=\overline{1,l}$. On the economic system market, goods supply vector has the form

$$\psi=\{\psi_i\}_{i=1}^{n}, \quad \psi_i>0, \quad i=\overline{1,n}.$$

The set of possible price vectors is a cone $K_+^n\subset \bar{R}_+^n$, being a subcone of the cone $\bar{R}_+^n$. Let us build the cone K_+^n. Let $C=\|c_{ik}\|_{i,k=1}^{n,l}$ be a certain non-negative matrix of the dimension $n\times l$ satisfying conditions

$$\sum_{k=1}^{n}c_{ki}>0, \quad i=\overline{1,l}, \quad \min_{k,s\, c_{ks}\neq 0}c_{ks}=1.$$

The matrix C determines the cone K_+^n by the rule

$$K_+^n=\left\{p\in\bar{R}_+^n, \sum_{k=1}^{n}c_{ki}p_k>0, i=\overline{1,l}\right\}. \tag{16}$$

Consider the case of insatiable consumers. Suppose that random fields of information evaluation by consumers satisfy the condition: for every i-th consumer the random field of information evaluation $\eta_i^0(p,\omega_i)$ by the i-th consumer on a probability space $\{\Omega_i,F_i,P_i\}, i=\overline{1,l}$, satisfies the inequality

$$\eta_i^0(p,\omega_i)\ge m_iC_i, \quad (p,\omega_i)\in K_+^n\times\Omega_i, \quad m_i>0, \quad i=\overline{1,l}, \tag{17}$$

where $C_i=\{c_{ki}\}_{k=1}^{n}$, and components $\eta_{ik}^0(p,\omega_i)$ of the random field of information evaluation by the i-th consumer

$$\eta_i^0(p,\omega_i)=\{\eta_{ik}^0(p,\omega_i)\}_{k=1}^{n}$$

satisfy the condition: $\eta_{ik}^0(p,\omega_i)=0$ if and only if $c_{ki}=0$.

Suppose that consumers operate independently and their random fields of choice have the form

$$\xi_i(p)=\frac{D_i\eta_i^0(p,\omega_i)}{\langle p,\eta_i^0(p,\omega_i)\rangle},\quad i=\overline{1,l}.$$

In the next Theorem, we assume that the above formulated conditions hold for random fields of consumers choice on the cone built above.

Note that if certain components of the vector $C_i=\{c_{ki}\}_{k=1}^n$ equal zero, then the i-th consumer does not consume goods numbered by these components.

Let $\mu_i(p)=\{\mu_{ki}(p)\}_{k=1}^n$ be a continuous realization of the random field of information evaluation by consumer

$$\eta_i(p,\omega_i)=\eta_i^0(p,\omega_i)=\{\eta_{ik}^0(p,\omega_i)\}_{k=1}^n.$$

Therefore, $\mu_{ki}(p)=\eta_{ik}^0(p,\omega_i)$ for some ω_0. As earlier, denote

$$\gamma_{ik}(p)=\frac{\mu_{ki}(p)p_k}{\sum_{j=1}^n\mu_{ji}(p)p_j},\quad k=\overline{1,n},\quad i=\overline{1,l},$$

components of the demand vector

$$\gamma_i(p)=\{\gamma_{ik}(p)\}_{k=1}^n,\quad i=\overline{1,l}.$$

Introduce into consideration a set

$$C_\delta=\left\{p\in K_+^n,\sum_{s=1}^n c_{si}p_s\ge\delta,i=\overline{1,l},\sum_{i=1}^n p_i\psi_i=\sum_{i=1}^l D_i\right\}$$

and notations $R_1=\max_k\psi_k,R_0=\min_k\psi_k,D_0=\sum_{i=1}^l D_i$.

Theorem 5. *Suppose that random fields of information evaluation by consumers on the cone* K_+^n, *given by the* Formula (16), *are continuous with probability* 1, *satisfy the* Condition (17), *and a number* δ *satisfies the inequality*

$$0<\delta\le\min\left\{\min_{i,k}\frac{D_i}{\psi_k},\frac{D_0}{nR_1}\min_i\sum_{k=1}^n c_{ki}\right\}.$$

Then for every continuous on K_+^n *demand matrix* $\|\gamma_{ik}(p)\|_{i=1,k=1}^{l,n}$, *i.e., with probability* 1, *there exists a corresponding price vector* $\bar p$ *satisfying the set of equations*

$$\sum_{i=1}^l\frac{\mu_{ki}(p)p_k}{\sum_{j=1}^n\mu_{ji}(p)p_j}D_i=p_k\psi_k,\quad k=\overline{1,n}.\tag{18}$$

Proof. Introduce into consideration a map

$$\Gamma(p)=\{\Gamma_k(p)\}_{k=1}^n,$$

$$\Gamma_k(p)=\frac{1}{\psi_k}\sum_{i=1}^l\frac{\mu_{ki}(p)p_kD_i}{\sum_{j=1}^n\mu_{ji}(p)p_j},\quad k=\overline{1,n},$$

and show that it maps the non-empty set C_δ into itself.

If δ satisfies the conditions of the Theorem, then the set C_δ is not empty and the inequality

$$\min_{\{k,s,f_{ks}=1\}} \frac{D_s}{\psi_k} \geq \delta$$

holds, where

$$f_{ks} = \begin{cases} 1, & \text{if } c_{ks} \neq 0, \\ 0, & \text{if } c_{ks} = 0. \end{cases}$$

Check the validity of the inequalities

$$\sum_{k=1}^{n} c_{ki} \Gamma_k(p) \geq \delta, \quad i = \overline{1,l}.$$

Really,

$$\sum_{k=1}^{n} c_{ki} \Gamma_k(p) = \sum_{k=1}^{n} c_{ki} \frac{1}{\psi_k} \sum_{s=1}^{l} \frac{\mu_{ks}(p) p_k D_s}{\sum_{j=1}^{n} \mu_{js}(p) p_j}$$

$$\geq \delta \sum_{k=1}^{n} c_{ki} \sum_{s=1}^{l} \frac{f_{ks} \mu_{ks}(p) p_k}{\sum_{j=1}^{n} \mu_{js}(p) p_j} = \delta \sum_{s=1}^{l} \frac{\sum_{k=1}^{n} c_{ki} f_{ks} \mu_{ks}(p) p_k}{\sum_{j=1}^{n} \mu_{js}(p) p_j}.$$

Because of assumptions about the matrix elements c_{ki}, for $s = i$

$$\frac{\sum_{k=1}^{n} c_{ki} f_{ki} \mu_{ki}(p) p_k}{\sum_{j=1}^{n} \mu_{ji}(p) p_j} \geq \min_{\{k,i,f_{ki}=1\}} c_{ki} \frac{\sum_{k=1}^{n} \mu_{ki}(p) p_k}{\sum_{j=1}^{n} \mu_{ji}(p) p_j} = 1.$$

Therefore,

$$\sum_{k=1}^{n} c_{ki} \Gamma_k(p) \geq \delta, \quad i = \overline{1,l}.$$

The map $\Gamma(p)$ is a continuous map of the convex compact set C_δ into itself. By the Schauder Theorem [10], there exists a fixed point of the map $\Gamma(p)$.

□

Theorem 6. *Assume that random fields of information evaluation by consumers on the cone* K_+^n *given by the* Formula (16) *are continuous with probability* 1, *satisfy the* Condition (17), *and a number* δ *satisfies conditions of the* Theorem 5. *If* $\sum_{i=1}^{l} f_{ki} > 0, k = \overline{1,n}$, *then for every continuous demand matrix* $\left\| \gamma_{ik}(p) \right\|_{i=1,k=1}^{l,n}$ *on* K_+^n, *i.e., with probability* 1, *there exists corresponding it a strictly positive price vector* $p^\varepsilon \in C_\delta$, *solving the set of equations*

$$\sum_{i=1}^{l} \gamma_{ik}^{\varepsilon}(p) D_i = p_k \psi_k, \quad k = \overline{1,n}, \tag{19}$$

where

$$\gamma_{ik}^{\varepsilon}(p) = \frac{\mu_{ki}(p) p_k + \varepsilon f_{ki}}{\sum_{j=1}^{n} \mu_{ji}(p) p_j + \varepsilon \sum_{j=1}^{n} f_{ji}}, \quad k = \overline{1,n}, \quad i = \overline{1,l}, \quad 0 < \varepsilon < 1.$$

Proof. Introduce into consideration a map

$$\Gamma^{\varepsilon}(p)=\{\Gamma_k^{\varepsilon}(p)\}_{k=1}^{n}.$$

$$\Gamma_k^{\varepsilon}(p)=\frac{1}{\psi_k}\sum_{i=1}^{l}\frac{(\mu_{ki}(p)p_k+\varepsilon f_{ki})D_i}{\sum_{j=1}^{n}\mu_{ji}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{ji}},\quad k=\overline{1,n},$$

and show that it maps the non-empty set C_δ into itself.

If δ satisfies the conditions of the Theorem, then the set C_δ is not empty and the inequality

$$\min_{\{k,s,f_{ks}=1\}}\frac{D_s}{\psi_k}\geq\delta,$$

holds. Check the validity of the inequalities

$$\sum_{k=1}^{n}c_{ki}\Gamma_k^{\varepsilon}(p)\geq\delta,\quad i=\overline{1,l}.$$

Really,

$$\begin{aligned}\sum_{k=1}^{n}c_{ki}\Gamma_k^{\varepsilon}(p)&=\sum_{k=1}^{n}c_{ki}\frac{1}{\psi_k}\sum_{s=1}^{l}\frac{(\mu_{ks}(p)p_k+\varepsilon f_{ks})D_s}{\sum_{j=1}^{n}\mu_{js}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{ji}}\\&\geq\delta\sum_{k=1}^{n}c_{ki}\sum_{s=1}^{l}\frac{f_{ks}(\mu_{ks}(p)p_k+\varepsilon f_{ks})}{\sum_{j=1}^{n}\mu_{js}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{js}}\\&=\delta\sum_{s=1}^{l}\frac{\sum_{k=1}^{n}c_{ki}f_{ks}(\mu_{ks}(p)p_k+\varepsilon f_{ks})}{\sum_{j=1}^{n}\mu_{js}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{js}}.\end{aligned}$$

Due to assumptions about the matrix elements c_{ki}, for $s=i$

$$\frac{\sum_{k=1}^{n}c_{ki}f_{ki}(\mu_{ki}(p)p_k+\varepsilon f_{ki})}{\sum_{j=1}^{n}\mu_{ji}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{ji}}\geq\min_{\{k,i,f_{ki}=1\}}c_{ki}\frac{\sum_{k=1}^{n}(\mu_{ki}(p)p_k+\varepsilon f_{ki})}{\sum_{j=1}^{n}\mu_{ji}(p)p_j+\varepsilon\sum_{j=1}^{n}f_{ji}}=1.$$

Therefore,

$$\sum_{k=1}^{n}c_{ki}\Gamma_k^{\varepsilon}(p)\geq\delta,\quad i=\overline{1,l}.$$

The map $\Gamma^{\varepsilon}(p)$ is a continuous map of the convex compact set C_δ into itself. By the Schauder Theorem [10], there exists a fixed point $p^{\varepsilon}=\{p_k^{\varepsilon}\}_{k=1}^{n}$ of the map $\Gamma^{\varepsilon}(p)$ whose components satisfy inequalities

$$p_k^{\varepsilon}\geq\frac{\varepsilon\min_i D_i\sum_{i=1}^{l}f_{ki}}{R_1\left(\mu+\varepsilon\max_i\sum_{j=1}^{n}f_{ji}\right)}>0,\quad k=\overline{1,n},$$

where

$$\mu = \max_i \sup_{p \in P \cap K_+^n} \sum_{s=1}^n \mu_{si}(p) p_s < \infty.$$

□

Theorem 7. *Let the conditions of the Theorem* 5 *and the inequalities*

$$\sum_{i=1}^l f_{ki} > 0, \quad k = \overline{1,n},$$

hold. Then there exists an equilibrium price vector $p^0 \in C_\delta$ *for which the demand does not exceed the supply, i.e., the set of inequalities*

$$\sum_{i=1}^l \frac{\mu_{ki}(p^0)}{\sum_{s=1}^n \mu_{si}(p^0) p_s^0} D_i \le \psi_k, \quad k = \overline{1,n}, \tag{20}$$

hold. Every equilibrium price vector satisfies the set of Equation (18).

Proof. Consider the auxiliary set of equations

$$\sum_{i=1}^l \gamma_{ik}^\varepsilon(p) D_i = p_k \psi_k, \quad k = \overline{1,n}, \tag{21}$$

built after the set of Equation (18), where

$$\gamma_{ik}^\varepsilon(p) = \frac{\mu_{ki}(p) p_k + \varepsilon f_{ki}}{\sum_{j=1}^n \mu_{ji}(p) p_j + \varepsilon \sum_{j=1}^n f_{ji}}, \quad k = \overline{1,n}, \quad i = \overline{1,l}, \quad 0 < \varepsilon < 1.$$

Every component of a solution $p^\varepsilon = \{p_k^\varepsilon\}_{k=1}^n$ for the set of Equation (21) is strictly positive by the Theorem 6. From the fact that the strictly positive vector p^ε satisfies the set of Equation (21), the validity of the set of inequalities

$$\sum_{i=1}^l \frac{\mu_{ki}(p^\varepsilon)}{\sum_{s=1}^n \mu_{si}(p^\varepsilon) p_s^\varepsilon + \varepsilon \sum_{s=1}^n f_{si}} D_i \le \psi_k, \quad k = \overline{1,n}, \tag{22}$$

follows. The sequence p^ε for $\varepsilon \to 0$ is compact because it belongs to the compact set C_δ. Due to the continuity of $\sum_{s=1}^n \mu_{si}(p) p_s$ on the set C_δ and the inequality

$$\sum_{s=1}^n \mu_{si}(p) p_s > m_i \delta, \quad p \in C_\delta,$$

one can go to the limit in the set of Inequalities (22), as $\varepsilon \to 0$. Denote one of the possible limit points of the sequence p^ε by p^0. Then p^0 solves the set of inequalities

$$\sum_{i=1}^l \frac{\mu_{ki}(p^0)}{\sum_{s=1}^n \mu_{si}(p^0) p_s^0} D_i \le \psi_k, \quad k = \overline{1,n}. \tag{23}$$

It is obvious that the vector p^0 belongs to the set C_δ.

□

5. Conclusion

Section 1 lists the main results. In Section 2, the theory of technological mappings from the CTM class is constructed, which contains the well-known technological mappings of Leontiev and Neumann. The main statement of this section is the Theorem of the existence of a continuous strategy of the firm behavior as arbitrarily close in terms of profit to the optimal one. Section 3 contains a number of Theorems, the main content of which is the statement about the existence of economic equilibrium under fairly general assumptions about the structure of supply and demand. Section 4 contains theorems on the existence of economic equilibrium under the condition of arbitrary assumptions about the structure of supply and demand and under the condition that each consumer has a positive income.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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