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A Note on the Inverse Connected *p*-Median Problem on Block Graphs

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Abstract

Recently, the inverse connected *p*-median problem on block graphs $G(V,E,w)$ under various cost functions, say rectilinear norm, Chebyshev norm, and bottleneck Hamming distance. Their contributions include finding a necessary and sufficient condition for the connected *p*-median problem on block graphs, developing algorithms and showing that these problems can be solved in $O(n\log n)$ time, where n is the number of vertices in the underlying block graph. Using similar technique, we show that some results are incorrect by a counter-example. Then we redefine some notations, reprove Theorem 1 and redescribe Theorem 2, Theorem 3 and Theorem 4.

Keywords

Location Theory, Block Graphs, Inverse Optimization, Connected *p*-Median

1. Introduction and Problem Formulation

In recent years, there has been an increasing interest in *connected p-center and p-median problems* where the subgraph induced by the selected set is connected. Yen [1] studied the connected *p*-center problem on block graphs. Bai *et al.* [2] considered the connected *p*-median problem on cactus graphs and showed that the problem can be solved in polynomial time. In this paper we consider the inverse connected *p*-median problem on block graphs. We shall follow the notations and terminologies given in Kang *et al.* [3], Nguyen and Hung [4]. Let $G=(V,E,w,l)$ be a finite, connected, undirected graph with vertex set V of *order* $n=|V|$ and edge set E of size $m=|E|$, where each vertex $v\in V$ is associated with a nonnegative weight $w(v)$ and each edge $(v_i,v_j)\in E$ is associated with a certain cost or length $l(v_i,v_j)$. For convenience, we denote

$G=(V,E,w)$ as the graph that $l(e)=1$ for all edge $e \in E$.

For any two vertices of $u,v \in G$, a *path* from u to v is vertex-edge alternative sequence: $u=x_1,e_1,x_2,e_2,\cdots,x_s,e_s,x_{s+1}=v$ such that the x_i are all distinct and $e_i=x_i x_{i+1}$ for $i=1,2,\cdots,s$. The number of edges of a path is its *length*. Let $d(u,v)=\sum_{i=1}^{s} l(e_i)$ be the length of a shortest path in G between u and v, called the *distance* of two vertices u and v. Furthermore, each edge $e=(u,v)$ can be considered as a continuous interval, where a point ρ in e is identified by a parameter $\lambda \in [0,1]$ such that $d(u,\rho)=\lambda l(e)$ and $d(v,\rho)=(1-\rho)l(e)$. We can also define the distance between two points similarly to the distance between two vertices. The classical p-median location model is to find the set of p points on G, say $S=\{\rho_1,\rho_2,\cdots,\rho_p\}$, so as to minimize the median function

$$F(S)=\sum_{v\in V} w(v)d(v,S),$$

where $d(v,S)=\min_{j=1}^{p} d(v,\rho_j)$. By the dominating property of vertex set, we know that there exists an optimal solution to the p-median problem that is exactly the subset of V. Hence, we focus on the set $S=\{v_1,v_2,\cdots,v_p\} \subset V$ hereafter. For the sake of modern location model, the new facilities are required to be connected to a network for communication/security reasons. This fact motivates the so-called connected p-median problem on G, where the set S is the connected set on the underlying graph. For two vertices set $S=\{v_1,v_2,\cdots,v_p\}$ and $S'=\{u_1,u_2,\cdots,u_q\}$, we define $d(S,S')=\min\{d(v_i,u_j) \mid v_i \in S, u_j \in S'\}$.

Given a graph G, a vertex u is called a *cut vertex* of G if $\kappa(G-\{u\})>\kappa(G)$, where $\kappa(G)$ denotes the number of components of G. A connected subgraph H of G is called a *block* of G if H is maximal and it contains no cut vertices. A graph G is a *block graph* if all blocks of G are cliques and any two distinct blocks B_1 and B_2 have at most one common vertex, where a *clique* in a graph is a complete subgraph maximal under inclusion.

Given a block graph G and a set S_p of p connected vertices, we denote by $\mathcal{B}(S_p)$ the set of connected subgraphs induced by deleting all vertices in S_p and all edges in blocks containing at least one vertex of S_p. A vertex v is said to be in the border of S_p if there does not exist any pair v' and v'' in S_p such that v is an intermediate vertex of the shortest path connecting them. Resultantly, we denote $\mathcal{D}(S_p)$ the set of all vertices in the border of S_p. Furthermore, let $\mathcal{F}(S_p)$ be the set of vertices, which are adjacent to some vertices in $\mathcal{D}(S_p)$ and are not in S_p. Meanwhile, we define $\mathcal{F}(S_p)(v)$, for v in $\mathcal{D}(S_p)$, as the set of all vertices in $\mathcal{F}(S_p)$ that is adjacent to v. If a connected subgraph in $\mathcal{B}(S_p)$ contains a vertex $v \in \mathcal{F}(S_p)$, then this graph is defined as $\mathcal{B}(S_p)(v)$. For a vertex $v \in \mathcal{D}(S_p)$, let $\overline{\mathcal{B}}(S_p)(v)$ be the subgraph induced by $\{v\} \cup \bigcup_{v' \in \mathcal{F}(S_p)(v)} \mathcal{B}(S_p)(v')$.

Modifying vertex weights of a graph at minimum total cost so that a predetermined set of p connected vertices becomes a connected p-median on the perturbed graph. This problem is the so-called inverse connected p-median problem on graphs. Nguyen and Hung [4] consider the problem of a block graph with uniform edge lengths under various cost functions. To solve the problem, they first find an optimality criterion for a set that is a connected p-median. Based on this criterion, they can formulate the problem as a convex or quasi-convex univariate optimization problem. Finally, they develop combinatorial algorithms that solve the problems under the three cost functions in $O(n\log n)$ time, where n is the number of vertices in the underlying block graph. In the scope of their paper, the following cost functions are considered.

1) Rectilinear norm:

$$C(p_v, q_v) = \sum_{v\in V}(c_v p_v + c_v q_v).$$

2) Chebyshev norm:

$$C(p_v, q_v) = \max_{v\in V}\{c_v p_v, c_v q_v\}.$$

3) Bottleneck Hamming distance:

$$C(p_v, q_v) = \max_{v\in V}\{c_v H(p_v), c_v H(q_v)\},$$

where the Hamming distance $H(\cdot)$ is identified by $H(x)=0$ if $x=0$ and $H(x)=1$ otherwise. The following results are established in Nguyen and Hung [4].

Lemma 1. [4] *If there exists a vertex* $v\in\mathcal{D}(S_p)$ *and vertex* $u\in\mathcal{F}(S_p)$ *such that* $W(\mathcal{B}(S_p)(u)) > W(\overline{\mathcal{B}}(S_p)(v))$, *then* S_p *is not a connected p-median of G.*

Theorem 1. [4] (Optimality Criterion) *The set* S_p *is a connected p-median of the block graph G if and only if* $W(\mathcal{B}(S_p)(u)) \le W(\overline{\mathcal{B}}(S_p)(v))$ *for all* $v\in\mathcal{D}(S_p)$ *and* $u\in\mathcal{F}(S_p)$.

Theorem 2. [4] *The inverse connected p-median problem on a block graph can be solved in* $O(n\log n)$ *time.*

Theorem 3. [4] *The inverse connected p-median problem on block graphs under Chebyshev norm can be solved in* $O(n\log n)$ *time.*

Theorem 4. [4] *The inverse connected p-median problem under bottleneck Hamming distance can be solved in* $O(n\log n)$ *time.*

2. Counterexample

As we observe, Lemma 1 and Theorem 1 in Nguyen and Hung [4] are incorrect. In this note, we point out these incorrect results by a counterexample.

Counterexample 1. *Let us consider the block graph G in* **Figure 1**, *where the prespecified set of connected vertices is* $S_4=\{v_1,v_2,v_3,v_4\}$ *and the weights are labeled on the corresponding vertices.*

According to the symbols and definitions in Nguyen and Hung [4], one can deduce the following results. $\mathcal{D}(S_4)=\{v_1,v_2,v_4\}$, $\mathcal{F}(S_4)=\{v_5,v_6,v_9,v_{10},v_{11},v_{13}\}$. Moreover, the subgraph $\mathcal{B}(S_4)(v_{13})$ is induced by $\{v_{11},v_{12},v_{13},v_{14}\}$ and the subgraph $\overline{\mathcal{B}}(S_4)(v_4)$ is induced by $\{v_4,v_{11},v_{12},v_{13},v_{14}\}$. Based on the weights of vertices, one obtains the weights of subgraphs are calculated as in **Table 1**.

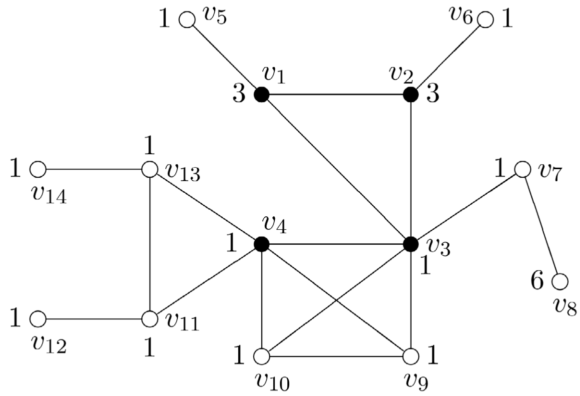

Figure 1. The block graph *G*.

From **Table 1**, one can see that $W\left(\mathcal{B}\left(S_p\right)\left(v_j\right)\right) \le W\left(\bar{\mathcal{B}}\left(S_p\right)\left(v_i\right)\right)$ for all $v_i \in \mathcal{D}\left(S_p\right)$ and $v_j \in \mathcal{F}\left(S_p\right)$. If Lemma 1 and Theorem 1 are correct, then the set S_4 is a connected *p*-median of the block graph. Let $S_4' = \{v_7, v_2, v_3, v_4\}$. Since $F\left(S_4'\right) = 20 < F\left(S_4\right) = 23$, the set S_4 is not a connected *p*-median of the block graph, this instance is indeed a counterexample for Lemma 1 and Theorem 1.

In addition, the proofs in Lemma 1 and Theorem 1 in Nguyen and Hung [4] are also incorrect. Let $S_4'' = \{v_{11}, v_2, v_3, v_4\}$. According to the symbols and definitions in Nguyen and Hung [4], one can deduce that the subgraph $\mathcal{B}\left(S_4\right)\left(v_{11}\right)$ is induced by $\{v_{11}, v_{12}, v_{13}, v_{14}\}$ and the subgraph $\bar{\mathcal{B}}\left(S_4\right)\left(v_1\right)$ is induced by $\{v_4, v_5\}$. Then

$$W\left(\mathcal{B}\left(S_4\right)\left(v_{11}\right)\right) = 4 \text{ and } W\left(\bar{\mathcal{B}}\left(S_4\right)\left(v_1\right)\right) = 4.$$

Since

$$F\left(S_4''\right) = 25 \ne F\left(S_4\right) + W\left(\bar{\mathcal{B}}\left(S_4\right)\left(v_1\right)\right) - W\left(\mathcal{B}\left(S_4\right)\left(v_{11}\right)\right) = 23,$$

the deduction in Lemma 1 and Theorem 1 are clearly incorrect.

3. Erratum

The Lemma 1 and Theorem 1 in Nguyen and Hung [4] are incorrect mainly because of the following notation definitions: $\mathcal{B}\left(S_p\right)$, $\mathcal{D}\left(S_p\right)$, $\mathcal{F}\left(S_p\right)$, $\mathcal{F}\left(S_p\right)(v)$, $\mathcal{B}\left(S_p\right)(v)$ and $\bar{\mathcal{B}}\left(S_p\right)(v)$. In this article, we will preserve the symbol definition of $\mathcal{B}\left(S_p\right)$ and $\mathcal{D}\left(S_p\right)$, redefine symbols: $\mathcal{B}\left(S_p\right)(v)$ and $\bar{\mathcal{B}}\left(S_p\right)(v)$, and do not use $\mathcal{F}\left(S_p\right)$ and $\mathcal{F}\left(S_p\right)(v)$ anymore to avoid confusion with $F\left(S_p\right)$.

For a vertex $v \in \mathcal{D}\left(S_p\right)$, let $\bar{\mathcal{B}}\left(S_p\right)(v)$ be the subgraph induced by vertices that can only be served by *v*. In other words, for vertices $v_i \in \bar{\mathcal{B}}\left(S_p\right)(v)$, only vertex $v \in \mathcal{D}\left(S_p\right)$ satisfies $d\left(v_i, v\right) = d\left(v_i, S_p\right)$. Let $\mathcal{N}\left(S_p\right)$ be the set of vertices of $G - S_p$, which are adjacent to some vertices in S_p. If the shortest path from $v_i \in \mathcal{B}\left(S_p\right)$ to S_p passes through $u \in \mathcal{N}\left(S_p\right)$, then v_i is said to be served by S_p through *u*. For a vertex $u \in \mathcal{N}\left(S_p\right)$, let $\mathcal{B}\left(S_p\right)(u)$ be the subgraph induced by the vertices which are served by S_p through *u*.

Table 1. Weights of subgraphs in G.

v	v_1	v_2	v_4	v	v_5	v_6	v_9	v_{10}	v_{11}	v_{13}
$W\left(\bar{B}\left(S_p\right)(v)\right)$	4	4	5	$W\left(B\left(S_p\right)(v)\right)$	1	1	2	2	4	4

Example 1. *Let us consider the block graph G in Figure 1, where the prespecified set of connected vertices is $S_4=\{v_1,v_2,v_3,v_4\}$ and the weights are labeled on the corresponding vertices.*

According to the new symbols and definitions above, one can deduce the following results. $\mathcal{D}(S_4)=\{v_1,v_2,v_4\}$, $\mathcal{N}(S_4)=\{v_5,v_6,v_7,v_9,v_{10},v_{11},v_{13}\}$. Moreover, the subgraph $\bar{\mathcal{B}}(S_4)(v_3)$ is induced by $\{v_3,v_7,v_8\}$ and the subgraph $\mathcal{B}(S_4)(v_{13})$ is induced by $\{v_{13},v_{14}\}$.

Using the same method in the proof of Lemma 1 in Nguyen and Hung [4], the following lemma can be proved.

Lemma 2. *If there exists a vertex $v\in\mathcal{D}(S_p)$ and vertex $u\in\mathcal{N}(S_p)$ such that $W\left(\mathcal{B}\left(S_p\right)(u)\right)>W\left(\bar{\mathcal{B}}\left(S_p\right)(v)\right)$, then S_p is not a connected p-median of $G(V,E,w)$.*

There is no problem in the method and thought of proving Theorem 1 in Nguyen and Hung [4], but the description is not clear and concise enough. We will give a new proof of Theorem 1 in the following.

Theorem 5. (Optimality Criterion) *The set S_p is a connected p-median of the block graph $G(V,E,w)$ if and only if $W\left(\mathcal{B}\left(S_p\right)(u)\right)\le W\left(\bar{\mathcal{B}}\left(S_p\right)(v)\right)$ for all $v\in\mathcal{D}(S_p)$ and $u\in\mathcal{N}(S_p)$.*

Proof. If S_p is a connected p-median of the block graph G, then $W\left(\mathcal{B}\left(S_p\right)(u)\right)\le W\left(\bar{\mathcal{B}}\left(S_p\right)(v)\right)$ for all $v\in\mathcal{D}(S_p)$ and $u\in\mathcal{N}(S_p)$. This is the converse-negative proposition of Lemma 2, which is clearly true.

Conversely, we prove that if $W\left(\mathcal{B}\left(S_p\right)(u)\right)\le W\left(\bar{\mathcal{B}}\left(S_p\right)(v)\right)$ for all $v\in\mathcal{D}(S_p)$ and $u\in\mathcal{N}(S_p)$, then the set S_p is a connected p-median of the block graph G. Let S_p^* be a connected p-median such that $S_p^*\ne S_p$ and $d(S_p,S_p^*)$ is as small as possible. Furthermore, if $d(S_p,S_p^*)=0$, let's assume that $S_p^*\cap S_p$ contains as many vertices as possible. We take a vertex $v'\in\mathcal{D}(S_p^*)$ such that $v'\notin S_p$. As $\bigcup_{u\in\mathcal{N}(S_p)}\mathcal{B}(S_p)(u)=V\setminus S_p$, we know that there exists a vertex $u''\in\mathcal{N}(S_p)$ such that $v'\in\mathcal{B}(S_p)(u'')$. Hence $W\left(\bar{\mathcal{B}}\left(S_p^*\right)(v')\right)\le W\left(\mathcal{B}\left(S_p\right)(u'')\right)$. Similarly, we choose a vertex $v''\in\mathcal{D}(S_p)$ such that $v''\notin S_p^*$. There exists a vertex $u'\in\mathcal{N}(S_p^*)$ such that $v''\in\mathcal{B}(S_p^*)(u')$. Hence $W\left(\bar{\mathcal{B}}\left(S_p\right)(v'')\right)\le W\left(\mathcal{B}\left(S_p^*\right)(u')\right)$. By assumption of S_p, $W\left(\mathcal{B}\left(S_p\right)(u'')\right)\le W\left(\bar{\mathcal{B}}\left(S_p\right)(v'')\right)$. Hence, $W\left(\bar{\mathcal{B}}\left(S_p^*\right)(v')\right)\le W\left(\mathcal{B}\left(S_p^*\right)(u')\right)$. On the other hand, since S_p^* is a connected p-median, $W\left(\bar{\mathcal{B}}\left(S_p^*\right)(v')\right)\ge W\left(\mathcal{B}\left(S_p^*\right)(u')\right)$. Hence, $W\left(\bar{\mathcal{B}}\left(S_p^*\right)(v')\right)=W\left(\mathcal{B}\left(S_p^*\right)(u')\right)$. We set $S'_p=\left(S_p^*\setminus\{v'\}\right)\cup\{u'\}$. Obviously S'_p is connected. Using the same method in the proof of Lemma 1 in Nguyen and Hung [4], we have $F(S'_p)=F(S_p^*)$. Then S'_p is also a connected p-median. If $d(S_p,S_p^*)>0$, then $d(S_p,S'_p)<d(S_p,S_p^*)$. If $d(S_p,S_p^*)=0$, then $S_p\cap S'_p$

contains more vertices than $S_p \cap S_p^*$. This contradicts the choice of S_p^*. Therefore, $S_p = S_p^*$ and S_p is a connected *p*-median of the block graph *G*.

In the solution approach of the inverse connected *p*-median problem on block graphs in [4], we only need to change $\mathcal{F}(S_p)$, $\mathcal{B}(S_p)(v)$ and $\bar{\mathcal{B}}(S_p)(v)$ into the new definition of $\mathcal{N}(S_p)$, $\mathcal{B}(S_p)(v)$ and $\bar{\mathcal{B}}(S_p)(v)$, respectively. The inverse connected *p*-median problem on block graphs $G(V,E,w)$ under various cost functions, say rectilinear norm, Chebyshev norm, and bottleneck Hamming distance, can be solved by using the approach in [4]. However, the description of Theorem 2, Theorem 3 and Theorem 4 in Nguyen and Hung [4] is not complete and rigorous enough. Let's redescribe the three theorems in the following.

Theorem 6. *The inverse connected p-median problem on block graphs* $G(V,E,w)$ *under Rectilinear norm can be solved in* $O(n\log n)$ *time, where n is the number of vertices in the block graph.*

Theorem 7. *The inverse connected p-median problem on block graphs* $G(V,E,w)$ *under Chebyshev norm can be solved in* $O(n\log n)$ *time, where n is the number of vertices in the block graph.*

Theorem 8. *The inverse connected p-median problem on block graphs* $G(V,E,w)$ *under Bottleneck Hamming distance can be solved in* $O(n\log n)$ *time, where n is the number of vertices in the block graph.*

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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