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A Probabilistic Approach of the Poincaré-Bendixon Problem in $\mathbb{R}^d$

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Abstract

We present how a probabilistic model can describe the asymptotic behavior of the iterations, with applications for ODE and approach of the Poincaré-Bendixon's problem in $\mathbb{R}^d$.

Keywords

Perron-Frobenius's Measure, ODE, Poincaré-Bendixon Problem, Lorenz's Equations

1. Introduction

Let f a function which applies a set $C \subset \mathbb{R}^d$ in itself. We iterate f indefinitely. Sometimes, the process converges to some fixed point or to some cycle. But, in many cases, it is quite impossible to know, after a long time, the position of the iteration $f^{(n)}$.

One of the best deterministic method to study the asymptotic behavior of the iteration is to linearize f. It consists to find a function φ such as: $\varphi \circ f = \lambda\varphi$ where λ is linear.

When this linearization is possible, $\varphi \circ f^{(n)} = \lambda^n \varphi$ and we obtain asymptotic cases, which are the generalization of the unidimensional cases $|\lambda| >, =, < 1$. But, we meet some well-known difficulties: the function φ is solution of a functional equation; the basins of attraction around each fixed point of f may have fuzzy frontiers… If $\lambda^p = 1$, we have other difficulties called resonance in $\mathbb{R}^d$. It is often a good approach near each fixed point. But, in many cases, we don't know to treat mathematically the problem.

So, it seems very important to understand what happens when we iterate f indefinitely, especially if the *set* $C \subset \mathbb{R}^d$ *is bounded when* f *applies C in itself.* In this case, a probabilistic approach with invariant measures gives other information. It is the object of this paper which is the synthesis of a long work: some demonstrations, as the one of the proposition in Section 4.4, can be found in previous articles or book [1].

2. The Perron-Frobenius's Measure

Let P a measure on a measurable set $C \subset \mathbb{R}^d$ and f a measurable function. We note $P_f = P \circ f^{-1}$ the transform of P by f. We define the f-invariant measure P of Perron-Frobenius as in [2]:

P is invariant under f if, for all borelian set B, P verifies the Perron-Frobenius's equation (PF):

$$P_f(B) = P \circ f^{-1}(B) = P(B)$$

This measure P remains invariant when we iterate the measurable function f. Under very general conditions, the solution of this equation is unique.

This measure presents the same difficulties as we have seen with linearization methods for f: we will see that it depends of the fixed points and we meet the resonance's problems, but it gives us many information about the areas where the iteration belongs more frequently. This information is asymptotic when $n \to \infty$ but doesn't give us any result about the transient steps of null measure.

This invariant measure is generally difficult to study. For instance, in the very simple case where f is invertible, its density p verifies the functional equation:

$$p = p_f = \left| f^{-1} \right| p \circ f^{-1}$$

This is very complicated to solve. If f is not invertible, it is more difficult.

2.1. The Fourier-Laplace's Transform

Here, we seek an analytic approach of the f-invariant measure with the Fourier-Laplace's transform. We use the known property of this measure: for all positive P-measurable function g, we have the formula [2]:

$$\int g \circ f(x) \mathrm{d}P(x) = \int g(x) \mathrm{d}P(x).$$

For $g(x) = e^{yx}$, we write the Fourier-Laplace's transform $\phi(y) = \mathcal{L}(e^{yX}) = E(e^{yX})$ with the series $\phi(y) = \Sigma_n b_n y^n$ and $\phi_f(y) = \mathcal{L}(e^{yf(X)})$. If the measure is invariant:

$$\phi(y) = \phi_f(y)$$

Here, $y \in \mathbb{R}^d$ or $\mathbb{C}^d$. If $y = it$, we have the characteristic function of the measure P.

Hypothesis

All along the paper, we suppose that the set $C \subset \mathbb{R}^d$ is bounded. f applies C in C and f is at least C^∞.

Anymore, the series $\phi(y) = \Sigma_n b_n y^n$ is convergent because it is bounded by an exponential series with the diameter of C.

We translate the distribution with a small fixed vector $a \in \mathbb{R}^d$, $X \mapsto X + a$.

So: $\phi(y,a)=E\left(e^{y(X+a)}\right)$

by f -transform: $\phi(y,a)\mapsto\phi_f(y,a)=E\left(e^{yf(X+a)}\right)$.

If the measure is invariant: $\phi(y,a)=\phi_f(y,a)$.

Proposition:

The resolving equation R_a of PF is: $\theta_f(y,a)=\phi(y,a)-\phi_f(y,a)=0$.

If f is a C^∞ -function, then, for $\forall a\in C$ and $\forall y$:

$$\theta_f(y,a)=\Sigma_n b_n\partial^n\left(e^{ya}-e^{yf(a)}\right)/\partial a^n=0.$$

Let $e^n(y,a)=\partial^n\left(e^{ya}-e^{yf(a)}\right)/\partial a^n$ be the gap of order n:

$$\theta_f(y,a)=\Sigma_n b_n e^n(y,a)=0$$

At $a=0$: $\theta_f(y)=\Sigma_n b_n e^n(y,0)\equiv 0$.

As $\theta_f(y,a)\equiv 0$ is an identity, $\partial^p\partial^q\theta_f(y,a)/\partial y^p\partial a^q\equiv 0$ for all p and q.

■ If the random variable $X\in C\subset\mathbb{R}^d$ has a measure *P* with density $p(x)$, the translated random variable $X+a$ has the same translated density. Using the convergent series: $\phi(y)=\Sigma_n b_n y^n$, we have for all small translation $a\in C\subset\mathbb{R}^d$ of the random vector *X*, the translated density:

$$p(x-a)=\mathcal{L}^{-1}\left(e^{ta}\phi(t)\right)=\mathcal{L}^{-1}\left(\Sigma_n b_n t^n e^{ta}\right).$$

So, we can write $p(x-a)$ as a distribution in the sense of Schwartz:

$$p(x-a)=\Sigma_n b_n\partial^n\delta(x-a)/\partial a^n.$$

$$\text{As } E\left(e^{y(X+a)}\right)=\Sigma_n b_n y^n e^{ya}=\Sigma_n b_n\partial^n e^{ya}/\partial a^n.$$

$$\text{And } E\left(e^{yf(X+a)}\right)=\int e^{yf(x+a)}\mathrm{d}P(x)=\int e^{yf(x)}p(x-a)\mathrm{d}x$$

$$E\left(e^{yf(X+a)}\right)=\Sigma_n b_n\partial^n\left(\int e^{yf(x)}\delta(x-a)\mathrm{d}x\right)/\partial a^n$$

$$E\left(e^{yf(X+a)}\right)=\Sigma_n b_n\partial^n e^{yf(a)}/\partial a^n.$$

By difference, we get: $\theta_f(y,a)=E\left(e^{y(X+a)}\right)-E\left(e^{yf(X+a)}\right)=0$. ■

Remarks

- We observe that $\partial^n\left(e^{yf(a)}\right)/\partial a^n=H_n(y,a)e^{yf(a)}$ where $H_n(y,a)$ is a Bell-polynomial in *y* with degree *n*. We can note the gap:

$$e^n(y,a)=\partial^n\left(e^{ya}-e^{yf(a)}\right)/\partial a^n=y^n e^{ya}-H_n(y,a)e^{yf(a)}$$

And: $e^n(y,0)=y^n-H_n(y)$ is a polynomial in *y* with degree *n*.

- We obtain $\phi_f(y)$ by putting $H_n(y)$ instead of y^n in the series of $\phi(y)=\Sigma_n b_n y^n$.
- We study the problem near a fixed point 0 of f : $f(0)=0$, then: $\phi_f(0)=\phi(0)=1$, and: $b_0=1$. But, the other b_n are unknown.

2.2. Consequences

The general solution of the linear equation $\theta_f(y)=0$ has the form $b\varphi(y)$ where b is an arbitrary constant real. So, we can write $\phi(y)=1+b\varphi(y)$ with the arbitrary constant b. It means that $\phi(y)=1$, for all $\varphi(y)=0$. We have a lattice distribu-

tion of probability for $\varphi(y)=0$.

The solution of the Perron-Frobenius's equation is a particular case of the equation $\theta_f(y)=0$.

First, we show the effect of an iteration on $e^n(y,0)=0$ and on $\theta_f(y,0)=0$.

Proposition

Iteration $a_\ell \mapsto f_\ell(a)$ *acts as a derivation on* $\theta_f(y,0)=0$ *and on* $e^n(y,0)=0$ *or on* $H_n(y,0)=0$ *in the sense:*

$$a_\ell \mapsto f_\ell(a) \Rightarrow \theta_f(y,0) \mapsto \partial\theta_f(y,a)/\partial a_\ell\big|_{a=0}$$

$$\textit{and } e^n(y,0)=0 \mapsto e^{n+1_\ell}(y,0)$$

$f^{(n)} \mapsto f^{(n+1)}$ *induces* $e^n(y,0) \to e^{n+1}(y,0)$ *or* $H_n(y,0) \to H_{n+1}(y,0)$.

By induction, all the coordinates of $n \in N^d$ *in* $e^n(y,0)$ *are:* $n_1 = \cdots = n_\ell = \cdots = n_d$.

■ The demonstration in [1] is based on the mean's formula for $a \to 0$.

For example, we study the impact of $a_\ell \mapsto f_\ell(a)$ on $D=\theta_f\left(y,\overline{a_\ell}, f_\ell(a)\right)$.

As $\theta_f\left(y,\overline{a_\ell},a_\ell\right)=0$:

$$D=\theta_f\left(y,\overline{a_\ell}, f_\ell(a)\right)-\theta_f\left(y,\overline{a_\ell},a_\ell\right)$$

$$\text{So: } D=\left(f_\ell(a)-a_\ell\right)\left(\partial\theta_f\left(y,\overline{a_\ell},a_\ell+r\left(\left(f_\ell(a)-a_\ell\right)\right)\right)\Big/\partial a_\ell\right)$$

$$\text{When } a \to 0 \quad f_\ell(a)-a_\ell \sim a_\ell(\lambda_\ell-1)$$

$$D \sim a_\ell(\lambda_\ell-1)\left(\partial\theta_f(y,0)/\partial a_\ell\right)$$

Then, if we iterate f, that means $a \mapsto f(a)$, we obtain:

$$D \sim \prod_{\ell=1}^{\ell=d} a_\ell(\lambda_\ell-1)\left(\partial\theta_f(y,0)/\partial a\right).$$

If this quantity is null for all *a*, then $\partial\theta_f(y,0)/\partial a=0$. For similar raisons, if $e^n(y,a)=0$, then: $e^n\left(y, f_\ell(a)\right) \sim a_\ell(\lambda_\ell-1)\partial e^n(y,0)/\partial a_\ell$. ■

3. Solution of the Resolvent R_0

Now, we choose a sufficiently large index $n \in N^d$, with $n=n_1=\cdots=n_\ell=\cdots=n_d$

Lemma

For a fixed $b_n \neq 0$, *under non-resonance conditions, if a solution of* $\theta_n^*(y)=\Sigma_{m\le n} b_m^* e^m(y)=0$ *exists, the zeros of* $e^n(y)$ *are zeros of* $\theta_n^*(y)$.

■ The solution of this equation is obtained as the following:

We choose a sufficiently large index $n \in N^d$ such as: $\varphi_n(y) \to \varphi(y)$ uniformly.

Then: $\theta_{nf}(y)=\Sigma_{m\le n} b_m e^m(y)$ verifies uniformly:

$$\left|\theta_{nf}(y)-\theta_f(y)\right|<\epsilon.$$

As $\theta_f(y)=0$, we search an approximation $\theta_n^*(y)=0$, and estimators b_m^* such as we have:

$$\theta_n^*(y)=\Sigma_{m\le n} b_m^* e^m(y)=0.$$

- For $y = 0$: $b_0 = 1$. As $\theta_n^*(y)$ is a polynomial, the condition $\theta_n^*(y) = 0$ implies ether all the coefficients of $\theta_n^*(y)$ are null or the solution is valid only for the y verifying $\theta_n^*(y) = 0$. But, as the term of highest degree of $\theta_n^*(y)$ is: $(1-\lambda^n)b_n y^n$, we must have, under non-resonance conditions and for all $b_n \neq 0$, $e^n(y) = 0$. (Because all the other gaps $e^m(y)$ have a lower total degree for all $m < n$).

Then, zeros of $e^n(y)$ are zeros of $\theta_n^*(y)$. ■

Theorem

Under the non-resonance condition, we can find a unique convergent solution of $\theta_n^*(y) = \Sigma_{m \le n} b_m^* e^m(y) = 0$, *up to an arbitrary constant b*:

$$\phi_n(y) = 1 - be^n(y).$$

We obtain a lattice distribution defined by the zeros of $e^n(y)$.

In the repellent case where $\lambda^n \gg 1$, *we have*:

$$\phi_n(y) \sim 1 - bH_n(y).$$

Then, the distribution of the real zeros of the polynomials $H_n(y)$ *gives the distribution of the Perron-Frobenius's measure when* $n \to \infty$.

We obtain a lattice distribution defined by the zeros of $H_n(y)$.

■ We note the polynomials $\phi_n^*(y) = 1 + \Sigma_{0<m\le n} b_m^* y^m = 1 + b_n \varphi^n(y)$.

$$\text{and } \phi_{fn}^*(y) = 1 + \Sigma_{0<m\le n} b_m^* H_m(y)$$

$$\text{So: } \theta_n^*(y) = \phi_n^*(y) - \phi_{fn}^*(y).$$

- We search a solution under the condition $e^n(y) = 0$.

1) Now, for all y verifying $e^n(y) = 0$, can we find a solution of $\theta_n^*(y) = 0$?

For all $m < n$, we note:

$$A_{n-1}^*(y) = \Sigma_{m<n} b_m^* e^m(y) = \theta_n^*(y) - b_n e^n(y).$$

If $\theta_n^*(y) = 0$:

$$A_{n-1}^*(y) = \Sigma_{m<n} b_m^* e^m(y) = -b_n e^n(y) = -(1-\lambda^n) b_n y^n - b_n \Sigma_{0<k<n} h_{nk} y^k$$

Where all the coefficients of $e^m(y) = y^m - H_m(y)$ are known because $H_m(y) = \Sigma_{0<k\le m} h_{mk} y^k$ is defined by the coefficients of the Bell's polynomials.

So, we study in $A_{n-1}^*(y)$ all the terms of y^m with degree $m < n$:

$$A_{n-1}^*(y) = \Sigma_{m<n} b_m^* \left(y^m - \Sigma_{0<k\le m} h_{mk} y^k\right) = -b_n \Sigma_{0<k<n} h_{nk} y^k$$

for a fixed arbitrarily $b_n^* = b_n \neq 0$.

2) We obtain a finite triangular system of linear equations: it can be solved step by step and we can identify in a unique way all the unknown coefficients b_m^* in function of the fixed b_n and the coefficients h_{mk} of $H_m(y) = \Sigma_{0<k\le m} h_{mk} y^k$ with $m \le n \in N^d$.

3) This solution is unique for all $b_n^* = b_n \neq 0$ arbitrarily fixed, near to the solution of $\theta_f(y) = 0$, as the b_m^* converge to the b_m. So, we have constructed the polynomials $\phi_n^*(y) - 1$ and $\phi_{fn}^*(y) - 1$ and we can write $\phi_n^*(y) = 1 + b_n \varphi^n(y)$ where b_n is arbitrary. That means $\phi_n^*(y) = 1$ when $\varphi^n(y) = 0$; then, we can choose now $\varphi^n(y) = e^n(y)$. ■

Different cases can happen according to $\lambda^n \gg 1$ or $\lambda^n \ll 1$.

If all the coordinates of $|\lambda|$ are less than 1, the process converges to the fixed point.

If some of them are less than 1, but others are greater than 1, we have a hyperbolic situation under no resonance conditions.

When $y^n \lambda^n \gg y^n$, we can write for large *n*:

$$\phi_n(y) \sim 1 - bH_n(y). \blacksquare$$

And now we have to study the zeros of $H_n(y)$.

Remark (demonstration in [1])

Under the condition that the set *C* is rectangular, if $q(y)$ is the density of real zeros of $H_n(y)$ when $n \to \infty$, then the invariant density $p(x)$ of the Perron-Frobenius's measure is:

$$p(x) = (-x)\partial q(x)/\partial x.$$

4. Study of the Zeros of $H_n(y)$ in the Repellent Case

We suppose f is C^∞, without resonance, and $C \subset \mathbb{R}^d$ bounded. The problem is reduced to find the asymptotic distribution of the zeros of $H_n(y)$ in the repellent case. Here, the distribution of the f-invariant measure *P* is given in general by the distribution of the real zeros of $H_{n-1}(y)$ when $n \to \infty$.

4.1. The Plancherel-Rotach's Method

We will see soon that all the real zeros of $H_n(y)$ are distinct when the steepest descent's method [3] can be applied to $H_n(y)$ and we get an estimation of the asymptotic distribution of these real zeros.

- First, we use the steepest descent's method as Plancherel and Rotach do [4]. We recall that the polynomial:

$$H_{n-1}(y) = e^{-yf(a)} \partial^n e^{yf(a)} / \partial a^n \Big|_{a=0} = \partial^n e^{yf(a)} / \partial a^n \Big|_{a=0}$$

can be represented by the Cauchy's integral:

$$H_{n-1}(y) = K \oint_\Gamma \frac{e^{yf(a)}}{a^n} \mathrm{d}a = K \oint_\Gamma e^{yf(a) - n\ln a} \mathrm{d}a$$

where Γ is a closed polydisk around the fixed point 0 of f, $a \in \mathbb{C}^d$, *K* is a finite non-null function, without importance in the context [3]. We take

$n = n_1 = n_\ell = n_d$.

We note the integrand $n\gamma(a) = yf(a) - n\ln a$

And we call $\gamma(a)$ *the Plancherel-Rotach's function*.

- Second, with Plancherel and Rotach, we use the steepest descent's method. We search the critical point of $\gamma(a)$. Under the numerous conditions of the general position, the critical point a_c maximizing $e^{n\gamma(a)}$ gives the solution. *The critical point* a_c is defined by the equation:

$$n\partial\gamma(a)/\partial a = y\partial f(a)/\partial a - n/a = 0.$$

(A sufficient condition to get this maximum is that the hessian matrix of $\gamma(a)$, which is Hermitian, is definite negative at a_c). Let $s = y/n$ with $s_\ell = y_\ell/n$, then:

$$\partial\gamma(a)/\partial a = s\partial f(a)/\partial a - 1/a = 0 .$$

The critical point must be isolated from the other critical points and at a finite distance. Some coordinates of a_c can be real, the others are complex. Then:

$$H_{n-1}(y) = K'\left(\exp(\gamma(a_c)) - \exp\left(\gamma\left(\overline{a_c}\right)\right)\right).$$

($\overline{a_c}$ is the conjugate of a_c). We notice that K' or the real part $\Re e(\gamma)$ of $\gamma(a)$ cannot annul $H_{n-1}(y)$, but, among the solutions, we have to choose $\Re(\gamma)$ maximum. Then, only the imaginary part $\Im\mathrm{m}(\gamma)$ of $\gamma(a)$ can nullify $H_{n-1}(y)$.

Proposition

Under the conditions of the general position, the critical point a_c *of the PR-function* $\gamma(a)$ *gives the real zeros of* $H_n(y)$. *For all complex coordinates* a *of* a_c:

$$n\Im\mathrm{m}(\gamma) = \Im\mathrm{m}(yf(a) - n\ln a) = k\pi .$$

As each iteration f_ℓ *acts as a derivation on* $H_{n-1}(y)$, *we see*:

$$n\Im\mathrm{m}(\gamma_\ell(a)) = \Im\mathrm{m}(s_\ell f_\ell(a) - \ln a_\ell) = \pi k_\ell/n .$$

We obtain asymptotically $\Im\mathrm{m}(\gamma_\ell(a)) \to \pi\kappa_\ell$ *when* $n \to \infty$.

The $k_\ell/n \to \kappa_\ell \in [0,1]$ *have an identic independent uniform distribution on* $[0,1]$.

In the unidimensional case, the repartition of the zeros is:

$$q(s)\mathrm{d}s = \Im\mathrm{m}(f(a))\mathrm{d}s/\pi .$$

■ We can tie these distributions of the PF-equation to each fixed point $f(0) = 0$. Then, we have local solutions. All these distributions can be masked in various situations. The principle of the maximum of the real part $\Re e(\gamma)$ of $\gamma(a)$ provides a method to define the fuzzy frontiers of the different domains of attraction.

As everybody knows, the steepest descent's method is difficult to use, but it indicates a very large variety of behaviors [3].

In the case of unidimensional function, the repartition of the zeros verifies:

$$\Im\mathrm{m}(\gamma(a)) = \kappa\pi \quad \text{with} \quad \partial\gamma/\partial a = 0$$

$$\text{So:} \quad q(s)\mathrm{d}s = \mathrm{Prob}\{1 \text{ zero} \in (s, s+\mathrm{d}s)\} = \mathrm{d}\kappa$$

$$q(s)\mathrm{d}s = \Im\mathrm{m}(\mathrm{d}\gamma/\mathrm{d}s)\mathrm{d}s/\pi = \Im\mathrm{m}(f(a))\mathrm{d}s/\pi$$

$$\text{Because:} \quad \mathrm{d}\gamma/\mathrm{d}s = \partial\gamma/\partial s + \partial\gamma/\partial a \cdot \partial a/\partial s = f(a). \blacksquare$$

Remark

If the n_ℓ are not equal, we take $\mu = n_1 + \cdots + n_\ell + \cdots + n_d$, and we fix:

$z_\ell = n_\ell / \mu$. Then, if $y_\ell = n_\ell s_\ell = \mu z_\ell s_\ell$ and $n_\ell \ln a_\ell = \mu z_\ell \ln a_\ell$, the Plancherel-Rotach's function is:

$$\mu\gamma(a) = \mu\Sigma_\ell z_\ell \left(s_\ell f_\ell(a) - \ln a_\ell\right) = \mu\Sigma_\ell z_\ell \gamma_\ell(a)$$

If $n = n_1 = \cdots = n_\ell = \cdots = n_d$, we have: $z_\ell = 1$.

4.2. Real or Imaginary Coordinates of the Solutions

The reality or the imaginary of the coordinates of a_c may vary with the orientation of *y*. The right framework to analyze this question seems to be the Morse's theory.

4.3. Examples

- Let the logistic map [4]: $f(a) = \lambda a - a^2/2$; and $\gamma(a) = s\left(\lambda a - a^2/2\right) - \ln a$;

$$\partial\gamma/\partial a = s\left(\lambda a - a^2\right) - 1 = 0$$

we put $\lambda\sqrt{s} = 2\cos\vartheta$, we have: $2\cos\vartheta a\sqrt{s} - sa^2 - 1 = 0$ with roots: $a\sqrt{s} = e^{\pm i\vartheta}$

$$\text{and: } \Im\text{m}\left(f(a)\right) = \Im\text{m}\left(\lambda a - a^2/2\right) = \sin 2\vartheta / s.$$

$$q(s)\,\mathrm{d}s = \Im\text{m}\left(f(a)\right)\mathrm{d}s/\pi = (1 - \cos 2\vartheta)\,\mathrm{d}\vartheta/\pi$$

$$\text{So: } q(s) = (\lambda/2\pi)\sqrt{1/s - \lambda^2/4}.$$

If we put $t = \cos\vartheta = \dfrac{\lambda\sqrt{s}}{2}$

Then *t* follows: $W(t)\,\mathrm{d}t = (2/\pi)\sqrt{1-t^2}\,\mathrm{d}t$.

We recover directly a well-known result: Let $H_n(y,a) = \partial^n\left(e^{y\left(\lambda a - a^2/2\right)}\right)\Big/\partial a^n$ where $e^{y\left(\lambda a - a^2/2\right)}$ is (with easy transformations) like the generatrix function $e^{\left(2ta - a^2\right)}$ of the Hermite polynomials $H_n(t)$. The law of the zeros of $H_n(x)$ is known as the semi-circular Wigner's law: $W(t)\,\mathrm{d}t = (2/\pi)\sqrt{1-t^2}\,\mathrm{d}t$.

- Then, the density of the logistic corresponding to *q*(*s*) is:

$$p(s) = -s\mathrm{d}q/\mathrm{d}s = -s(2/\pi)\mathrm{d}\left(\frac{\lambda}{4\sqrt{s}}\sqrt{1 - \frac{s\lambda^2}{4}}\right)\Bigg/\mathrm{d}s$$

$$p(s) = \lambda\Big/\left(2\pi\sqrt{4s - s^2\lambda^2}\right).$$

We deduce that the density of the logistic map follows a Beta (1/2, 1/2) low in a more general situation than in the Ulam-Von Neumann's case [5].

- The map $a_1 = \lambda a + \dfrac{a^2}{2}$, for $\lambda > 1$. We have neglected this important case in our previous papers. In general, this iteration tends to infinite. The corresponding Hermite polynomials $H_n(x)$ are always positive except if $x = 0$ for the odd index *n*. It seems that this iteration can serve as a parameter in multidimensional case. So, we will say that "$x = 0$ half the time" and arbitrary for even indexes *n*.

- m-Hermitian case: $f(a) = \lambda a - a^m/m$.

  The Plancherel-Rotach's function is: $\gamma(a) = s(\lambda a - a^m/m) - \ln a$.

  With the critical point a defined by the trinomial equation:

$$\mathrm{d}\gamma(a)/\mathrm{d}a = s(\lambda a - a^m) - 1 = 0$$

studied by H. Fell.

Consequence

We take now a quadratic function f in $\mathbb{R}^d$ with $f(0) = 0$. We write the PR function $\gamma(a)$ for every fixed point 0 of $f(a)$:

$$f(a) = \lambda a + Qa^2/2$$

the hessian of sQ is symmetric. For all s such as sQ is non-degenerate, it exists an orthogonal transformation T: $a = Tu$, with: $T'sQT = D$, the diagonal matrix of eigenvalues of sQ and:

$$\begin{aligned}\ln a &= \sum_{\ell=1}^{\ell=d} \ln a_\ell = \ln \prod_{\ell=1}^{\ell=d} a_\ell = \ln \mathrm{Vol}(a) \\ &= \ln \mathrm{Vol}(u) = \ln \prod_{\ell=1}^{\ell=d} u_\ell = \ln u = \sum_{\ell=1}^{\ell=d} \ln u_\ell.\end{aligned}$$

Because the volume $\mathrm{Vol}(a) = \mathrm{Vol}(u)$ is invariant under an orthogonal transformation.

We note $D_\ell = K_\ell^2$ if $\ell = 1, \cdots, p$ and $D_\ell = -K_\ell^2/2$ if $\ell = p+1, \cdots, d$.

Then, the P.R. function $\gamma(a)$ becomes:

$$\begin{aligned}\gamma(u) &= sf(Tu) - \ln Tu = s\lambda Tu + Du^2/2 - \ln u \\ &= \sum_{\ell=1}^{\ell=d} \Lambda_\ell u_\ell + \sum_{\ell=1}^{\ell=p} K_\ell^2 u_\ell^2/2 - \sum_{\ell=p+1}^{\ell=d} K_\ell^2 u_\ell^2/2 - \sum_{\ell=1}^{\ell=d} \ln u_\ell \\ &= \sum_{\ell=1}^{\ell=p} \left(\Lambda_\ell u_\ell + K_\ell^2 u_\ell^2/2 - \ln u_\ell\right) + \sum_{\ell=p+1}^{\ell=d} \left(\Lambda_\ell u_\ell - K_\ell^2/2u_\ell^2 - \ln u_\ell\right)\end{aligned}$$

where $\Lambda u = s\lambda Tu$.

If we note: $\gamma_+(u_\ell) = \Lambda_\ell u_\ell + K_\ell^2 u_\ell^2/2 - \ln u_\ell$

and $\gamma_-(u_\ell) = \Lambda_\ell u_\ell - K_\ell^2 u_\ell^2/2 - \ln u_\ell$

So: $\gamma(u) = \sum_{\ell=1}^{\ell=p} \gamma_+(u_\ell) + \sum_{\ell=p+1}^{\ell=d} \gamma_-(u_\ell)$.

And, applying the logistic calculus to each $\gamma_+(u_\ell)$ and $\gamma_-(u_\ell)$, we obtain p conditions $\Lambda_\ell u_\ell = 0$ half the time and *d-p* random independent variables following a Beta (1/2, 1/2) low. But, we may have other fixed points: $a(1-\lambda) = Qa^2$.

Remark

- We can extend these results to a C^∞ function f with the Morse-Palais Lemma as in [6], (p.174 et seq.), if the hessian is definite.

5. A differential Equation as a Repellent Iteration

We consider ordinary differential equation [1]:

$$\mathrm{d}a/\mathrm{d}t = F(a)$$

where $a \in C \subset \mathbb{R}^d$ or $\mathbb{C}^d$, $t \in \mathbb{R}^+$, $F(a)$ is a C^∞-application of $a \in C$ in C. The domain C is supposed bounded. The problem is to find a function $a(t)$ verifying this equation with an initial condition: $a(t_0) = a_0$.

We use the theorical solution of Caratheodory $a(t)$ for $t > t_0$:

$$a(t) = a_0 + \int_{t_0}^{t} F(a(u))\mathrm{d}u = a_0 + S(a_0, a(t)).$$

The differential iteration

We associate the differential iteration $f(a)$ belonging in the bounded domain *C*:

$$f(a) = a + \delta F(a)$$

where $\delta = t/n$ is the path. When we iterate n times, we have:

$$a_n = f^{(n)}(a(t_0)).$$

The method gives the solution a_n by iterating *n* times $f(a)$ from a starting point $a(t_0)$ with the path $\delta = t/n$ and this solution $a_n \to a(t)$ when $n \to \infty$:

For $n > p$: $a_n = f^{(n)}(a_0) = a(t_0) + \delta\left(\sum_{p=0}^{p=n-1} F(a_p)\right)$ $a_n = a(t_0) + \delta S_n(a_0)$.

Then, when $n \to \infty$: $\lim_{n\to\infty} f^{(n)}(a_0) = a(t) = a_0 + \int_{t_0}^{t} F(a(u))\mathrm{d}u$

with $\delta S_n(a_0) \to S(a(t), a_0) = \int_{t_0}^{t} F(a(u))\mathrm{d}u$.

The fixed points of a differential iteration are the zeros α of *F*: $F(\alpha) = 0$.

5.1. The Invariant Measure of a Differential Iteration

Now, we submit a probabilistic version of the Poincaré-Bendixon's problem in $\mathbb{R}^d$.

Proposition

Under the previous hypothesis, all the non-null measures verify:

$$E\left(\int_0^1 yS(a_0, X(t)) e^{yX + vyS(a_0 X(t))}\mathrm{d}v\right) = 0.$$

Then, we have asymptotic random cycles around each fixed point. For all these cycles, the times of return in each very small borelian set around a point of a cycle are constant in probability. Along each cycle, the conditional probability has a constant density.

■ With $f(a) = a + \delta F(a)$ for every measurable function *F*. Then, for $a_n = f^{(n)}(a_0) = a(t_0) + \delta\left(\sum_{p=0}^{p=n-1} F(a_p)\right)$ with $\delta = t/n$, we must have the resolving equation in the neighborhood each fixed point for one or *n* iterations:

$$\phi(y) = E(e^{yX}) = E(e^{yf(X)}) = E\left(e^{yf^{(n)}(X)}\right).$$

That means, especially for $f^{(n)}$:

$$\theta_{nf}(y) = E\left(e^{yX} - e^{yf^{(n)}(X)}\right) = E\left(e^{yX}\left(1 - e^{y\delta S_n(a_0, X)}\right)\right) = 0$$

$$\text{As: } \delta S_n(a_0) \to \int_{t_0}^{t} F(a(u))\mathrm{d}u = S(a_0, a(t)).$$

By continuity:

$$\theta_{nf}(y) \to E\left(e^{yX}\left(1 - e^{yS(a_0, X(t))}\right)\right) = E\left(\int_0^1 \mathrm{d}\left(e^{yX + vyS(X(t))}\right)\Big/\mathrm{d}v\,\mathrm{d}v\right) = 0$$

But $\theta_f(y)=E\left(e^{yX}-e^{yf^{(n)}(X)}\right)\to E\left(e^{yX}-e^{y(X(t))}\right)=0$.

In consequence, if $E\left(\int_0^1\left(\mathrm{d}\left(e^{yX+vyS(X(t))}\right)/\mathrm{d}v\right)\mathrm{d}v\right)=0$, we have non-null measures verifying $\theta_f(y)=0,\forall y$. In other words, $a(t)=a(t_0)$ for the invariant non-null measure and some t_0. Under this condition, the asymptotic behavior is random periodic cycles with an unknown almost period $T=t-t_0$.

But, when we have many fixed points, the complete solution is more difficult because we meet some problems with domains of domination (see Section 4) and transitions from a domain of a fixed point to an another. ■

Remarks

- Theoretically, if we know the probability's measure, we can define some statistics (mean, standard deviation…).
- We can try to extend these results to PDE equations and obtain other new results, as in the following:

Let the PDE: $\partial a/\partial u=F(a)$

Where $a\in C\subset\mathbb{R}^d$ or $\mathbb{C}^d$, $u\in\mathbb{R}^p$ with $d>p$.

After transformation of the PDE into iterations, suppose that one can use the Caratheodory's solution for the PDE:

$$a(u)=a(u_0)+\int F(a(v))\mathrm{d}v.$$

And we see that the only asymptotic solutions for a non-null measure are periodic cycles with the unknown almost period $T=u-u_0$.

5.2. Examples

1) Suppose that F has a hessian definite negative, then, when $\delta\to 0$, it is easy to verify that the critical point verifies: $ay=1$, with an approximation of

$$H_n(y)=\partial^n yF(a)e^{ya+vy\delta f(a)}/\partial a^n\Big|_{a=0}\quad\text{for}\quad\delta\to 0.$$

The critical point a_c is real and we don't have a probabilistic solution.

2) Suppose we have a linearity in b: Let $\boldsymbol{a}=(a,b)$ with $a\in\mathbb{R}^d$ and $b\in\mathbb{R}$.

We write: $\mathrm{d}\boldsymbol{a}/\mathrm{d}t=F(\boldsymbol{a})$ as: with $F(\boldsymbol{a})=(f(a)b+g(a),h(a)b+k(a))$

$$\mathrm{d}a/\mathrm{d}t=f(a)b+g(a);\quad \mathrm{d}b/\mathrm{d}t=h(a)b+k(a)$$

where: $g(0)=k(0)=0$ in order to have $F(\mathbf{0})=\mathbf{0}$.

We write the Plancherel-Rotach's function with then $y=(y,z)$, $y\in\mathbb{R}^d$ and $z\in\mathbb{R}$

$$n\gamma(\boldsymbol{a})=y\left(a+\delta\left(f(a)b+g(a)\right)\right)+z\left(b+\delta\left(h(a)b+k(a)\right)\right)$$

Putting $z=\delta z'$ and $b'=\delta b$ such as $zb=z'b'$ for $\delta>0$, we obtain when $\delta\to 0$:

$$n\gamma(\boldsymbol{a}')\to y\left(a+f(a)b'\right)+z'b'-n\ln b'-n\ln a$$

Where $\boldsymbol{a}'=(a,b')$ (The change $b'=\delta b$ doesn't modify the equation $\theta_f(y)=0$). And the critical point is defined by:

$$\partial\gamma/\partial b' = yf(a)b' + z'b' - n = 0$$

$$\partial\gamma/\partial a = ya + yb'\partial f(a)/\partial a - n = 0$$

$$\text{So: } ya\left(yf(a)+z'\right) + ny\partial f(a)/\partial a - n\left(yf(a)+z'\right) = 0 .$$

The imaginary critical points give the distribution of the cycles. Under general conditions, this distribution doesn't depend on $g(a), h(a), k(a)$ but only on $f(a)$.

6. Critical frequencies

Asymptotically, we have random cycles. Let $a(t)$ be a point on a such asymptotic cycle and a very small invariant borelian around this point. So, we have many large times to return in this borelian. In the differential iteration, we have many and large $\tau = (t+kT)/n$ which give the same $a(t)$ where T is a random quasi-period [6].

Proposition

When the number of iterations $n \to \infty$ *and if the la hessian of yF is definite negative, the approximation with defines* $s = y/n$ *in function of the critical point* a:

$$s + ts\partial F(a)/\partial a - 1/a = 0$$

where $1/a = (1/a_\ell, \ell = 1, 2, \cdots, d)$.

If s_a *is a particular solution and if* $\vec{s}$ *is an eigenvector of* $-\partial F(a)/\partial a$ *for the eigenvalue* $1/t$*, the general solution is disjunctive*:

$$s = s_a \text{ if } t \neq -1/\lambda_a \text{ or } s = \vec{s} \text{ if } t \neq -1/\lambda_a .$$

The eigenvalue $1/t$ *can be interpreted as a critical asymptotic frequency.*

■ Contrary to the previous Section 5, we don't write the critical point a as a function of s, but s as a function of a. For fixed a on an asymptotic cycle, we recognize the linear affine equation of s depending on the parameter t. We have to find a particular solution s_a:

$$s_a + ts_a\partial F(a)/\partial a - 1/a = 0 .$$

$$\text{Formally: } s_a = \left(Id + t\partial F(a)/\partial a\right)^{-1}(1/a) .$$

This solution s_a is valid for all $t \neq -1/\lambda_a$ where λ_a is eigenvalue of $\partial F(a)/\partial a$ at the critical point a. If $t \neq -1/\lambda_a$, s_a is a particular solution of the equation and the general solution will be $s = s_a + \vec{s}$, then:

$$s_a + \vec{s} + t(s_a + \vec{s})\partial F(a)/\partial a - 1/a = 0$$

$$\text{And: } \vec{s} = -t\vec{s}\partial F(a)/\partial a .$$

As $t \neq -1/\lambda_a$, $\vec{s} = 0$. The general solution is disjunctive and shows a discontinuity at the eigenvalues λ_a. ■

Remark: calculation of s_a

s_a is obtained with $\left(Id + \tau\partial F(a)/\partial a\right)^{-1}$ for all $\tau \neq -1/\lambda_a$ which doesn't belong to the spectrum of $-\partial F(a)/\partial a$ with the series development of τ.

Remark: the Fredholm alternative

Here, we have the Fredholm alternative: either we have s_a for all $\tau \neq -1/\lambda_a$ or $\vec{s}$ for $\tau \neq -1/\lambda_a > 0$. Suppose we start with $0 < \tau < -1/\lambda_a$, but τ is increasing: what happens when $\tau \to -1/\lambda_a$? What is the physical interpretation? Can we connect this phenomenon to some physical constants or boundaries?

7. Case Where the Hessian Is Degenerated: The Lorenz's Equation

Generally, the hessian is not definite negative. The Lorenz's equation [7] is a particularly important example because the differential iteration can be broken down into three independent iterations which have a remarkable feature: a partial linearity; an iteration with a negative hessian which induces a probabilistic solution and another with a positive hessian. It is an ideal example to clarify the previous results.

However, as there is an interpenetration of the distributions related to each fixed point, the connection between the various results remains delicate. The probabilistic presentation seems to be the least bad: it gives the probability of presence except at the places where the domination changes; in this case, we go from a basin to an another.

7.1. The Iteration at Its Repellent Fixed Points

The vectors of this equation are written in bold notations:

$$\mathrm{d}\boldsymbol{a}/\mathrm{d}t = F(\boldsymbol{a}) \quad \text{where} \quad \boldsymbol{a} = (a,b,c):$$

$$\mathrm{d}a/\mathrm{d}t = \sigma(b-a)$$

$$\mathrm{d}b/\mathrm{d}t = \rho a - b - ac$$

$$\mathrm{d}c/\mathrm{d}t = -\beta c + ab.$$

The differential equation applies a bounded set C in itself for $\delta > t > 0$ (the phenomenon is occurring between a cold sphere at -50° and hot sphere, the earth, at $+15^\circ$ as the terrestrial atmosphere is modelled by Lorenz).

The differential iteration $\boldsymbol{a}_1 = f(\boldsymbol{a})$ associated with a given path $\delta = t/n$ is:

$$a_1 = a + \delta\sigma(b-a)$$

$$b_1 = b + \delta(\rho a - b - ac)$$

$$c_1 = c + \delta(-\beta c + ab).$$

This iteration is *quadratic*, but has a linearity in a.

We recall the known results concerning the fixed points:

The fixed points are zeros of $F(\boldsymbol{a}) = 0$. If $\rho > 1$ and $\alpha = \sqrt{\beta(\rho-1)}$, it exists three fixed points: the point $\mathbf{0} = (0,0,0)$, and two others symmetric with respect to the axis of *c*:

$$\boldsymbol{\alpha}_+ = (\alpha,\alpha,\alpha^2/\beta) \quad \text{and} \quad \boldsymbol{\alpha}_- = (-\alpha,-\alpha,\alpha^2/\beta).$$

At **0**, the eigenvalue's equation λ of the linear part is:

$$(\beta+\lambda)\left[(\sigma+\lambda)(1+\lambda)-\sigma\rho\right]=0,$$

But, at $\boldsymbol{\alpha}_+$ or at $\boldsymbol{\alpha}_-$:

$$\lambda(\beta+\lambda)(1+\sigma+\lambda)-\alpha^2(2\sigma+\lambda)=0$$

Coefficients β,σ,ρ are such as these three fixed points are repellent; that means we have to study the distributions around each fixed point. We don't speak here about attractive cycles, resonances, and some particular values of the parameters, etc. It remains many points to clarify.

7.2. Analysis of the Hessian

Projecting $f(\boldsymbol{a})$ onto an axis $\boldsymbol{y}=(x,y,z)$, we write:

$$\boldsymbol{y}f(\boldsymbol{a})=L(\boldsymbol{a})+\delta Q(\boldsymbol{a})$$

where $L(\boldsymbol{a})$ is linear for $\boldsymbol{a}$:

$$L(\boldsymbol{a})=x\left(a+\delta\sigma(b-a)\right)+y\left(b+\delta(\rho a-b)\right)+zc(1-\delta\beta)$$

$$L(\boldsymbol{a})=aL_1+bL_2+cL_3$$

with: $L_1=x(1-\delta\sigma)+\delta\rho y$

$$L_2=\delta\sigma x+y(1-\delta)$$

$$L_3=z(1-\delta\beta)$$

and $Q(\boldsymbol{a})$ is quadratic: $Q(\boldsymbol{a})=(zb-yc)a$.

The hessian $Q(\boldsymbol{a})$ is degenerated and not definite negative. But, Q doesn't change when we translate the origin from a fixed point to an another.

First, we examine the matrix of $Q(\boldsymbol{a})$:

$$Q=\begin{bmatrix}0 & z & -y\\ z & 0 & 0\\ -y & 0 & 0\end{bmatrix}$$

Let $\mu=\sqrt{y^2+z^2}$ the positive eigenvalue of the characteristic equation of Q:

$$\mu\left(\mu^2-y^2-z^2\right)=0$$

The matrix of the eigenvectors T is orthogonal and constant for all $\boldsymbol{a}$.

$$T=\frac{1}{\mu\sqrt{2}}\begin{bmatrix}0 & \mu & \mu\\ y\sqrt{2} & -z & z\\ z\sqrt{2} & y & -y\end{bmatrix}$$

Corresponding to the diagonal matrix of the eigenvectors: $\Lambda=\begin{bmatrix}0 & 0 & 0\\ 0 & -\mu & 0\\ 0 & 0 & \mu\end{bmatrix}$.

7.3. Change of Basis Near 0

- We calculate in the basis of eigenvectors directly with the Hermite's polynomials. As T is orthogonal, the transposed T' is also its inverse: $T'=T^{-1}$.

Then, the application $\boldsymbol{u}=T\boldsymbol{a}$ with $\boldsymbol{u}=(u,v,w)$ transforms:

$$yf(\boldsymbol{a}) \mapsto G(\boldsymbol{u}) = yf(T'\boldsymbol{u})$$

$$Q(\boldsymbol{a}) \mapsto Q(T'\boldsymbol{u}) = \delta\mu(w^2 - v^2)$$

$$L(\boldsymbol{a}) \mapsto LT'\boldsymbol{u} .$$

Now, in the basis $\boldsymbol{u}$, the function $yf(T'\boldsymbol{u}) = G(\boldsymbol{u})$ is factorized into three independent functions:

$$G(\boldsymbol{u}) = g_1(u) + g_2(v) + g_3(w)$$

with: $g_1(u) = l_1 u$; $g_2(v) = l_2 v - \delta\mu v^2$; $g_3(w) = l_3 w + \delta\mu w^2$.

Where: $l_1 = (\delta\sigma x + y(1-\delta) + z(1-\delta\beta))/\sqrt{2}$

$$l_2 = (x - \delta\sigma x + \delta\rho y)y/\mu - (\delta\sigma x + y(1-\delta) - z(1-\delta\beta))z/\mu\sqrt{2}$$

$$l_3 = (x - \delta\sigma x + \delta\rho y)z/\mu - (\delta\sigma x + y(1-\delta) - z(1-\delta\beta))y/\mu\sqrt{2}$$

- To calculate l_1, l_2 et l_3, we form:

$$L(a) = a(x - \delta\sigma x + \delta\rho y) + b(\delta\sigma x + y(1-\delta)) + zc(1-\delta\beta)$$

with: $L_1 = x(1-\delta\sigma) + \delta\rho y$; $L_2 = \delta\sigma x + y(1-\delta)$; $L_3 = z(1-\delta\beta)$

Then: $l\boldsymbol{u} = (l_1, l_2, l_3)\boldsymbol{u} = LT'\boldsymbol{u} = (L_1, L_2, L_3)\dfrac{1}{\mu\sqrt{2}}\begin{bmatrix} 0 & y\sqrt{2} & z\sqrt{2} \\ \mu & -z & y \\ \mu & z & -y \end{bmatrix}\boldsymbol{u}$.

- We get 3 independent iterations:

. the first iteration g_1 is linear;

. the second g_2 is a random iteration;

. the third g_3 remains positive, except if $l_3 = 0$ half the time.

- Let the resolving gap $e^n(\boldsymbol{y}) = \partial\left(\partial^n\left(e^{yf(\boldsymbol{a})}\right)/\partial\boldsymbol{a}^n\right)\partial\delta\Big|_{a=0} = 0$

For $\forall t \le \delta$, putting $\boldsymbol{a} = T'\boldsymbol{u}$, we have:

$$e^n(\boldsymbol{u}) = T^n\partial\left(\partial^n\left(e^{yf(T'\boldsymbol{u})}\right)/\partial\boldsymbol{u}^n\right)\partial\delta\Big|_{\boldsymbol{u}=0} = 0$$

$$\partial^n\left(e^{yf(T'\boldsymbol{u})}\right)/\partial\boldsymbol{u}^n = \partial^n\left(e^{g_1(u)}\right)/\partial u^n \cdot \partial^n\left(e^{g_2(v)}\right)/\partial v^n \cdot \partial^n\left(e^{g_3(w)}\right)/\partial w^n .$$

This gives: $\partial^n\left(e^{g_1(u)}\right)/\partial u^n = l_1^n e^{g_1(u)}$

$$\partial^n\left(e^{g_2(v)}\right)/\partial v^n = H_n(g_2(v))e^{g_2(v)}$$

$$\partial^n\left(e^{g_3(w)}\right)/\partial w^n = H_n(g_3(w))e^{g_3(w)}$$

And: $e^n(\boldsymbol{u}) = \partial l_1^n H_n(g_2(v))H_n(g_3(w))\left(e^{yf(T'\boldsymbol{u})}\right)\partial\delta\Big|_{u=0} = 0$.

Proposition

The solution around the fixed point $\mathbf{0}$ *consists of the intersection of the family of random surfaces defined by:*

$$l_2/2\sqrt{\mu} \mapsto low\ \beta(1/2, 1/2)$$

with the surfaces $\sigma x - y - z\beta = 0$ *and* $(-\sigma x + \rho y)z + (\sigma x - y + z\beta)y/\sqrt{2} = 0$.

■ With the same calculations of encodings and interchanging the derivations, we have:

$$\partial l_1^n/\partial\delta = 0;\ \partial H_n(g_2(v))/\partial\delta = 0;\ \partial H_n(g_3(w))/\partial\delta = 0$$

We study separately the three expressions:

- First: $\partial l_1^n / \partial \delta = n(\partial l_1/\partial \delta) l_1^{n-1} = 0$.

  Either $\partial l_1/\partial \delta = \sigma x - y - z\beta = 0$, or: $l_1 \sim (y+z)/\sqrt{2} = 0$

- Second: the polynomial $H_n(g_3(w))$ when $w=0$ is a Hermite's polynomial $H_n(x)$ where x is $x = il_3/(\sqrt{2\delta\mu})$. This polynomial $i^n H_n(il_3/(\sqrt{2\delta\mu}))$ is always positive half the time. In a general way:

$$\partial H_n(x)/\partial\delta = nH_{n-1}(x)\partial x/\partial\delta = 0\text{. So: } \mathrm{d}\left(l_3/\sqrt{2\delta\mu}\right)/\mathrm{d}\delta = 0,$$

And $l_3 \sim \left(xz\sqrt{2}+(y-z)y\right)/\mu\sqrt{2} = 0$ half the time.

- Third: in the case of $H_n(g_2(w))$, in addition to the solution $l_2 = 0$, we have to find the possible invariant distribution of $H_n\left(l_2/(\sqrt{2\delta\mu})\right) = 0$.

  Let the integrand of $n\gamma(w) = g_2(w) - n\ln w$.

  When $\delta \to 0$, $l_2 \sim \left(xy\sqrt{2}+(y-z)z\right)/\sqrt{2}\mu$ with $\mu = \sqrt{y^2+z^2}$.

By normalization of the coordinates $\boldsymbol{x} = (x,y,z) = \delta n\boldsymbol{s} = (\delta nr, \delta ns, \delta nt)$, we obtain:

$$l_2 \sim n\delta\left(rs\sqrt{2}+(s-t)t\right)/2\left(s^2+t^2\right)^{\frac{1}{2}} = n\delta l_2(\boldsymbol{s})$$

$$\delta\mu = n\delta^2\left(s^2+t^2\right)^{1/2} = n\delta^2\mu(\boldsymbol{s})$$

$$n\gamma(v) = n\left(\delta l_2(\boldsymbol{s})v - \mu(\boldsymbol{s})(\delta v)^2 - \ln\delta v + \ln\delta\right)$$

Putting $\delta v = v$, we have: $n\gamma(v) = n\left(l_2(\boldsymbol{s})v - \mu(\boldsymbol{s})v^2 - \ln v\right)$.

We search the critical point: $\mathrm{d}\gamma(v)/\mathrm{d}v = l_2(\boldsymbol{s}) - 2\mu(\boldsymbol{s})v - 1/v = 0$

The imaginary roots are: $v(\boldsymbol{s}) = l_2(\boldsymbol{s})/4\mu(\boldsymbol{s}) \pm i\sqrt{1/2\,\mu(\boldsymbol{s}) - l_2(\boldsymbol{s})^2/16\mu(\boldsymbol{s})^2}$.

Under the condition: $l_2(\boldsymbol{s})^2 < 8\mu(\boldsymbol{s})$:

$$l_3 \sim \left(rt\sqrt{2}+(s-t)s\right)/\mu(\boldsymbol{s})\sqrt{2} = 0 \quad \text{half the time,}$$

Implies: $l_2(\boldsymbol{s}) = -(s-t)^2/\sqrt{2}\left(s^2+t^2\right)^{1/2}$

The condition becomes: $(s-t)^4/\left(s^2+t^2\right)^{3/2} < 16$

$l_1 = 0$ implies $s+t=0$, then: $s<8$.

In any case, we observe that the conditions $l_3 = l_1 = 0$ allow us to express r et t depending on s and we can write that the density of zeros of s is now:

$$q(s)\mathrm{d}s = \mathrm{Prob}(1 \text{ zero between } s, s+\mathrm{d}s) = \left|\Im m f(v(s))\right|\mathrm{d}s/\pi$$

$$q(s)\mathrm{d}s = l_2(\boldsymbol{s})\sqrt{8\mu(\boldsymbol{s}) - l_2(\boldsymbol{s})^2}/8\pi\mu(\boldsymbol{s})\mathrm{d}s = \mathrm{d}\kappa.$$

Then, κ follows a uniform low on (0, 1) with: $s+t=0$ (or: $\sigma x - y - z\beta = 0$) and: $xy\sqrt{2}+(y-z)z = 0$.

We also remark that the normalization doesn't affect the coefficients of the orthogonal matrix:

$$T(x,y,z) = T(\delta nr, n\delta s, n\delta t) = T(r,s,t). \blacksquare$$

7.4. Analysis near α_+ and α_-

We now verify similar results the two other fixed points $\boldsymbol{\alpha}_+$ and $\boldsymbol{\alpha}_-$.

We search the distributions around the two other fixed points. To pass from the fixed point **0** to the fixed point $\boldsymbol{\alpha}_+$ or $\boldsymbol{\alpha}_-$, we have just to put in the iteration instead of $\boldsymbol{a}=(a,b,c)$: $\boldsymbol{a}'+\boldsymbol{\alpha}_+=(a'+\alpha,b'+\alpha,c'+\alpha^2/\beta)$ or $\boldsymbol{a}''-\boldsymbol{\alpha}_+=(a''-\alpha,b''-\alpha,c''+\alpha^2/\beta)$.

- *Calculation for* $\boldsymbol{\alpha}_+$

So, for $\boldsymbol{a}'+\boldsymbol{\alpha}_+=\boldsymbol{a}$; $\boldsymbol{a}_1'=\boldsymbol{a}'+\boldsymbol{\alpha}_+$ and $\boldsymbol{a}_1=f(\boldsymbol{a})$ where $\boldsymbol{a}_1=(a_1,b_1,c_1)$ becomes

$$\boldsymbol{a}_1=\boldsymbol{a}_1'+\boldsymbol{\alpha}_+=f(\boldsymbol{a})=f(\boldsymbol{a}'+\boldsymbol{\alpha}_+);$$

then: $\boldsymbol{a}_1'=\boldsymbol{a}'+\delta F(\boldsymbol{a}'+\boldsymbol{\alpha}_+)$

As: $F(\boldsymbol{a})=(\sigma(b-a),(\rho a ? b-ac),(-\beta c+ab))$

$\boldsymbol{a}_1=f(\boldsymbol{a})$ becomes for $\boldsymbol{a}+\boldsymbol{\alpha}_+$:

$$a_1'=a+\delta\sigma(b-a)=a_1$$

$$b_1'=b+\delta(\rho a-b-ac)+\delta(-\alpha c-a\alpha^2/\beta)=b_1+\delta(-\alpha c-a\alpha^2/\beta)$$

$$c_1'=c+\delta(-\beta c+ab)+\delta\alpha(a+b)=c_1+\delta\alpha(a+b)$$

The projection of $f(\boldsymbol{a})$ on an axis $\boldsymbol{y}=(x,y,z)$ can be written:

$$\boldsymbol{y}f(\boldsymbol{a}')=xa_1+yb_1+\delta y(-\alpha c+a\alpha^2/\beta)+zc_1+z\delta\alpha(a+b)$$

$$\boldsymbol{y}f(\boldsymbol{a}')=\boldsymbol{y}f(\boldsymbol{a})+\delta\left(a\left(z\alpha-y\alpha^2/\beta\right)+z\alpha b-y\alpha c\right)$$

and $Q(\boldsymbol{a})$ is invariant: $\boldsymbol{y}f(\boldsymbol{a}')=L'(\boldsymbol{a})+\delta Q(\boldsymbol{a})$

$L(\boldsymbol{a})$ is linear for $\boldsymbol{a}$: $L'(\boldsymbol{a})=L(\boldsymbol{a})+\delta\left(a\left(z\alpha-y\alpha^2/\beta\right)+z\alpha b-y\alpha c\right)$

$$L'(\boldsymbol{a})=aL_1'+bL_2'+cL_3'$$

with: $L_1'=L_1+\delta\left(z\alpha-y\alpha^2/\beta\right)$; $L_2'=L_2+\delta z\alpha$; $L_3'=L_3-\delta y\alpha$.

Then T and Λ remain invariant. The following is only a calculus.

We calculate l_1', l_2' et l_3', with

$$L(\boldsymbol{a})=a(x-\delta\sigma x+\delta\rho y)+b(\delta\sigma x+y(1-\delta))+zc(1-\delta\beta):$$

where $L_1=x(1-\delta\sigma)+\delta\rho y$; $L_2=\delta\sigma x+y(1-\delta)$; $L_3=z(1-\delta\beta)$

$$\boldsymbol{l}'\boldsymbol{u}=(l_1',l_2',l_3')\boldsymbol{u}=LT'\boldsymbol{u}$$

And:
$$=\left(L_1+\delta\left(z\alpha-y\alpha^2/\beta\right),L_2+\delta z\alpha,L_3-\delta y\alpha\right)\frac{1}{\mu\sqrt{2}}\begin{bmatrix}0 & y\sqrt{2} & z\sqrt{2}\\ \mu & -z & y\\ \mu & z & -y\end{bmatrix}.$$

The results are modified; if $\boldsymbol{l}=(l_1,l_2,l_3)$ is related to 0 and $\boldsymbol{l}'=(l_1',l_2',l_3')$ to α_+

$$l_1'=l_1+\delta\alpha(z-y)/\sqrt{2}$$

$$l_2'=l_2+\delta\alpha\left((z-y\alpha/\beta)y\sqrt{2}-z(z+y)\right)/\mu\sqrt{2}$$

$$l_3'=l_3+\delta\alpha\left((z-y\alpha/\beta)z\sqrt{2}+y(z+y)\right)/\mu\sqrt{2}$$

The following calculations remain the same with these modifications.

- *Calculation for* $\boldsymbol{\alpha}_-$

When $\boldsymbol{a}$ becomes $\boldsymbol{a}''+\boldsymbol{\alpha}_-$ the calculation is the same with the coordinates:

$$a_1''=a+\delta\sigma(b-a)=a_1$$

$$b_1'' = b + \delta(\rho a - b - ac) + \delta(\alpha c - a\alpha^2/\beta) = b_1 + \delta(\alpha c - a\alpha^2/\beta)$$
$$c_1'' = c + \delta(-\beta c + ab) - \delta\alpha(a+b) = c_1 - \delta\alpha(a+b).$$

It remains the problems of domination and frontiers between the various distributions attached at each fixed point.

Remark

We have to go back to the original coordinates. And the solution gives only the probabilities of presence...

8. Conclusions

After this study, we can say, under good conditions, that an EDO is deterministic near the origin of the process, but have random or fixed cycles after a very long time.

With this probabilistic method, we obtain some new results, but we meet also many new difficulties due to the particular steepest descent's method used to study the Plancherel-Rotach's function. Many things have to be lightened

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Cirier, G. (2017) Les itérations polynomiales bornées dans $\mathbb{R}^d$. *Editions. Univ. Européenne. Balti.* (some demonstrations must be improved). https://my.editions-ue.com

[2] Lasota, A. and Mackey, M.C. (1991) Chaos, Fractals and Noise. 2nd Edition, Springer-Verlag, Berlin.

[3] Delabaere, E. and Howls, C.J. (2002) Global Asymptotics for Multiple Integrals with Boundaries. *Duke Mathematical Journal*, **112**, 199-264. https://doi.org/10.1215/S0012-9074-02-11221-6

[4] Plancherel, M. and Rotach, W. (1929) Sur les valeurs asymptotiques des polynômes d'Hermite $H_n(x) = (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2/2}\right)$ *Commentarii Mathematici Helvetici*, **1**, 227-254. https://doi.org/10.1007/BF01208365

[5] Ulam, S.M. and von Neumann, J. (1947) On Combination of Stochastic and Deterministic Processes. *Bull*, **53**, 1120.

[6] Cirier, G. (2021) Asymptotic Frequencies of Bounded Differential Equations. *Journal of Mathematics and Statistics*, **7**, 11.

[7] Lorenz, E.N. (1963) Deterministic Nonperiodic Flow. *Journal of the Atmospheric Sciences*, **20**, 130-141. https://doi.org/10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2