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On a Class of Semigroup Graphs

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Abstract

Let $G=\Gamma(S)$ be a semigroup graph, *i.e.*, a zero-divisor graph of a semigroup S with zero element 0. For any adjacent vertices x, y in G, denote $C(x,y)=\{z\in V(G) \mid N(z)=\{x,y\}\}$. Assume that in G there exist two adjacent vertices x, y, a vertex $s\in C(x,y)$ and a vertex z such that $d(s,z)=3$. This paper studies algebraic properties of S with such graphs $G=\Gamma(S)$, giving some sub-semigroups and ideals of S. It constructs some classes of such semigroup graphs and classifies all semigroup graphs with the property in two cases.

Keywords

Zero-Divisor Semigroup, Sub-Semigroup, Zero-Divisor Graph

1. Introduction

Throughout, G is a simple and connected graph. For a vertex x of G, the neighborhood of x is denoted as $N(x)$, which is the set of all vertices adjacent to x. Denote also $\overline{N(x)}=N(x)\cup\{x\}$. The cardinality of $N(x)$ is denoted by *deg*(x). The vertex x is called an *end vertex* if $deg(x)=1$, and an *isolated vertex* if $deg(x)=0$ ([1]). Throughout, S is a commutative semigroup with 0. Recall that for a commutative semigroup (or a commutative ring) S with 0, the zero-divisor graph $\Gamma(S)$ is an undirected graph whose vertices are the zero-divisors of $S\setminus\{0\}$, and with two vertices a,b adjacent in case $ab=0$ ([2]-[6]). If $G\cong\Gamma(S)$ for some semigroup S with zero element 0, then G is called a *semigroup graph*.

Some fundamental properties and possible algebraic structures of S and graphic structures of $\Gamma(S)$ were established in [3] [4] [5] among others. For example, it was proved that $\Gamma(S)$ is always connected, and the diameter of $\Gamma(S)$ is less than or equal to 3. If $\Gamma(S)$ contains a cycle, then its core, *i.e.*, the union of the cycles in $\Gamma(S)$, is a union of squares and triangles, and any vertex not in the core is an end vertex which is connected to the core by a single edge. In [7]-[11], the authors continued the study on the sub-semigroup structure and ideal structure of semigroups. Therefore, studying the interplay between the algebraic structures of S and the graph theoretic structures of $G=\Gamma(S)$ is still a fun problem.

For any adjacent vertices a,b in $V(G)$, denote $C(a,b)=\{x\in V(G) \mid N(x)=\{a,b\}\}$ and let T_a denote the set of all end vertices adjacent to a. In [12], we discuss the properties of $\Gamma(S)$ satisfies condition (K_p), here we assume $p=3$ and consider the following Δ assumed on $G=\Gamma(S)$

(Δ) There exist in G two adjacent vertices a,b, a vertex $s\in C(a,b)$ and a vertex z such that $d(s,z)=3$.

In this paper, we study algebraic properties of semigroup S and the graphic structures of $\Gamma(S)$ such that the condition (Δ) holds for $\Gamma(S)$. (We can further assume that triangles and rectangles coexist in the core $K[\Gamma(S)]$.) In particular, it is proved that $S\setminus[C(a,b)\cup T_a\cup T_b]$ is an ideal of S. Under some additional conditions, it is proved that $S\setminus C(a,b)$ may be an ideal or a sub-semigroup of S (Theorem 2.4). We also use Theorem 2.4 to construct some classes of semigroup graphs which satisfies the condition (Δ), and give a complete classification of such semigroup graphs in two cases.

We record a known result on finite semigroups to end this part (see e.g., [[13], Corollary 5.9 on page 25]). We also include a proof for the completeness.

Lemma 1.1. *Any finite nonempty semigroup S contains an idempotent element.*

Proof. Take any element x from S and consider the sequence $x,x^2,x^3,\cdots$. Since S is a finite set, there exist $m<n$ such that $x^m=x^n$. Let $r=n-m$, and take k such that $kr\geq m$. Then

$$x^{kr}=x^m\cdot x^{kr-m}=\left(x^r\cdot x^m\right)x^{kr-m}=x^r\cdot x^{kr}=x^r\left(x^r x^{kr}\right)=\cdots=\left(x^{kr}\right)^2.$$

□

2. Properties of *S*

Note that, for any $x\in S$, $Ann(x)=\{y \mid xy=0, y\in S\}$, thus for any vertex $x\in\Gamma(S)$, $N(x)\subseteq Ann(x)$, and $Ann(x)\subseteq \overline{N(x)}\cup\{0\}$,we have the following lemma.

Lemma 2.1. *Let S be a commutative semigroup with* 0, $\Gamma(S)$ *its zero-divisor graph. For any vertex $x\in\Gamma(S)$, if there exists a vertex $y\in\Gamma(S)$ such that $d(x,y)=3$, then $x^2\neq 0$ in S.*

Proof. As $d(x,y)=3$, there exist vertices $a,z\in\Gamma(S)$ such that $x-a-z-y$, $xz\neq 0$ and $ay\neq 0$. If $x^2=0$, then $x^2z=0$ and thus $xz\in Ann(x)$. Clearly $xz\in Ann(y)$. Then $xz\in Ann(x)\cap Ann(y)=\{0\}$, a contradiction. □

Part (1) of the following result is contained in [[7], Proposition 2.8]. Part (2) is contained in lemma 1.1 from [8]. And we prove it in a different way.

Proposition 2.2. *Let* $G=\Gamma(S)$ *be a zero-divisor graph of a semigroup S. For a vertex* $b\in V(G)$, *let* $T_b=\{x\in V(G)\,|\,xb=0,x\neq b,deg(x)=1\}$.

1) *If* $b^2\neq 0$, *then* $T_b\cup\{0\}$ *is a sub-semigroup of S.*

2) *If b is not an end vertex and* $T_b\neq\varnothing$, *then* $\{0,b\}$ *is an ideal of S.*

Proof. (1) We only need consider as $T_b\neq\varnothing$. If G contains no cycle, then G is either a two-star graph or a star graph by [[8], Theorem 1.3]. If G is a star graph, then $T_b=S\setminus\{b,0\}$. For all $x,y\in T_b$, we must have $xy\neq b$, since otherwise $0=xyb=b^2\neq 0$, a contradiction. This shows that $T_b\cup\{0\}$ is a sub-semigroup of S when G is a star graph. If G is a two-star graph or a graph with cycles, then $B\neq\varnothing$ where $B=\{x\in V(G)\,|\,deg(x)\geq 2,xb=0\}$. For all $x\in T_b$, we have

$$x^2\in Ann(b)=\{0\}\cup T_b\cup B$$

If $x^2\in B$, denote $x^2=v$. Then there exists $z\in S\setminus\{b\}$ such that $zv=0$. Since $x^2z=vz=0$, we have $xz\in Ann(x)=\{0,x,b\}$. Clearly, $xz\neq 0$. If $xz=x$, then $v=x^2=x^2z=vz=0$, a contradiction. If $xz=b$, then $0=xzb=b^2\neq 0$, another contradiction. So we must have $x^2\in T_b\cup\{0\}$. If $|T_b|\geq 2$, then exists a vertex $y\in T_b$ such that $x\neq y$. If $xy\in B$, denote $xy=v$. Then there exists $z\in S\setminus\{b\}$ such that $zv=0$. As $xyz=0$, we have $xz\in Ann(y)=\{0,y,b\}$ and $yz\in Ann(x)=\{0,x,b\}$, and thus $xz=y$ and $yz=x$. Then $x^2=xyz=vz=0$. On the other hand, $xy=x^2z=0$, a contradiction. So $xy\in T_b\cup\{0\}$, and hence $T_b\cup\{0\}\leq S$.

(2) Since $T_b\neq\varnothing$, there exists $x\in T_b$ such that $by\in Ann(x)=\{0,b,x\}$ for all $y\in S$. By assumption, b is not an end vertex and thus there exists $z\in S\setminus\{x\}$ such that $bz=0$. Then $by\neq x$ since otherwise, $by=x$ and it implies $0=bzy=zx\neq 0$, a contradiction. This completes the proof. □

Remark 2.3. *In Proposition* 2.2(1), *the conclusion can not hold if* $b^2=0$.

For a vertex v of a graph G, if v is not an end vertex and there is no end vertex adjacent to v, then v is said to be an *internal vertex*. We now prove the main result of this section.

Theorem 2.4. *Let* $G=\Gamma(S)$ *be a semigroup graph satisfying condition* (Δ). *Then* $\{0,a,b\}$ *is an ideal of S, and* $S\setminus[C(a,b)\cup T_a\cup T_b]$ *is an ideal of S. Furthermore,*

1) *If both a and b are internal vertices, then* $S\setminus C(a,b)$ *is an ideal of S.*

2) *If a is an internal vertex, while b is not an internal vertex and* $b^2\neq 0$, *then* $S\setminus C(a,b)$ *is a sub-semigroup of S.*

Proof. Fix some $s\in C(a,b)$ and let $B=\{x\,|\,x\in S,x\notin C(a,b),d(s,x)=2\}$, $L=\{y\,|\,y\in S,d(s,y)=3\}$. By assumption $L\neq\varnothing$, $C(a,b)\neq\varnothing$, and $T_a\cup T_b\subset B$. Notice that there is no end vertex in $B\setminus(T_a\cup T_b)$. By [[3], Theorem 2.3] or by [[5], Theorem 1(2)], $S=\{0,a,b\}\cup C(a,b)\cup B\cup L$ and it is a disjoint union of four nonempty subsets. By Lemma 2.1. we have $c^2\neq 0$, $\forall c\in C(a,b)$, and hence $Ann(c)=\{0,a,b\}$. Clearly, $a^2\in Ann(c)$, $b^2\in\{0,a,b\}$ and

$$\{0,a,b\}(B\cup L)\subseteq Ann(c)=\{0,a,b\}.$$

This shows that $\{0,a,b\}$ is an ideal of *S*.

For any *y* in *L*, there exists a vertex $x\in B$ such that $xy=0$. Then $yS\subseteq Ann(x)$ while $C(a,b)\cap Ann(x)=\varnothing$. Hence $LS\cap C(a,b)=\varnothing$. Furthermore, for any $s\in S$, $ys\in\{0,a,b\}\cup L\cup B$. If $ys\in\{0,a,b\}\cup B$, then it is clear that $ys\notin T_a\cup T_b$ whether $sy=x$ or not. Thus $LS\cap(T_a\cup T_b)=\varnothing$, and hence

$$(C(a,b)\cup T_a\cup T_b)\cap LS=\varnothing.$$

For any vertex x_1 in $B\setminus(T_a\cup T_b)$, $x_1\notin C(a,b)$ and it has degree greater than one. Hence for any $x_1\in B\setminus(T_a\cup T_b)$ and any $x_2\in S$, there exists a vertex $u\in B\cup L$ such that $x_1u=0$. Then $x_1x_2\in Ann(u)$ and it implies $x_1x_2\notin C(a,b)$. Thus $\left[(B\setminus(T_a\cup T_b))S\right]\cap C(a,b)=\varnothing$. Finally, by [[5], Theorem 4], the core of *G* together with 0 forms an ideal of *S*. Thus these arguments show that $S\setminus\left[C(a,b)\cup T_a\cup T_b\right]$ is an ideal of *S*.

1) If both *a* and *b* are internal vertices, then $S\setminus C(a,b)=S\setminus\left[C(a,b)\cup T_a\cup T_b\right]$. In this case, $S\setminus C(a,b)$ is clearly an ideal of *S*.

2) Now assume that *b* is not an internal vertex, and $b^2\neq 0$. Again let T_b be the set of end vertices adjacent to *b*. By the above discussion, we already have $\left(\left[\{0,a,b\}\cup L\cup(B\setminus T_b)\right]S\right)\cap C(a,b)=\varnothing$. Since $b^2\neq 0$, we have $T_b^2\leq T_b\cup\{0\}$ by Theorem 2.2(1). These facts show that $S\setminus C(a,b)$ is a sub-semigroup of *S*, and it completes the proof. □

Remarks 2.5. *In Theorem* 2.4, *if there is no* $z\in V(G)$ *such that* $d(s,z)=3$, *then the theorem may not hold. An example is contained in Example* 3.1.

3. Some Examples and Complete Classifications of the Graphs in Two Cases

In this section, we use Theorem 2.4 to study the correspondence of zero-divisor semigroups and several classes of graphs satisfying the four necessary conditions of [[5], Theorem 1] as well as the general assumption of Theorem 2.4.

Example 3.1. *Consider the graph G in* **Figure 1**, *where both U and V consist of end vertices. We claim that each graph in* **Figure 1** *is a semigroup graph.*

In fact, first notice that $d(y_i,V)=3$, $C(a,b)=\{y_1,\cdots,y_m\}$, and $C(a,d)=\{x_1,\cdots,x_n\}$. By Theorem 2.4, if *G* has a corresponding semigroup $S=V(G)\cup\{0\}$, then the subset $S\setminus(\{y_1,\cdots,y_m\}\cup U)$ must be an ideal of *S*. If further $a^2\neq 0$, then $S\setminus\{y_1,\cdots,y_m\}$ is a sub-semigroup of *S*. Also by [[7], Theorem 2.1], $S\setminus(\{y_1,\cdots,y_m\}\cup U\cup V)$ is a sub-semigroup of $S\setminus(\{y_1,\cdots,y_m\}\cup U)$, and thus a sub-semigroup of *S*.

For $m=2,n=2$, $U=\{u\}$ and $V=\{v,\overline{v}\}$, it is not very hard to construct a semigroup *T* such that $\Gamma(T)=G-\{y_1,y_2\}$ following the way mentioned above.

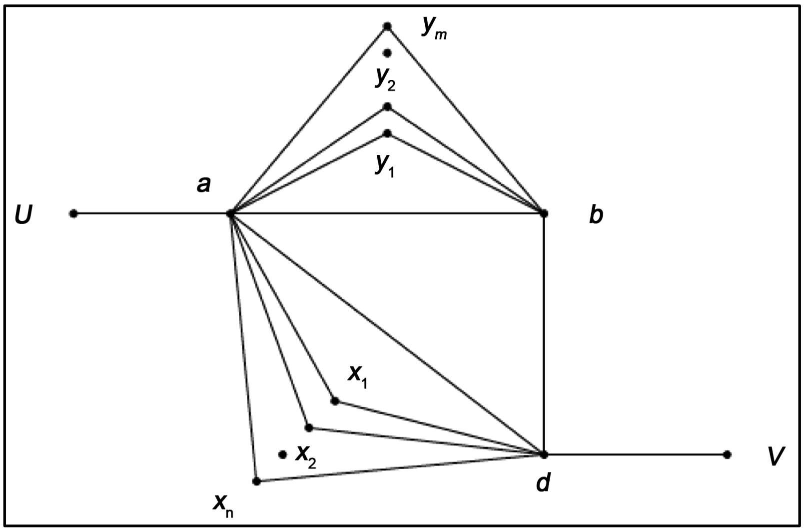

Figure 1. A class of semigroup graph with both U and V consist of end vertices.

Then after a rather complicated calculation, we succeed in adding two vertices y_1, y_2 to this table such that $\Gamma(S) = G$. The multiplication on S is listed in Table 1 and the detailed verification for the associativity is omitted here:

Notice that $S \setminus \{x_1, x_2\}$ is not a sub-semigroup of S since $uv = x_1$. Notice also that $S \setminus (U \cup \{x_1, x_2, \cdots, x_n\})$ is a sub-semigroup of S.

We remark that the construction in Table 1 can be routinely extended for all $n \geq 1$, $m \geq 1$, $|U| \geq 0$ and $|V| \geq 0$, where each of $m, n, |U|, |V|$ could be a finite or an infinite cardinal number. In other words, each graph in Figure 1 has a corresponding semigroup for any finite or infinite $n \geq 1$, $m \geq 1$, $|U| \geq 0$ and $|V| \geq 0$.

Remark 3.2. *Consider the graph G in Figure 1 and assume that* $n \geq 1$, $m \geq 1$, $|U| \geq 0$, $|V| \geq 1$.

1) *If we add an end vertex w which is adjacent to b, then the resulting graph* $\overline{G}$ *has no corresponding zero-divisor semigroup, even if* $U = \varnothing$.

2) *If we add a vertex w such that* $N(w) = \{b, d\}$, *then the resulting graph H has no corresponding zero-divisor semigroup, even if* $U = \varnothing$.

Proof. (1) Assume $v \in V$. We only need consider the case when $U = \varnothing$. Suppose that $\overline{G}$ is the zero-divisor graph of a semigroup S with $V[\Gamma(S)] = V(\overline{G})$. By Proposition 2.2(2), we have $bx_1 = bv = b$ and $dy_1 = dw = d$. Clearly, $aw, av \in Ann(x_1) \cap Ann(y_1) = \{a, 0\}$, and thus $aw = a$ and $av = a$. As $awv = av = a$, we have $wv \in [Ann(b) \cap Ann(d)] \setminus Ann(a)$. That means $wv = a$ and $a^2 \neq 0$. We have $y_1 wv = y_1 a = 0$ and $x_1 wv = x_1 a = 0$. Thus $y_1 w = x_1 w = d$ and $y_1 v = x_1 v = b$ by Lemma 2.1. Consider $x_1 y_1 v$. We have $b = x_1 b = x_1 (y_1 v) = y_1 (x_1 v) = y_1 b = 0$, a contradiction. The contradiction shows that $\overline{G}$ has no corresponding semigroup.

(2) Assume $v \in V$. We only need consider the case $U = \varnothing$. Suppose that H is the zero-divisor graph of a semigroup S. We have
$bS \subseteq Ann(y) \cap Ann(w) \subseteq \{0, b\}$. Clearly,

Table 1. The associative multiplication table of S in Example 3.1.

$\cdot$	a	d	b	x_1	x_2	y_1	y_2	u	v	$\overline{v}$
a	a	0	0	0	0	0	0	0	a	a
d	0	d	0	0	0	d	d	d	0	0
b	0	0	0	b	b	0	0	b	b	b
x_1	0	0	b	x_1	x_2	b	b	x_1	x_1	x_1
x_2	0	0	b	x_2	x_2	b	b	x_2	x_2	x_2
y_1	0	d	0	b	b	d	d	y_1	b	b
y_2	0	d	0	b	b	d	d	y_2	b	b
u	0	d	b	x_1	x_2	y_1	y_2	u	x_1	x_1
v	a	0	b	x_1	x_2	b	b	x_1	v	v
$\overline{v}$	a	0	b	x_1	x_2	b	b	x_1	v	v

$wv \in [Ann(d) \cap Ann(b)] \setminus Ann(a) \subseteq \{a, w\}$ since $wva = wa = a$. If $wv = a$, then we have $wvx_1 = ax_1 = 0$ and $wvy_1 = ay_1 = 0$, which means $wx_1, wy_1 \in Ann(v) \subseteq \{0, v, d\}$. As $wx_1, wy_1 \in Ann(a) \setminus \{0\}$, we have $wx_1 = wy_1 = d$. Then $0 = dx_1 = wy_1x_1 = y_1d = d$, a contradiction. Now assume $wv = w$ and consider wx_1. $wx_1 \in Ann(a) \cap Ann(d) \cap Ann(b) \subseteq \{0, a, b, d\}$. We claim $wx_1 \neq d$ since otherwise, $d = wx_1 = wv \cdot x_1 = wx_1 \cdot v = dv = 0$, a contradiction. In a similar way we prove $wy_1 \in \{a, b\}$. Moreover, $wy_1 \cdot x_1 = y_1 \cdot wx_1 = 0$ whether $wx_1 = a$ or $wx_1 = b$. Thus $wy_1 = a$. As $x_1^2 w = y_1^2 w = x_1 y_1 w = 0$, we have $x_1y_1, x_1^2, y_1^2 \in Ann(a) \cap Ann(w) \subseteq \{b, d, 0\}$, but $x_1y_1 \neq 0$. Now consider x_1y_1. We conclude $x_1y_1 = d$ since otherwise, $x_1y_1 = b$ and it implies $b = bx_1 = x_1y_1x_1 = x_1^2 y_1 \neq b$, a contradiction. Finally, $x_1y_1 = d$ implies $d = dy_1 = y_1x_1y_1 = x_1y_1^2 \in \{0, b\}$, a contradiction. This completes the proof.

□

Now come back to the structure of semigroup graphs G satisfying the main assumption in Theorem 2.4. We use notations used in its proof. The vertex set of the graph was decomposed into four mutually disjoint nonempty parts, *i.e.*, $V(G) = \{a, b\} \cup C(a, b) \cup B \cup L$, where after taking a c in $C(a, b)$

$$B = \{v \in V(G) \mid v \notin C(a, b), d(c, v) = 2\}, L = \{v \in V(G) \mid d(c, v) = 3\}.$$

(For example, for the graph G in Figure 1, $C(a, b) = \{y_j\}$, $B = U \cup \{d\} \cup \{x_i\}$, $L = V$. In particular, L consists of end vertices.) By [[5], Theorem 1(4)], for each pair x, y of nonadjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subseteq \overline{N(z)}$. Then we have the following observations:

(1) *No two vertices in L are adjacent in G.* Thus a vertex of L is either an end vertex or is adjacent to at least two vertices in B. In particular, the subgraph induced on L is a completely discrete graph.

(2) A vertex in B is adjacent to either a or b. *If a vertex k in B is adjacent to a vertex l in L, then k is adjacent to both a and b.* Thus B consists of four parts: end vertices in T_a that are adjacent to a, end vertices in T_b that are adjacent to b, ver-

tices in B_2 that are adjacent to both *a* and *b*, and vertices in B_1 that are adjacent to one of a,b and at the same time adjacent to another vertex in *B*. By Example 3.1, the structure of the induced subgraph on $B_1 \cup B_2$ seems to be complicated. In the following, we will give a complete classification of the semigroup graphs G with $|B_1 \cup B_2| \le 2$.

First, consider the case $|B_1 \cup B_2| = 1$.

Theorem 3.3. *Let G be a graph satisfying condition* (Δ). *Assume further that* $|B \setminus (T_a \cup T_b)| = 1$. *Then G is a semigroup graph if and only if the following conditions hold*:

1) $1 \le |C(a,b)| \le \infty$, $1 \le |W| \le \infty$ *and W consists of end vertices, where* $W = \{s \in V(G) \mid d(c_1, s) = 3\}$.

2) *either* $T_a = \varnothing$ *or* $T_b = \varnothing$. (*see* Figure 2 *with* $V = \varnothing$.)

Proof. As $|B \setminus (T_a \cup T_b)| = 1$, $B \setminus (T_a \cup T_b) = B_2$. By the previous observations, we need only prove the following two facts.

1) If $|T_a| \ge 0$ and $T_b = \varnothing$, then G is a subgraph of Figure 1 with $C(a,d) = \varnothing$. (see also Figure 2 with $V = \varnothing$.) We claim that G is a semigroup graph. In fact, if $U = \varnothing$, delete the three rows and the three columns involving x_1, x_2 and u in Table 1 to obtain an associative multiplication on $S_1 = S \setminus (U \cup \{x_1, x_2, \cdots, x_n\})$. Clearly, $\Gamma(S_1) = G$ for $|C(a,b)| = 2 = |V|$, $|C(a,d)| = 0 = |U|$ in Figure 1. Also, the table can be extended for any finite or infinite $|C(a,b)| \ge 1$ and $|V| \ge 0$ while $|U| = 0$. If $|U| > 0$, then we work out a corresponding associative multiplication table listed in Table 2, for $C(a,b) = \{y_1, y_2\}$, $U = \{u_1, u_2\}$, $W = \{v_1, v_2\}$ in Figure 2.

Clearly, the table can be extended for all finite or infinite $|C(a,b)| \ge 1$, $|U| \ge 1$ and $|V| \ge 1$. This completes the proof. □

(2) If both $|T_a| > 0$ and $|T_b| > 0$, then we conclude that G is not a semigroup graph.

In fact, in this case, G is a graph in Figure 2, where $|W| \ge 1$, $|U| \ge 1$, $|V| \ge 1$. Assume $u \in U$, $v \in V$, $w \in W$ and $c \in C(a,b)$. We now proceed to prove that such a graph does not have a corresponding semigroup.

Suppose that G is the zero-divisor graph of a semigroup S with $V[\Gamma(S)] = V(G)$. By Proposition 2.2(2), we have $du = dv = dc_1 = d$. Then $uc_1 d = ud = d$, which implies $uc_1 \in [Ann(a) \cap Ann(b)] \setminus Ann(d) \subseteq \{c_i, d\}$.

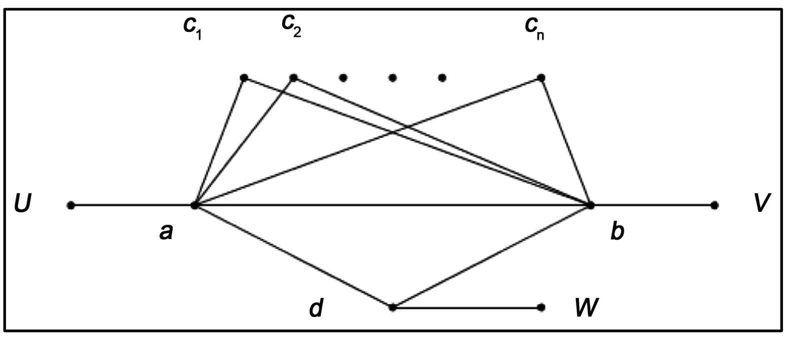

Figure 2. A class of graph satisfying condition (Δ) and $|B/(Ta \cup Tb)| = 1$.

Table 2. The associative multiplication table of S for $|U| > 0$.

$\cdot$	a	b	d	y_1	y_2	u_1	u_2	v_1	v_2
a	0	0	0	0	0	0	0	a	a
b	0	a	0	0	0	a	a	b	b
d	0	0	d	d	d	d	d	0	0
y_1	0	0	d	d	d	d	d	a	a
y_2	0	0	d	d	d	d	d	a	a
u_1	0	a	d	d	d	y_1	y_1	b	b
u_2	0	a	d	d	d	y_1	y_1	b	b
v_1	a	b	0	a	a	b	b	v_1	v_1
v_2	a	b	0	a	a	b	b	v_1	v_1

Assume $uc_1 = d$. Then $uc_1 w = dw = 0$, and thus $c_1 w = a$ by Lemma 2.1. As $c_1 wv = av = a$, we have $wv \in [Ann(d) \cap Ann(b)] \setminus Ann(c_1) \subseteq \{d\}$, thus $wv = d$. Then $a = wvc_1 = dc_1 = d$, a contradiction.

So $uc_1 = c_i$, and therefore $wuc_1 = wc_i \neq 0$. We have

$$wu \in [Ann(a) \cap Ann(d)] \setminus Ann(c_1) \subseteq \{d\},$$

and thus $wu = d$. Then $b = bw = buw = bd = 0$, a contradiction. This completes the proof. □

A natural question arising from Example 3.1 is if L only consists of end vertices. The following example shows this is not the case.

Example 3.4. *Consider the graph G in Figure 3, where* $C(a,b) = \{c_1, c_2, \cdots, c_m\}$, $L = \{y_1, y_2, \cdots, y_n\}$, $B = \{x_1, x_2\} \cup V$ ($m \geq 1$, $n \geq 1$, $|V| \geq 0$) *and V consists of end vertices adjacent to b. Notice that each of* m, n *and* $|V|$ *could be finite or infinite. We conclude that each graph in Figure 3 has a corresponding zero-divisor semigroup.*

Proof. We need only work out a corresponding associative multiplication table for $|V| = m = n = 2$. We use Theorem 2.4 and list the associative multiplication in Table 3. Clearly, the table can be extended for all finite or infinite $m, n \geq 1$, and $|V| \geq 0$.

This completes the proof.

□

We have three remarks to Example 3.4.

(1) *Let* $n \geq 1, m \geq 1$. *If we add to G in Figure 3 an end vertex u such that* $au = 0$, *then the resulting graph* $\bar{G}$ *has no corresponding zero-divisor* semigroup.

Proof. (1) Suppose to the contrary that $\bar{G}$ is the zero-divisor graph of a semigroup P with $V[\Gamma(P)] = V(\bar{G})$. By Proposition 2.2(2), we have $a^2 \in \{0, a\}$ and $b^2 \in \{0, b\}$. First, we have $v_1 y_1 \in Ann(b) \cap Ann(x_1) \cap Ann(x_2) = \{a, b, 0\}$ and similarly, $uy_1, c_1 y_1 \in \{a, b\}$. Then $v_1 y_1 = a$ and $a^2 = a$ since $av_1 y_1 = ay_1 = a$. On the other hand, $auy_1 = 0$ and it implies $uy_1 = b$. Similarly,

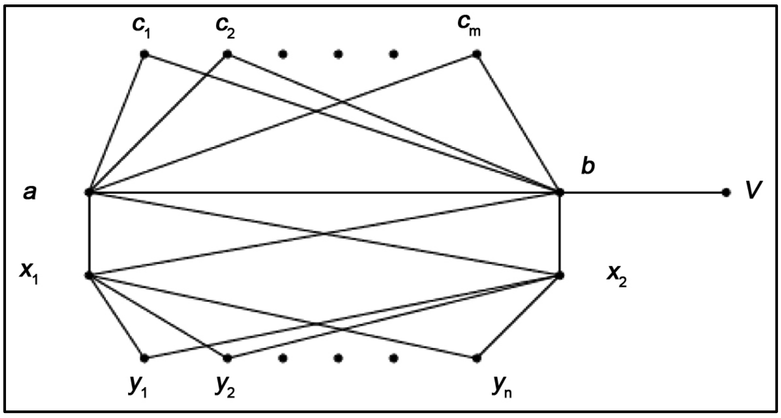

Figure 3. A class of semigroup graph with $B_2 = \{x_1, x_2\}$.

Table 3. The associative multiplication table of S in Example 3.4.

$\cdot$	a	b	c_1	c_2	v_1	v_2	x_1	x_2	y_1	y_2
a	a	0	0	0	a	a	0	0	a	a
b	0	0	0	0	0	0	0	0	b	b
c_1	0	0	x_1	x_1	x_1	x_1	x_1	x_1	b	b
c_2	0	0	x_1	x_1	x_1	x_1	x_1	x_1	b	b
v_1	a	0	x_1	x_1	x_1	v_1	x_1	x_1	a	a
v_2	a	0	x_1	x_1	x_1	v_1	x_1	x_1	a	a
x_1	0	0	x_1	x_1	x_1	x_1	x_1	x_1	0	0
x_2	0	0	x_1	x_1	x_1	x_1	x_1	x_2	0	0
y_1	a	b	b	b	a	a	0	0	y_1	y_1
y_2	a	b	b	b	a	a	0	0	y_1	y_1

we have $c_1y_1 = b$. Consider c_1uy_1. We have $b = ub = u(c_1y_1) = c_1(uy_1) = c_1b = 0$, a contradiction. This completes the proof. □

(2) *Let* $n \geq 1, m \geq 1$. *If we add to* G *in* **Figure 3** *an end vertex* y *such that* $yx_1 = 0$, *then the resulting graph* $\overline{G}$ *has no corresponding zero-divisor semigroup, whether or not* $T_b = \varnothing$.

Proof. (2) Assume $\{y_1, y\} \subseteq L$, where y is an end vertex adjacent to x_1. Suppose to the contrary that $\overline{G}$ is the zero-divisor graph of a semigroup P with $V[\Gamma(P)] = V(\overline{G})$. First, $x_2y \in Ann(a) \cap Ann(x_1) \cap Ann(y_1) = \{x_1, 0\}$. Thus $x_2y = x_1$, and hence $x_1^2 = 0$. By Proposition 2.2(2), we have $c_1x_1 = x_1$ and therefore, $c_1^2x_1 = x_1$. Thus $c_1^2 \in \{c_i, x_2 \mid i\}$. We have $c_1^2y_1 = 0$ since $c_1y_1 \in Ann(a) \cap Ann(x_1) \cap Ann(x_2) = \{a, b, 0\}$. Since $c_1^2 = x_2$, $c_1y \in \{a, b, x_1\}$ and $c_1^2y = x_2y = x_1$, it follows that $c_1y = x_1$. Finally, $c_1yy_1 = x_1y_1 = 0$ and by Lemma 2.1, we have $c_1y_1 = x_1$, contradicting $c_1y_1 \in \{a, b\}$. This completes the proof.

□

(3) *Let* $n \geq 1, m \geq 1$ *and assume* $V = \varnothing$ *in* **Figure 3**. *If further we add to* G *an*

edge connecting x_1 *and* x_2, *then the resulting graph* $\overline{G}$ *has no corresponding zero-divisor semigroup.*

Proof. Suppose to the contrary that $\overline{G}$ is the zero-divisor graph of a semigroup P with $V[\Gamma(P)] = V(\overline{G})$. By Lemma 2.1, we have $c_1x_1 \in Ann(y_1) = \{x_1, x_2, 0\}$ and similarly, $c_1x_2 \in \{x_1, x_2\}$. Then we have $c_1^2x_1 \neq 0$ and $c_1^2x_2 \neq 0$, which means $c_1^2 \in [Ann(a) \cap Ann(b)] \setminus [Ann(x_1) \cup Ann(x_2)] = \{c_i \mid i = 1, 2, \cdots, m\}$ since $c_1^2 \neq 0$ by Lemma 2.1. Similarly, we have $y_1^2 \in \{y_i \mid i = 1, 2, \cdots, n\}$. Clearly, we have $c_1y_1 \in Ann(a) \cap Ann(x_1) \subseteq \{a, b, x_1, x_2, 0\}$. Then as $c_1^2y_1 = c_iy_1 \neq 0$ for some $i \in \{1, 2, \cdots, m\}$, we have $c_1y_1 \in \{x_1, x_2\}$. Finally, $0 = (c_1y_1)y_1 = c_1y_1^2 = c_1y_i \neq 0$ (for some $i \in \{1, 2, \cdots, n\}$), a contradiction. This completes the proof. □

Combining the above results, we now classify all semigroup graphs satisfying the main assumption of Theorem 2.4 with $|B_1 \cup B_2| = 2$:

Theorem 3.5. *Let G be a graph satisfying condition* (Δ). *Assume further* $|B \setminus (T_a \cup T_b)| = 2$.

(1) *If* $B_2 = B \setminus (T_a \cup T_b)$, *then G is a semigroup graph if and only if G is a graph in* Figure 3, *where* $1 \leq m \leq \infty$, $1 \leq n \leq \infty$ *and* $0 \leq |V| \leq \infty$.

(2) *If* $|B_2| = 1$, *then G is a semigroup graph if and only G is a graph in* Figure 1, *where* $n = 1$, $1 \leq m \leq \infty$ $0 \leq |V| \leq \infty$, $0 \leq |U| \leq \infty$.

Proof. (1) By Example 3.4, each graph in Figure 3 is a semigroup graph. Clearly, $B_2 = B \setminus (T_a \cup T_b)$ and it consists of two vertices. Conversely, the result follows from [[7], Theorem 2.1] and the three remarks after Example 3.4.

(2) If $|B_2| = 1$, then assume $B \setminus (T_a \cup T_b) = \{x_1, x_2\}$, where $a - x_2 - b$. In this case, $x_1 - x_2$ in *G*. If $x_1 - a$ in *G*, then there is no end vertex adjacent to x_1. In this subcase, *G* is a semigroup graph if and only if $T_b = \varnothing$ by Example 3.1 and Remark 3.2(1), the case of $|C(a,d)| = 1$. The other subcase is $x_1 - b$ in *G*, and it is the same with the above subcase. This completes the proof. □

It is natural to ask the following *question*: Can one give a complete classification of semigroup graphs $G = \Gamma(S)$ with $|B_1 \cup B_2| = n$ for any $n \geq 3$? At present, it seems to be a rather difficult question.

Add two end vertices to two vertices of the complete graph K_n to obtain a new graph, and denote the new graph as $K_n + 2$. By [[14], Theorem 2.1], $K_n + 2$ has a unique zero-divisor semigroup *S* such that $\Gamma(S) \cong K_n + 2$ for each $n \geq 4$. Having Theorem 2.4 in mind, it is natural to consider graphs obtained by adding some caps to $K_n + 2$.

Example 3.6. *Consider the graph G in* Figure 4. *The subgraph* G_1 *induced on the vertex subset* $S^* = \{a, b, x_1, x_2, y_1, y_2\}$ *is the graph* $K_4 + 2$, *i.e.,* K_4 *together with two end vertices* y_1, y_2. *Then* G_1 *has a unique corresponding zero-divisor semigroup* $S = S^* \cup \{0\}$ *by* [[15], Theorem 2.1]. *We can work out the corresponding associative multiplication table, and list it in* Table 4.

(1) *If we add to* G_1 *a vertex c such that* $N(c) = \{a, b\}$, *then the resulting graph* H_1 *has no corresponding zero-divisor semigroup.*

(2) *If we add to* G_1 *a vertex d such that* $N(d) = \{a, x_1\}$, *then the resulting*

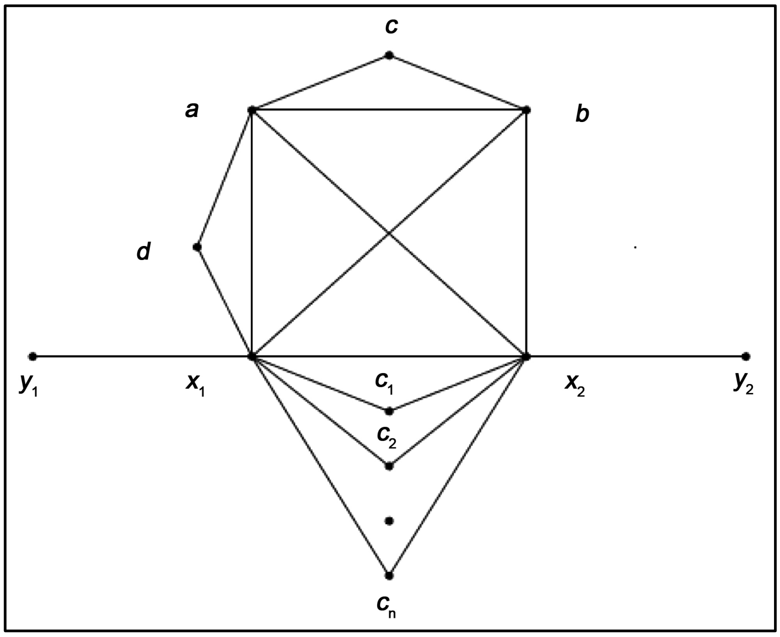

Figure 4. "caps" added to $K_4 + 2$.

Table 4. The associative multiplication table of $K_4 + 2$.

·	a	b	x_1	x_2	y_1	y_2
a	a	0	0	0	a	a
b	0	0	0	0	x_2	x_1
x_1	0	0	0	0	0	x_1
x_2	0	0	0	0	x_2	0
y_1	a	x_2	0	x_2	y_1	a
y_2	a	x_1	x_1	0	a	y_2

graph H_2 has no corresponding zero-divisor semigroup.

(3) *If we add to G_1 vertices c_i ($i \in I$) such that $N(c_i) = \{x_1, x_2\}$, then the resulting graph H has corresponding zero-divisor semigroups, where I could be any finite or infinite index set.*

In each of the above three cases, we say that *a cap is added to the subgraph* $K_4 + 2$.

Proof. (1) Suppose that H_1 is the zero-divisor graph of a semigroup S_1 with $V[\Gamma(S_1)] = V(H_1)$. Then by Theorem 2.4, S is an ideal of $S_1 = S \cup \{c\}$. Thus we only need check the associative multiplication of S_1 based on the table of S already given in **Table 4**. First, we have $cx_2 = x_2$ by Proposition 2.2(2). Consider $y_1 bc$. Clearly, $0 = 0y_1 = (cb)y_1 = c(by_1) = cx_2 = x_2$, a contradiction. This completes the proof.

(2) Suppose that H_2 is the zero-divisor graph of a semigroup $S_2 = S \cup \{d\}$ with $V[\Gamma(S_2)] = V(H_2)$. If $x_1^2 \neq 0$, then by Theorem 2.4(2), S is a sub-semigroup of S_2. Then $\Gamma(S) = K_4 + 2$, and it implies $x_1^2 = 0$ by **Table 4**, a contradiction.

Table 5. The associative multiplication table of $K_4 + 2$ with some caps on x_1, x_2.

$\cdot$	a	b	x_1	x_2	y_1	y_2	c_1	c_2
a	a	0	0	0	a	a	a	a
b	0	b	0	0	b	b	b	b
x_1	0	0	x_1	0	0	x_1	0	0
x_2	0	0	0	x_2	x_2	0	0	0
y_1	a	b	0	x_2	y_1	c_1	c_1	c_1
y_2	a	b	x_1	0	c_1	y_2	c_1	c_1
c_1	a	b	0	0	c_1	c_1	c_1	c_1
c_2	a	b	0	0	c_1	c_1	c_1	c_1

In the following we assume $x_1^2 = 0$.

By Lemma 2.1, we have $d^2 \neq 0$, and thus $ay_1, ay_2 \in Ann(d) = \{a, x_1, 0\}$. Clearly $ay_1 \neq 0$ and we can have $ay_2 = a$. (Otherwise, $ay_2 = x_1$ and we have $0 = x_1 y_1 = ay_2 y_1 = (ay_1) y_2 \neq 0$, a contradiction.) Then $ay_1 y_2 \neq 0$, and thus $y_1 y_2 \in [Ann(x_1) \cap Ann(x_2)] \setminus Ann(a)$. It means $y_1 y_2 = a$ and $a^2 \neq 0$. Clearly $by_1 y_2 = 0$, and thus $by_1 = x_2$, $by_2 = x_1$ by Lemma 2.1. Similarly, $cy_1 y_2 = ca = 0$ and thus $cy_1 = x_2$, $cy_2 = x_1$. Finally, consider bcy_1. We have $0 = bx_2 = b(cy_1) = c(by_1) = cx_2 = x_2$, a contradiction. This completes the proof.

(3) Suppose that H is the subgraph of G in Figure 4 induced on the vertex set $S^* \cup \{c_i \mid i \in I\}$. Assume that H is the zero-divisor graph of a semigroup P with $V[\Gamma(P)] = V(H)$. Clearly, it dose not satisfy the condition of Theorem 2.4. For $|I| = 2$, we work out an associative multiplication table and list it in Table 5:

The table can be easily extended for any finite or infinite index set I. □

We remark that in Example 3.6, replace K_4 by K_n for any $n \geq 5$, the results still hold. There exists no difficulty to generalize the proofs to the general cases. Thus we have proved the following general result.

Theorem 3.7. *Assume $n \geq 4$ and let $G = K_n + 2$ be the complete graph K_n together with two end vertices. Add some (finite or infinite) caps to the subgraph K_n to obtain a new graph H such that G is a subgraph of H. Then H is a semigroup graph if and only if each of the gluing vertices is adjacent to an end vertex in G.*

Fund

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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