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Boundedness Types of Perturbations on the Growth of Semigroups

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Abstract

We study types of boundedness of a semigroup on a Banach space in terms of the Cesáro-average and the behavior of the resolvent at the origin and also exhibit a characterization of type Hille-Yosida for the generators of φ^j-bounded strongly continuous semigroups. Furthermore, these results are used to investigate the effect of the Perturbation on the type of the growth of sequences.

Keywords

Cesáro Average, C_0-Semigroups, Boundedness, Perturbation Stability, Hille-Yosida, Growth of Sequences

1. Introduction

Let $(A_j, D(A_j))$ be the sequence of generators of a C_0-semigroup $T^j := T^j(1+\varepsilon)_{\varepsilon\ge -1}$ on a Banach space *X*. Let $\varphi^j : \mathbb{R}^+ \to \mathbb{R}^+$ be a given function. That will say that the semigroup T^j is φ^j-bounded, if

$$\left\|\sum_j T^j(1+\varepsilon)\right\| \le M\sum_j \varphi^j(1+\varepsilon)$$

holds for some $M > 0$ and each $\varepsilon \ge -1$. If $\varphi^j(1+\varepsilon) = M\left(1+(1+\varepsilon)^d\right)$ for some $M, d \ge 0$, then the semigroup is polynomially bounded and is bounded when $d = 0$. It is known that there exist $\omega^j \in \mathbb{R}$ and $M \ge 0$ such that $\left\|\sum_j T^j(1+\varepsilon)\right\| \le M\sum_j e^{\omega^j(1+\varepsilon)}$ for all $\varepsilon \ge -1$. When the growth bounds of the semigroup $\omega_0^j := \inf\left\{\omega^j \in \mathbb{R}, \exists M \ge 1, \forall\varepsilon \ge -1, \left\|\sum_j T^j(1+\varepsilon)\right\| \le M\sum_j e^{\omega^j(1+\varepsilon)}\right\}$, are positive, the semigroup is exponentially bounded. We are concerned with the types of boundedness of T^j and its perturbation when φ^j is exponentially bounded, *i.e.*, for all $\delta > 0$, $\int_0^\infty e^{-\delta(1+\varepsilon)}\sum_j \varphi^j(1+\varepsilon)\,d(1+\varepsilon) < \infty$.

Many results are shown in this article. The first one states that if the Cesáro average of T^j and that of its adjoint are φ^j-bounded, then so does the semigroup.

Next, give a Hille-Yosida type characterization of generators of φ^j-bounded C_0-semigroups. Furthermore, Theorem 3.2 gives a sufficient condition that improves the results of (Shiand Feng [1]) and (Eisner [2]). Notice that a closer result including integrability conditions of each powers of the resolvent was given in (Batty *et al.* [3], Theorem 6.6) under more assumptions on φ^j.

Finally, we consider a sequence perturbation $\left(B_j, D\left(B_j\right)\right)$ and establish the previous results for the perturbed semigroup $S(1+\varepsilon)_{\varepsilon\geq -1}$ generated by $A_j + B_j$.

Throughout this article A_j stands for a closed densely defined linear sequence operators on X with domain $D\left(A_j\right)$ and spectre $\sigma\left(A_j\right)$. The pseudo-spectral bounded of sequence A_j is defined by $s_0\left(A_j\right) := \inf\left\{\omega^j > s\left(A_j\right) : \exists C_{\omega^j} \text{ such that } \left\|\sum_j R\left(\lambda_j + A_j\right)\right\| \leq \sum_j C_{\omega^j} \text{ whenever } \operatorname{Re}\left(\lambda_j\right) > \omega^j\right\}$, where $s\left(A_j\right)$ denotes the spectral bounded of sequence A_j given by $s\left(A_j\right) := \sup\left\{\operatorname{Re}\left(\lambda_j\right) : \lambda_j \in \sigma\left(A_j\right)\right\}$, with the convention $s\left(A_j\right) = -\infty$ if $\sigma\left(A_j\right) = \phi^j$.

2. Boundedness Types of a Semigroup Interms of Cesáro-Boundedness of the Semigroup and Its Adjoint

It is shown in (Zwart [4]) that if T^j is a C_0-semigroup on a Banach space X and $\left(T^j\right)'$ is its adjoint, then for each $\varepsilon > -1$, $x^j \in X$, $\left(x^j\right)' \in X'$ and, $\varepsilon \geq 0$,

$$\left|\sum_j \left\langle T^j (1+\varepsilon) x^j ; \left(x^j\right)' \right\rangle\right| \leq \left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon} \sum_j \left\|T^j(s)x^j\right\|^{(1+\varepsilon)} \mathrm{ds}\right)^{\frac{1}{1+\varepsilon}} \left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon} \sum_j \left\|\left(T^j\right)'(s)\left(x^j\right)'\right\|^{\frac{1+\varepsilon}{\varepsilon}} \mathrm{ds}\right)^{\frac{\varepsilon}{1+\varepsilon}} \quad (1)$$

The following theorem is a consequence of this inequality and recovers (Zwart [4], Theorem 2.1, 2.2), (Van Casteren and Jan [5], Theorem 3.1, (iii ⇔ iv)), (Casteren and Jan [6], Proposition 3.1) and (Guo and Zwart [7], Theorem 8.2).

Theorem 2.1. Let T^j be a C_0-semigroup on a Banach space X, $1+\varepsilon, \frac{1+\varepsilon}{\varepsilon} > 1$ with $\frac{\varepsilon}{1+\varepsilon} + \frac{1}{1+\varepsilon} = 1$. Let φ^j and ϕ^j be measurable positive functions. If for each $x^j \in X$, and $\left(x^j\right)' \in X'$

$$\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon} \sum_j \left\|T^j(s)x^j\right\|^{(1+\varepsilon)} \mathrm{ds} \leq \sum_j \varphi^j(1+\varepsilon)\left\|x^j\right\|^{(1+\varepsilon)} \quad \forall \varepsilon > -1 \quad (2)$$

and

$$\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon} \sum_j \left\|\left(T^j\right)'(s)\left(x^j\right)'\right\|^{\frac{1+\varepsilon}{\varepsilon}} \mathrm{ds} \leq \sum_j \phi^j(1+\varepsilon)\left\|\left(x^j\right)'\right\|^{\frac{1+\varepsilon}{\varepsilon}} \quad \forall \varepsilon > -1 \quad (3)$$

hold, then

$$\left\|\sum_j T^j(1+\varepsilon)\right\| \le \sum_j \left(\varphi^j(1+\varepsilon)\right)^{\frac{1}{1+\varepsilon}} \left(\phi^j(1+\varepsilon)\right)^{\frac{\varepsilon}{1+\varepsilon}} \quad \forall \varepsilon > -1 \tag{4}$$

We indicate that one cannot omit the Condition (3). (Van Casteren and Jan [5], Example (2)), gives a polynomially bounded group, $T^j(s)_{s\in\mathbb{R}}$ of linear sequence operators acting on $X = L^2(\mathbb{R})$, while its Cesáro average is bounded.

3. Boundedness Types of a Semigroup in Terms of the Resolvent

In order to deal with the converse when T^j is φ^j-bounded, consider the set Λ of all continuous functions $\varphi^j : [0,\infty) \to [1,\infty)$ such that

$$\begin{cases} \text{there exist a constant } M \ge 1 \text{ such that,} \\ \text{for each } \varepsilon > -\beta \quad \sup_{\varepsilon \ge -1} \sum_j \varphi^j(1+\varepsilon) e^{-(1+\varepsilon)(\varepsilon+\beta)} \le M \sum_j \varphi^j\left(\dfrac{1}{\varepsilon+\beta}\right) \end{cases} \tag{5}$$

Note that if the functions $\varphi^j, \psi^j \in \Lambda$, and $\varepsilon \ge 0$, then $\left(\varphi^j + (1+\varepsilon)\psi^j\right) \in \Lambda$ see (Boukdir [8]).

Many classical functions are contained in Λ: The bounded functions φ^j for which there exists $M \ge 1$ such that $1 \le \varphi^j(1+\varepsilon) \le M$, the polynomial functions, $\varphi^j(1+\varepsilon) = \left(1+(1+\varepsilon)^d\right)$ for $d \ge 0$. Indeed,

$$\begin{aligned} \varphi^j(1+\varepsilon) e^{-(\varepsilon+\beta)(1+\varepsilon)} &= \left(1+(1+\varepsilon)^d\right) e^{-(\varepsilon+\beta)(1+\varepsilon)} \le 1+(1+\varepsilon)^d e^{-(\varepsilon+\beta)(1+\varepsilon)} \\ &\le 1+\frac{d^d e^{-d}}{(\varepsilon+\beta)^d} \le M\left(1+\frac{1}{(\varepsilon+\beta)^d}\right) = M\varphi^j\left(\frac{1}{\varepsilon+\beta}\right) \end{aligned}$$

for each $\varepsilon \ge -1, \varepsilon_1 > -\beta$ and $M = \max\left\{1, d^d e^{-d}\right\}$. Also, the function $\varphi^j(1+\varepsilon) = e^{d(1+\varepsilon)^{\frac{1}{d}}}$ with $d > 1$. In certainty, since $d(1+\varepsilon)^{\frac{1}{d}} - (1+\varepsilon)^2 \le (d-1)(1+\varepsilon)^{\frac{-1}{d-1}} \le (d-1)(1+\varepsilon)^{\frac{-1}{d}}$ for each $\varepsilon > -1$ then

$$\varphi^j(1+\varepsilon) e^{-(1+\varepsilon)(\varepsilon+\beta)} = e^{d(1+\varepsilon)^{\frac{1}{d}} - (\varepsilon+\beta)(1+\varepsilon)} \le e^{d(\varepsilon+\beta)^{\frac{-1}{d}}} = \varphi^j\left(\frac{1}{\varepsilon+\beta}\right).$$

Theorem 3.1. Let $\left(A_j, D(A_j)\right)$ be the sequence of operator on a Banach space *X*. Let $\varphi^j : [0,\infty) \to [1,\infty)$ be a continuous function with $\varphi^j(0) \ge 1$. Suppose the following assertions.

i) $\left(A_j, D(A_j)\right)$ is closed, densely defined, and for every $\lambda_j > 0$ one has $\lambda_j \in \varrho(A_j)$ and

$$\sup_{\lambda_j > (\varepsilon+\beta)} \left\| \sum_j \left(\lambda_j - (\varepsilon+\beta)\right)^n R(\lambda_j, A_j)^n \right\| \le M \sum_j \varphi^j\left(\frac{1}{\varepsilon+\beta}\right) \tag{6}$$

for each $\varepsilon > -\beta$, $n \in \mathbb{N} \cup \{0\}$ and some constant $M \ge 0$.

ii) $\left(A_j, D(A_j)\right)$ generates a φ^j-bounded C_0-semigroup T^j.

Then, (i) ⇒ (ii). Conversely, if in addition φ^j satisfies (5), then (ii) ⇒ (i).

Proof. (i) ⇒ (ii). Is deduced from the Hille-Yosida Theorem and the exponential formula, $T^j(1+\varepsilon)x^j = \lim_{n\to\infty}\left(\frac{n}{1+\varepsilon}\right)^n R\left(\frac{n}{1+\varepsilon}, A_j\right)^n x^j$ for each $x^j \in X$, with

$\lambda_j = \frac{n}{1+\varepsilon}$ and $(\varepsilon+\beta) = \frac{1}{1+\varepsilon}$.

(ii) ⇒ (i). Since A_j be the sequence of generates a C_0-semigroup, then it is closed and densely defined. The φ^j-boundedness of T^j and (5) imply that there exist $M_0, M_1 \ge 0$ such that

$$\begin{aligned}\left\|\sum\nolimits_j T^j(1+\varepsilon)\right\| &\le M_0 \sum\nolimits_j \varphi^j(1+\varepsilon) \\ &\le M_1 \sum\nolimits_j \varphi^j\left(\frac{1}{\varepsilon+\beta}\right) e^{(\varepsilon+\beta)(1+\varepsilon)} \quad \forall \varepsilon \ge -1, \forall \varepsilon > -\beta.\end{aligned} \tag{7}$$

Consequently, $\omega_0^j(T^j) \le 0$ and hence $\lambda_j \in \varrho(A_j)$ for all $\lambda_j > 0$. Let $n \in \mathbb{N} \cup \{0\}$ and $0 < \varepsilon+\beta < \lambda_j$. Then

$$\begin{aligned}&\left\|\sum\nolimits_j (-1)^n \left(\lambda_j - (\varepsilon+\beta)\right)^{n+1} R^{n+1}(\lambda_j, A_j)\right\| \\ &= \left\|\sum\nolimits_j \left(\lambda_j - (\varepsilon+\beta)\right)^{n+1} \int_0^\infty e^{-\lambda_j (1+\varepsilon)\frac{(1+\varepsilon)^n}{n!}} T^j(1+\varepsilon)\,d(1+\varepsilon)\right\| \\ &\le M_0 \sum\nolimits_j \left(\lambda_j - (\varepsilon+\beta)\right)^{n+1} \int_0^\infty e^{-(\lambda_j - (\varepsilon+\beta))(1+\varepsilon)} \frac{(1+\varepsilon)^n}{n!} \left(e^{-(\varepsilon+\beta)(1+\varepsilon)} \varphi^j(1+\varepsilon)\right) d(1+\varepsilon) \\ &\le M_1 \sum\nolimits_j \varphi^j\left(\frac{1}{\varepsilon+\beta}\right).\end{aligned}$$

Theorem 3.2. Suppose A_j be a closed and densely defined a sequence of operators in a Banach space X with $s(A_j) \le 0$. For a continuous function $\varphi^j : [0,\infty) \to [0,\infty)$ with $\varphi^j(0) \ge 1$, we study the following assertions.

a) For all $x^j \in X, y^j \in X'$, and $\varepsilon > -\beta$

$$\sup_{\beta<1} (1-\beta) \int_{-\infty}^{+\infty} \sum\nolimits_j \left\| R\left((1+\varepsilon)+is, A_j\right) x^j \right\|^2 ds \le \sum\nolimits_j \varphi^j\left(\frac{1}{\varepsilon+\beta}\right) \left\|x^j\right\|^2,$$

And

$$\sup_{\beta<1} (1-\beta) \int_{-\infty}^{+\infty} \sum\nolimits_j \left\| R\left((1+\varepsilon)+is, A_j'\right) y^j \right\|^2 ds \le \sum\nolimits_j \varphi^j\left(\frac{1}{\varepsilon+\beta}\right) \left\|y^j\right\|^2 \tag{8}$$

b) For each $x^j \in X, y^j \in X'$, and $\varepsilon > -\beta$

$$\sup_{\beta<1} (1-\beta) \int_{-\infty}^{+\infty} \sum\nolimits_j \left| \left\langle R^2\left((1+\varepsilon)+is, A_j\right) x^j, y^j \right\rangle \right| ds \le \sum\nolimits_j \varphi^j\left(\frac{1}{\varepsilon+\beta}\right) \left\|x^j\right\| \left\|y^j\right\| \tag{9}$$

c) A_j generates a φ^j-bounded C_0-semigroup T^j on X, for which

$$\sum\nolimits_j \left\|T^j(1+\varepsilon)\right\| \le \frac{e^2}{2\pi} \sum\nolimits_j \varphi^j(1+\varepsilon) \quad \text{for each} \quad \varepsilon \ge -1 \tag{10}$$

Then (a) ⇒ (b) ⇒ (c). In this case, the semigroup T^j is given by $T^j(0) = I_d$ and for $\varepsilon > -1$,

$$\sum\nolimits_j T^j(1+\varepsilon) x^j = \frac{1}{2\pi(1+\varepsilon)} \sum\nolimits_j \int_{-\infty}^{+\infty} e^{((1+\varepsilon)+is)(1+\varepsilon)} R^2\left((1+\varepsilon)+is, A_j\right) x^j ds, \tag{11}$$

for each $(1+\varepsilon) > s_0(A_j)$ and $x^j \in X$. Furthermore, if X is a Hilbert space and

φ^j satisfies (5), then (c) implies (a) with $(1+\varepsilon)(\varphi^j)^2\left(\frac{1}{\varepsilon+\beta}\right)$, instead of $\varphi^j\left(\frac{1}{\varepsilon+\beta}\right)$, for some $\varepsilon > -1$.

Proof. (a) ⇒ (b) It is obtained by applying the Cauchy-Schwarz inequality.

(b) ⇒ (c). Deduce from (Gomilko [9]) and ((p. 505) from (Chill & Tomilov [10]) that the assumption (b) implies that the sequence of operator A_j generates a C_0-semigroup T^j and for all $\varepsilon > -1$, $\sup_{\varepsilon>-1}\left\|e^{-(1+\varepsilon)^2}\sum_j T^j(1+\varepsilon)\right\| < \infty$, and for all $\varepsilon > -1$, $x^j \in X$ and $y^j \in X'$

$$\sum_j\left\langle T^j(1+\varepsilon)x^j, y^j\right\rangle = \frac{e^{(1+\varepsilon)^2}}{2\pi(1+\varepsilon)}\sum_j\int_{-\infty}^{+\infty} e^{is(1+\varepsilon)}\left\langle R^2\left((1+\varepsilon)+is, A_j\right)x^j, y^j\right\rangle ds \quad (12)$$

then the result is deduced by choosing $\varepsilon+\beta = \frac{1+\varepsilon}{2} = \frac{1}{1+\varepsilon}$.

Conversely. Let $0 < \varepsilon+\beta < 1+\varepsilon$. As in (7) the Parseval identity yields

$$\begin{aligned}
&\sum_j\left\|\int_{-\infty}^{+\infty} R\left((1+\varepsilon)+is, A_j\right)x^j\right\|^2 ds \\
&= 2\pi\int_0^{+\infty} e^{-2(1+\varepsilon)^2}\sum_j\left\|T^j(1+\varepsilon)x^j\right\|^2 d\left((1+\varepsilon)\right) \\
&\le \frac{e^4}{2\pi}M_1^2\sum_j(\varphi^j)^2\left(\frac{1}{\varepsilon+\beta}\right)\int_0^{\infty} e^{-2(1-\beta)(1+\varepsilon)}\left\|x^j\right\|^2 d(1+\varepsilon) \\
&= \frac{e^4 M_1^2}{4\pi(1-\beta)}\sum_j(\varphi^j)^2\left(\frac{1}{\varepsilon+\beta}\right)\left\|x^j\right\|^2,
\end{aligned}$$

and the identical reasoning for the dual case.

Remark 3.3. 1) The condition $\varphi^j(0) \ge 1$ cannot be omitted in the above Theorem 3.2. If not, the semigroup may not be strongly continuous at the origin.

2) It is not enough that the condition (9) be satisfied by some $(1+\varepsilon) > s_0(A_j)$.

An example due to (Selim Grigorevich Krein [11]) see also (Kaiser & Weis [12]), exhibits that there exists a closed, densely defined sequence of operators $(A_j, D(A_j))$ acting on a Hilbert space X such that the resolvent exists and uniformly bounded on $\{\lambda_j \in \mathbb{C} : \operatorname{Re}\lambda_j \ge 0\}$ and

$$\int_{-\infty}^{+\infty}\left(\sum_j\left\|R(is, A_j)x^j\right\|^2 + \sum_j\left\|R(is, A_j')x^j\right\|^2\right)ds < \infty \quad \forall x^j \in X,$$

but the sequence A_j of generators of semigroup is not strongly continuous at the origin.

3) If $\varphi^j(1+\varepsilon) = M\left(1+(1+\varepsilon)^d\right)$ in (9) we recover the result of (Eisner [2]) when $d \ge 0$, and (Gomilko [9]) with $d = 0$.

4) If $\varphi^j = e^{(d-1)(\varepsilon+\beta)^{\frac{-1}{d-1}}}$ for some $d > 1$ in (9), obtain (Laubenfels *et al.* [13] Corollary (3.5)), exactly, the semigroup satisfies

$$\sum_j\left\|T^j(1+\varepsilon)\right\| \le \frac{e^2}{2\pi}e^{d(1+\varepsilon)^{\frac{1}{d}}} \quad \text{for all} \quad \varepsilon \ge -1. \quad (13)$$

In order to give a generalization of (Eisner and Zwart [14], Theorem 2.1) supposes that the resolvent of the sequence of generator A_j is $(1+\varepsilon)$-integrabe, *i.e.*

$$\sum_j \left\| \int_{-\infty}^{+\infty} R\left((1+\varepsilon)+is, A_j\right) x^j \right\|^{(1+\varepsilon)} \mathrm{d}s < \infty \quad \text{for all} \quad x^j \in X , \tag{14}$$

and

$$\sum_j \left\| \int_{-\infty}^{+\infty} R\left((1+\varepsilon)+is, A_j'\right) y^j \right\|^{\frac{1+\varepsilon}{\varepsilon}} \mathrm{d}s < \infty \quad \text{for all} \quad y^j \in X' , \tag{15}$$

Note. We can deduce that:

$$\mathrm{e}^{d(1+\varepsilon)^{\frac{1}{d}}} \le \sum_j \varphi^j(1+\varepsilon) \quad \text{for all} \quad \varepsilon \ge -1 .$$

Proof. From (10) and (13).

Theorem 3.4. Let A_j be the sequence of generator of a C_0-semigroup T^j on a Banach space X such that $s(A_j) \le 0$ and its resolvent is $(1+\varepsilon)$-integrable for $\varepsilon > 0$. Let $\varphi^j : [0,\infty) \to [1,\infty)$ be a continuous function. If there exist $(1+\varepsilon)_0, M > 0$ such that

(i) $\sum_j \left\| R\left((1+\varepsilon)+is, A_j\right) \right\| \le M \sum_j \varphi^j\left(\frac{1}{1+\varepsilon}\right)$ for all $0 < (1+\varepsilon) < (1+\varepsilon)_0$;

(ii) $\sum_j \left\| R\left((1+\varepsilon)+is, A_j\right) \right\| \le M$ for all $(1+\varepsilon) \ge (1+\varepsilon)_0$, (16)

then

$$\sum_j \left\| T^j(1+\varepsilon) \right\| \le \sum_j \left(\varphi^j\right)^2 (1+\varepsilon) \tag{17}$$

for some $\varepsilon > -1$ and $(1+\varepsilon) > \frac{1}{(1+\varepsilon)_0}$.

Proof. As it is shown in Eisner (2007), from (16) deduce that $s_0(A_j) \le 0$ and for all $r \in \left(0, (1+\varepsilon)_0\right)$ there exists $M_1 \ge 0$ such that for each $x^j \in X$ and $y^j \in X'$

$$\sum_j \left\| R\left(r+i, A_j\right) x^j \right\|_{L^{(1+\varepsilon)}(\mathbb{R},X)} \le M_1 \sum_j \varphi^j\left(\frac{1}{r}\right) \left\| x^j \right\|$$

and

$$\sum_j \left\| R\left(r+i, A_j'\right) y^j \right\|_{L^{\frac{(1+\varepsilon)}{\varepsilon}}(\mathbb{R},X')} \le M_1 \sum_j \varphi^j\left(\frac{1}{r}\right) \left\| y^j \right\|$$

The Cauchy Schwarz inequality yields

$$\int_{-\infty}^{+\infty} \left| \sum_j \left\langle R^2\left(r+is, A_j\right) x^j, y^j \right\rangle \right| \mathrm{d}s \le M_1^2 \sum_j \left(\varphi^j\right)^2 \left(\frac{1}{r}\right) \left\| x^j \right\| \left\| y^j \right\| . \tag{18}$$

By the inverse formula we get

$$\begin{aligned} \sum_j \left| \left\langle T^j(1+\varepsilon) x^j, y^j \right\rangle \right| &\le \frac{\mathrm{e}^{r(1+\varepsilon)}}{2\pi(1+\varepsilon)} \int_{-\infty}^{+\infty} \sum_j \left| \left\langle R^2\left(r+is, A_j\right) x^j, y^j \right\rangle \right| \mathrm{d}s \\ &\le \frac{\mathrm{e}^{r(1+\varepsilon)}}{2\pi(1+\varepsilon)} M_1^2 \sum_j \left(\varphi^j\right)^2 \left(\frac{1}{r}\right) \left\| x^j \right\| \left\| y^j \right\| . \end{aligned}$$

For $(1+\varepsilon)$ large enough, one can choose $(1+\varepsilon)=\frac{1}{r}$.

Note. We can deduce that:

i) $\sum_j \varphi^j(1+\varepsilon) \le \frac{2\pi}{e^2}\sum_j (\varphi^j)^2 (1+\varepsilon)$ for all $\varepsilon \ge -1$.

ii) $e^{d(1+\varepsilon)^{\frac{1}{d}}} \le \frac{2\pi}{e^2}\sum_j (\varphi^j)^2 (1+\varepsilon)$ for all $\varepsilon \ge -1$.

Proof. i) From (10) and (17).

ii) From (13) and (17).

Corollary 3.5. Let A_j be the sequence of generator of a C_0-semigroup T^j on a Banach space X such that $s_0(A_j) \le 0$ and the resolvent is $(1+\varepsilon)$-integrable, for some $\varepsilon > 0$. If

$$\lim_{(1+\varepsilon)\to 0^+} \sqrt{(1+\varepsilon)} \left\| \sum_j R((1+\varepsilon), A_j) \right\| = 0 \tag{19}$$

and there exist constant $(1+\varepsilon)_0, M > 0$ such that

$\sum_j \left\| R((1+\varepsilon)+is, A_j) \right\| \le \left(1+\sum_j \left\| R((1+\varepsilon), A_j) \right\|\right)$, for all $0 < (1+\varepsilon) < (1+\varepsilon)_0$

$$\sum_j \left\| R((1+\varepsilon)+is, A_j) \right\| \le M \quad \text{for all} \quad (1+\varepsilon)_0 \le (1+\varepsilon). \tag{20}$$

Then the semigroup T^j is uniformly stable.

Proof. It is enough to choose $\sum_j \varphi^j\left(\frac{1}{1+\varepsilon}\right) = \left(1+\sum_j \left\| R((1+\varepsilon), A_j) \right\|\right)$

Remark 3.6. 1) By $\sum_j \left\| R(\lambda_j, A_j) \right\| \ge \sum_j \frac{1}{dist(\lambda_j, \sigma(A_j))}$, (19) and (20) are equivalent to $\sum_j \left\| R((1+\varepsilon)+is, A_j) \right\| \le M_1$ for each $\varepsilon \ge -1$ and $s \in \mathbb{R}$. Hence with the $(1+\varepsilon)$-integrability of the resolvent the uniform stability of the semigroup follows from (Eisner [15], Theorem 2.15).

2) Note that the Conditions (20) are satisfied by positivity-preserving semi-groups, acting in $L^{(1+\varepsilon)}(X, dx^j)$ for some $0 \le \varepsilon < \infty$ and $\omega_0^j = 0$, see (Davies [16], Lemma 9).

4. Boundedness Types of the Perturbed Semigroups

We exhibit that the conditions on A_j which ensure the φ^j-boundedness of the semigroup T^j are sufficient to obtain the same property for the perturbed $S := S(1+\varepsilon)_{\varepsilon\ge -1}$ of a semigroup sequence of generators $A_j + B_j$.

Let A_j be the sequence of generator of a C_0-semigroup T^j with $s_0(A_j) \le 0$. Peekingan other closed sequence operator $(B_j, D(B_j))$ such that $D(A_j) \subset D(B_j)$, and let $(B_j', D(B_j'))$ its dual.

Suppose that

$(\mathcal{H}_0)$ $\sum_j \left\| B_j R(\lambda_j, A_j) \right\| \le M < 1$ for all $\lambda_j \in \mathbb{C}^+$, and

$$(\mathcal{H}_1) \quad \sum_j \left\| R(\lambda_j, A_j) B_j y^j \right\| \le M \sum_j \left\| y^j \right\| \quad \text{for all} \quad \lambda_j \in \mathbb{C}^+, \quad y^j \in D(B_j). \tag{21}$$

Let $\lambda_j \in \mathbb{C}^+$. Since $s_0(A_j) \le 0$, and by $(\mathcal{H}_1)$ and the decomposition $\lambda_j - (A_j + B_j) = (\lambda_j - A_j)\left[I_d - R(\lambda_j, A_j) B_j\right]$, deduce that

$\lambda_j \in \varrho(A_j + B_j) = \varrho(A_j + B_j)'$. Furthermore the inverse $R(\lambda_j,(A_j + B_j))$ satisfies

$$\begin{aligned}\sum\nolimits_j \left\| R\left(\lambda_j,\left(A_j + B_j\right)\right) \right\| &= \sum\nolimits_j \left\| \sum\nolimits_{n=0}^{\infty} \left(R\left(\lambda_j, A_j\right) B_j \right)^n R\left(\lambda_j, A_j\right) \right\| \\ &\le (1-M)^{-1} \sum\nolimits_j \left\| R\left(\lambda_j, A_j\right) \right\| \end{aligned} \tag{22}$$

where $M := \sum_j \left\| R(\lambda_j, A_j) B_j \right\|$. From $(\mathcal{H}_0)$ we obtain that $\sum_j \left\| R(\lambda_j, A_j)' B_j' (y^j)' \right\| \le M \sum_j \left\| (y^j)' \right\|$ for each $(y^j)' \in D(B_j)'$, and similar arguments exhibit that

$$\sum\nolimits_j \left\| R\left(\lambda_j, A_j + B_j\right)' y^j \right\| \le (1-M)^{-1} \sum\nolimits_j \left\| R\left(\lambda_j, A_j\right)' y^j \right\|. \tag{23}$$

Note that if (21) holds then the resolvent of $(A_j + B_j)$ is $(1+\varepsilon)$-integrable when that of A_j is.

Proposition 4.1. Let $(A_j, D(A_j))$ be a closed and densely defined a sequence operator on a Banachspace *X* with $s(A_j) \le 0$. Let B_j be a closed sequence of operator such that $D(A_j) \subset D(B_j)$ and satisfy $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$. Let φ^j be a continuous function for which (8) holds, then $(A_j + B_j)$ generates a φ^j-bounded C_0-semigroup.

Proof. Since $D(A_j + B_j) = D(A_j) \cap D(B_j) = D(A_j)$ then $D(A_j + B_j)$ is dense. By the assumption $(\mathcal{H}_0)$, $(\mathcal{H}_1)$ and $s_0(A_j) \le 0$ we deduce that $s_0(A_j + B_j) \le 0$ and the Conditions (9) for $(A_j + B_j)$ are deduced from the Cauchy-Schwarz inequality, (22) and (23).

The following proposition organizes a connection between the φ^j-boundedness of the semigroup T^j and that of its perturbed *S*. Furthermore this result gives a generalization of (Kaiser and Weis [12], Theorem 3.1) and (Batty and Charles [17], Theorem 1).

Proposition 4.2. Let $(A_j, D(A_j))$ be a sequences of generator of a C_0-semigroup T^j on a Hilbert space *X*. Let B_j be a closed sequence of operator satisfying $D(A_j) \subset D(B_j)$ and for which the hypothesis $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$ hold. Let φ^j be a continuous function satisfying (5).

If T^j is φ^j-bounded, then $(A_j + B_j)$ generates a $(\varphi^j)^2$-bounded C_0-semigroup.

Proof. from (22), (23) and Theorem 3.2.

Assuming $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$, we will give sufficient conditions on A_j confirming that both T^j and *S* have the same boundedness types for large $(1+\varepsilon)$.

Proposition 4.3. Let T^j be a C_0-semigroup generated by the sequence of operator A_j for which $\mathbb{C}^+ \subseteq \varrho(A_j)$ and having $(1+\varepsilon)$-integrable resolvent for some $\varepsilon > 0$. Suppose that $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$ hold for some closed sequence of operator B_j. Let $\varphi^j : [0,\infty) \to [1,\infty)$ be a continuous function satisfying (16).

Then the semigroup *S* generated by $(A_j + B_j)$ is strongly continuous sequence on $(0,+\infty)$ and satisfy (17).

Proof. a direct consequence of Theorem 3.4.

5. Conclusion

As discussed above the type of the boundedness of a semigroup T^j in terms of increment of its Cesàro-average and that of its adjoint $\left(T^j\right)' := \left(T^j\right)'(1+\varepsilon)_{\varepsilon\geq-1}$ is φ^j-bounded, then the semigroup is bounded (see Theorem 2.1). Also, we introduced a Hille-Yosida type characterization of generators of φ^j-bounded C_0-semigroups (see Theorem 3.1). We presented some effect of a perturbation sequence operator A_j by sequence operator B_j, that satisfies some assumptions specified (see Proposition 4.1 and Proposition 4.2).

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Author Contributions

The authors approve and read the article.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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